Singularly perturbed elliptic systems modeling partial separation and their free boundaries

Susanna Terracini
joint works with Nicola Soave

Dipartimento di Matematica Giuseppe Peano, Università degli Studi di Torino, (I)

We investigate the asymptotic behavior, as $\beta \to +\infty$, of solutions to competition-diffusion system of type

$$\begin{cases} \Delta u_{i,\beta} = \beta u_{i,\beta} \prod_{j\neq i} u_{j,\beta}^2 & \text{in } \Omega, \\ u_{i,\beta} = \varphi_i \geq 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, 2, 3,$$

where $\varphi_i \in W^{1,\infty}(\Omega)$ satisfy the *partial segregation condition*

$$\varphi_1 \, \varphi_2 \, \varphi_3 \equiv 0 \quad \text{in } \overline{\Omega}.$$

For $\beta > 1$ fixed, a solutions can be obtained as a minimizer of the functional

$$J_\beta(\mathbf{u}, \Omega) := \int_\Omega \big(\sum_{i=1}^3 |\nabla u_i|^2 + \beta \prod_{j=1}^3 u_j^2 \big) \, dx$$

on the set of functions in $H^1(\Omega, \mathbf{R^3})$ with fixed traces on $\partial\Omega$. We prove *a priori* and *uniform in* β Hölder bounds. In the limit, we are lead to minimize the energy

$$J\mathbf{u}, \Omega) := \int_\Omega \sum_{i=1}^3 |\nabla u_i|^2 \, dx$$

over all partially segregated states:

$$u_1 \, u_2 \, u_3 \equiv 0 \quad \text{in } \overline{\Omega}$$

satisfying the given, partially segregated, boundary conditions above. We prove regularity of the free boundary up to a low-dimensional singular set.