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Quantity Competition in Networked Markets

Outflow and Inflow Competition

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12/25/2008

Abstract. This analysis investigates how Cournot competition operates in an economy in which a graph describes the set of feasible trading relationships. A general equilibrium model is presented in which prices and supply chains are simultaneously determined. In such economies primitives dictate whether an individual buys, sells or retails. This paper: (1) provides sufficient conditions for pure strategy equilibrium existence; (2) characterizes equilibrium prices, flows of goods and markups; (3) studies the welfare effects of changes in the network structure; and (4) provides necessary and sufficient conditions for such an economy to become competitive as the trading network grows large. The main results show that: (a) linked players with different marginal rates of substitution may not trade; (b) adding trading relationships may have negative consequences on individual and social welfare; (c) no economy in which a positive amount of goods is resold can ever be competitive; and (d) large well connected economies are competitive.

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1 Introduction

The analysis discusses how scarce resources are allocated in economies with limited trading relationships. In particular the paper investigates how Cournot competition operates in economies in which a graph describes the set of feasible trading relationships. A general equilibrium model is presented in which prices and supply chains are simultaneously determined. Prices must guarantee that each player's demand for goods clears the supply of goods to him. Equilibrium flows of goods in this model endogenously determine whether an individual buys, sells or retails, based on preferences, production possibilities and the position held in the network.

For this model I: (1) provide sufficient conditions for pure strategy equilibrium existence; (2) characterize equilibrium prices, flows and markups; (3) study the welfare effects of changes in the network structure; and (4) provide necessary and sufficient conditions for a networked economy to become competitive as the trading graph grows large. The main contributions of the analysis show that: (a) not all connected players with different marginal rates of substitution do necessarily trade; (b) adding trading relationships may have negative consequences on individual and social welfare; and (c) no economy in which a positive amount of goods is resold can ever be competitive independently of size and structure of the market, since middlemen always command a resale rent; and (d) large well connected economies are competitive. In the model presented competition amongst retailers will undermine, but not eliminate resale rents whenever intermediaries are needed to supply goods.

In a networked economy neither demand nor supply of goods are anonymous, since individuals can transact only with persons they are connected to. In fact individuals value goods not only based on their preferences, but also based on the preferences of individuals with which they can trade since goods can be resold in such markets. Traders selfishly rout flows in order to maximize their well being. Selfish behavior results in price discrimination of individuals that hold different positions in the trading network. Resale at a markup is pervasive. Such phenomenon does not only hinge on the scarcity of trading partners, but also on different prices that reign throughout the economy in equilibrium. Even when all traders are connected equilibrium resale persists and goods are traded at different prices by individuals in the economy.

Another distinctive feature of networked quantity competition models is that not all players with different marginal rates of substitution do necessarily trade even if linked. Retailers face a double marginalization problem since price distortions affect both the price of the goods they buy and the price of the goods they sell. Moreover such distortions are significant even between any two connected individuals who elect not to trade. In fact individuals with lower willingness to pay for a good do not necessarily benefit from selling to players with a higher willingness to pay, since such sale may raise the price paid for the goods purchased. This prediction differs from a standard Cournot setup in which any two players with different marginal utilities always elect to trade.

Adding trading relationships in these markets does not necessarily improve the social welfare in the economy. In fact since players price discriminate when choosing their flows, adding a link may lead goods to flow to markets in which the demand for them is lower in order to elicit higher prices in local markets in which the goods are most desired. Additionally increasing the set of trading partners of an individual may decrease his individual welfare even though he has the option not to trade on the newly added link. Indeed if trades on the new link raise demand of one of the two individuals, price discrimination by suppliers of that individual may lead to a decrease in amount of goods sold to him and to a drop in welfare.

For a networked economy to become competitive not only has the set of players to grow large, but the set of trading relationships has to grow as well. Sufficient conditions on the market structure are provided to guarantee that an economy becomes competitive as it grows large. It is shown that if an economy ever becomes competitive, as it grows large the amount of goods resold must vanish. In such economies if intermediaries are required to distribute goods they command a rent in markets of any size. Because resale only occurs at positive markups, no retail can occur in a competitive market when individuals compete on quantities. The limiting economy can be fully characterized even when it does not become competitive. In such instances a retail sector can persists even with infinitely many producers, retailers and consumers. Even though equilibrium markups and gross margins decrease as the economy grows large, they do not vanish in the retail sector of any limiting economy, since individuals become price takers as suppliers, but not as buyers. Social welfare increases as all local markets become more competitive and is bounded above by its limiting value.

Sufficient conditions are provided to ensure that a pure strategy equilibrium of the outflow competition model exists. As in most quantity competition models such conditions can be relaxed when the economy grows large. Weaker conditions are provided for large economies.

The consequences of allowing individuals to choose how much to buy are also explored. Similar results hold even though the distribution of rents differs. More rents flow to buyers rather than sellers and social welfare is generally higher than when players choose how much to sell. Again the equilibria will not be perfectly competitive in economies that are small or with significant intermediation, because of the price distortions inherent in such model.

Literature Review

A vast literature documents relevance of social networks in economics. A detailed and extensive survey about these studies is presented by Jackson in [14]. Several papers have been devoted to the analysis of networked markets. A recent literature discusses the exchange of goods in two-sided networked markets. In such models a market for a specified good has on one side producers and on the other side consumers. The network structure describes how the individuals on the two sides of the market can trade. Several papers discuss such setup and differ in the way trade on the network is modeled. Kranton and Minehart 2001 [21] model the competition amongst suppliers, by having them hold simultaneous ascending price auctions to which consumers linked to them can participate. The paper discusses network formation and shows that link patterns which lead to efficient outcomes are equilibria of the network formation game. Corominas-Bosch 2004 [6] models the trade on each link as resulting from a non-cooperative bargaining game taking place between the two connected individuals. The paper provides conditions on the network structure for the subgame perfect equilibrium of the bargaining game to coincide with the Walrasian outcome. Blume, Easley, Kleinberg and Tardos 2007 [3] study a networked two-sided market in which all trades are mediated by middlemen sitting on each link. In their model middlemen can make take it or leave it offers to both sides of the market thus capturing all the surplus and implementing the efficient allocation. In all these models the position that an individual has on the network exogenously determines whether he buys, sells or retails. The quantity competition model presented here differs from the basic setup of this literature, because the market is not two-sided and because equilibrium dictates which roles individuals have on the supply chains.

Kakade, Kerns and Orthiz [19] study the competitive equilibria of a networked market. They provide conditions for the existence of a competitive equilibrium in a networked market and develop polynomial algorithms for its computation. Individuals in such economies act as islands that trade competitively within themselves and are linked to other islands via a graph. As in quantity competition model proposed the flow of goods in the economy is endogenous. In their setup however players take prices of flows as a given.

Finally a related literature takes flows are exogenously determined and considers how selfish individuals would rout flows on a network with congestion to maximize their profits. Such literature usually takes a Bertrand approach to model how individuals on the network compete to rout flows. This literature has been developed to study how internet providers should price stream of information and to evaluate the welfare effect of such routing. Chawla and Roughgarden 2007 [8] study how oligopolistic routers should sell bandwidth if they are faced with capacity constraints. In this model the sources and the sinks of flows of information on the network are exogenous and routers located on the network compete à la Bertrand to supply bandwidth. Bounds on the equilibrium price correspondence are found. Flow patterns are characterized and welfare effects of this type of competition are derived. Acemouglu and Ozdaglar 2007 [2] study competition amongst profit-maximizing service providers in communication parallel-serial networks. Again sources and sinks of the flows on the network are exogenously determined. In this model however the capacity constraint on the flows is replaced with a latency function defining how long it takes to rout a certain amount of flows on a given edge. The paper shows that the welfare loss from competition amongst providers in a parallel-serial network may be large, because providers on the same path cannot coordinate their actions. They provide conditions on the latency function under which the welfare loss can be bounded. In the quantity competition model discussed by the paper wholesalers play a role similar to the one of the providers, but their role is endogenously determined by equilibrium. Also the capacity constraints are replaced by constraints on the flows that can originate from each individual.

Roadmap

Section 2 is devoted to the outflow competition model. It presents the basic setup (section 2.1), the model of oligopolistic outflow competition (section 2.2) and some examples of how such markets operate (section 2.3). Section 2.4 provides conditions for pure strategy equilibrium existence. The analysis proceeds with the characterization of equilibrium prices and flows (section 2.5), with the discussion of the welfare effects of changes in the network structure (section 2.6) and with the study of the complete network economy (section 2.7). Section 3 discusses competition and persistent price distortions in arbitrarily large economies and the welfare effects of replicating an economy. Section

4 presents the inflow competition model. All the proofs can be found in appendix.

2 Outflow Competition in Networked Markets

2.1 Basic Setup and Constrained Efficiency

Consider a pure exchange in which trades can occur only amongst persons that know each other. The economy consists of: a set of consumers, an undirected graph describing the "who knows who" relationships amongst consumers, and endowments and preferences for each player.

Let an undirected graph $G = (V, E)$ describe the relationships in the economy. The vertices of such graph V are all consumers in the economy and the edges of the graph E describe the feasible trading relationships. Therefore agent i and agent j can trade if and only if $ij \in E$. The assumption that the graph is undirected implies that if i can supply to j then j can supply to i. Thus $ij \in E$ if and only if $.ji \in E$. The set of neighbors of player $i \in V$ consist of all players linked to i and is denoted by:

$$V(i) = \{j \in V | \, ij \in E\}$$

There are two goods in the economy. For convenience call them consumption good q and money m. Each consumer $i \in V$ is endowed with a finite amount of the consumption good Q_i and with a large amount of money. Let Q denote the aggregate endowment of consumption good. Assume that preference are separable in the two goods, linear in money holdings and strictly increasing, strictly concave and three times continuously differentiable in the consumption good:

$$u_i(q) + m$$

Assumption A1 *For any player $i \in V$ assume that u_i is three times continuously differentiable on $[-Q, \infty)$. Additionally assume that $u_i' > 0$ and that $u_i'' < 0$.*

Utility on consumption is defined also on short trading positions, even though short sales are forbidden. The differentiability assumption on $[-Q, 0)$ interval can be easily relaxed. But different assumptions on the consequences of remaining with a short consumption trading position have to be invoked. The existence section discusses such issue in detail.

The flow of consumption good from individual i to individual j is denoted by q_j^i. Since trades can occur only amongst consumers that know each other, $q_j^i = 0$ whenever $ji \notin E$. The price of the flow q_j^i is denoted by p_j^i. The feasibility constraints of consumer i guarantee that i holds a nonnegative amount of consumption good after the trades occur:

$$q_i = Q_i + \sum_{k \in V(i)} [q_i^k - q_k^i] \geq 0$$

The non-negativity constraint on monetary holdings is neglected. It is assumed that money endowments are large enough for such constraint never to bind. Therefore since utility is linear in money, the monetary endowment will not affect decisions. The welfare of individual i given flows

$\mathbf{q} \in \mathbb{R}_+^E$ and prices $\mathbf{p} \in \mathbb{R}_+^E$ can be written as:

$$w_i(\mathbf{q}|\mathbf{p}) = u_i(q_i) + \textstyle\sum_{k \in V(i)} [p_k^i q_k^i - p_i^k q_i^k]$$

A player is said to resell good whenever he buys and sells a positive amount of goods. The amount of good resold by individual i is the total amount of goods that are bought to be sold. Thus it is defined by:

$$R_i = \min \left\{ \textstyle\sum_{k \in V(i)} q_i^k, \sum_{k \in V(i)} q_k^i \right\}$$

The interpretation given to the model is that of a pure exchange economy and abstracts from production. However, such simplification is made for expositional clarity alone. Indeed it is possible to view some suppliers in the network as firms producing consumption good through a process with capacity Q_i and cost function:

$$c_i(q) = u_i(Q_i) - u_i(Q_i - q)$$

In the pure exchange interpretation the opportunity cost of selling a good, simplifies to the value of forgone consumption.

Given the relationships in G an allocation is said to be *constrained efficient* if it maximizes the sum of individuals' welfare given set of feasible transfers. Because all consumers value money equally, all that matters for efficiency is the allocation of consumption good amongst consumers. Therefore any constrained efficient allocation solves:

$$\mathbf{q}^* \in \arg\max_{\mathbf{q} \in \mathbb{R}_+^E} \textstyle\sum_{i \in V} u_i(q_i) \quad \text{subject to}$$

$$\textstyle\sum_{i \in V} (q_i - Q_i) = 0 \quad \& \quad q_i \geq 0 \quad \text{for any } i \in V$$

If the solution to this problem lies in the interior of the domain, the constrain efficient allocation equalizes the marginal utility of consumption of any two consumers i and j that belong to the same component of the networked economy G.[1] That is $u_i'(q_i^*) = u_j'(q_j^*)$ if there is a path from i to j. If instead no feasible profile of flows can equalize marginal utilities across individuals belonging to the same component, the non-negativity constraint on consumption limits the extent of feasible redistribution and pins down the allocation. If the networked economy is connected any constrained efficient allocation is also efficient, since goods can be routed to all locations. But if the economy is not connected redistribution can only equalize marginal rates of substitution within each component of a graph, but not across components. Constrained efficiency always pins down an allocation. But, it does not pin down the flows of goods in the economy unless further assumptions are invoked.[2]

[1] Any maximal connected subgraph of G is a component of G. See [4].

[2] Assumption about the network structure and about cycling goods may identify the flows as well.

2.2 Outflow Competition

In this economy each consumer can sell to all other individuals he knows some of the consumption good that he is endowed with or has bought from others. Competition proceeds as follows. Initially each individual selects how much consumption he wants to supply to each neighbor. Then given the chosen flows market prices are determined at each node so that the demand of goods clears the supply of flows. A buyer will pay all of his inflows equally at the marginal value of the last unit bought, which is the marginal value of the last unit consumed. The inverse demand function at the node $i \in V$ satisfies for any $j \in V(i)$:

$$p_i^j(\mathbf{q}) = p_i(q_i) = u_i'(q_i) = u_i'(Q_i + \textstyle\sum_{k \in V(i)}[q_i^k - q_k^i]) \tag{1}$$

Such prices arise because, given the number of units for sale at each node, no higher price would clear the market.

The price paid by consumer i not only decreases when his inflows increase, but also increases when his outflows increase. Therefore increasing the amount of goods supplied to a neighbor leads both to an increase in the price paid for all units purchased and to a decrease in the price received for all units sold to that neighbor. The concavity of the utility map implies that $\partial p_i(q_i)/\partial q_i^j \leq 0$ and $\partial p_i(q_i)/\partial q_j^i \geq 0$ for any $j \in V(i)$. Changes in other flows on the network do not affect the price paid by i so long as his inflows and outflows remain unchanged.

The pricing equation implicitly assumes that each supplier can commit to deliver a flow of output to known buyers. Given the quantities delivered at each local market if suppliers were to compete locally on prices, equilibrium prices would still be determined by equation 1 since no supplier could benefit from a unilateral deviation in the price offered. A reduction in price would not affect the quantity sold, while an increase in the price reduces revenues because of falling sales. This remark was first made by Kreps and Sheinkman while studying Bertrand competition with quantity pre-commitment in [22]. The implicit assumption made favors suppliers when the price is determined in each local market, because the demand curve can be used to clear markets. Section 4 explores the consequences of the alternative setup in which buyers commit to inflows bought.

Suppliers take into account the effects that their flows have both on the prices they get for each unit supplied and on the price they pay for each unit bought. The problem of supplier $i \in V$ is to choose which quantities to supply to each of his neighbors given price effects on inflows and outflows:

$$\begin{aligned} \max_{q_k^i \geq 0 \text{ for } k \in V(i)} \quad & u_i(q_i) + \textstyle\sum_{k \in V(i)} \left[p_k(q_k) q_k^i - p_i(q_i) q_i^k \right] \\ \text{s.t.} \quad & q_i = Q_i + \textstyle\sum_{k \in V(i)}[q_i^k - q_k^i] \geq 0 \end{aligned}$$

If the nonnegativity constraint on consumption of player $i \in V$ does not bind, the first order condition for q_j^i in the networked economy requires that:

$$p_j(q_j) - u_i'(q_i) + \frac{\partial p_j(q_j)}{\partial q_j^i} q_j^i - \frac{\partial p_i(q_i)}{\partial q_j^i} \textstyle\sum_{k \in V(i)} q_i^k \leq 0$$

Such condition holds with equality whenever a positive quantity is supplied $q_j^i > 0$. It states that the marginal benefit of selling an additional unit, the price, must offset the marginal cost of forgone consumption, the marginal decrease of the price of that outflow and the marginal increase of the prices of all inflows. The first wedge it due to the fact that i is a Cournot supplier of j. While the second wedge is due to the fact i is a monopsonistic buyer at his market.

A necessary condition for a trade from i to j to occur $q_j^i > 0$ is that price player i receives for trading an additional unit to j exceeds the marginal benefit of consuming that unit.

$$p_j(q_j) > u_i'(q_i)$$

Since consumers account for price distortions when choosing their flows, the supply of consumption good is curtailed in each local market. Hence worst use for an individual of goods he owns is consumption and not trade. Thus when clearing each local market, no buyer would pay more than such benefit on the last unit he purchases. Pricing equation 1 implies that the marginal benefit of the last unit consumed is exactly the price each buyer pays for his inflows.

For clarity's sake let the individual welfare that consumer $i \in V$ derives from a profile of flows $\mathbf{q}$ be defined by:

$$w_i(\mathbf{q}) = u_i(q_i) + \textstyle\sum_{k \in V(i)} \left[u_k'(q_k) q_k^i - u_i'(q_i) q_i^k \right]$$

In what follows the expression *outflow equilibrium* will be used to refer to a pure strategy Nash equilibria of the outflow competition model. Specifically:

Definition 1 *A profile of flows $\mathbf{q} \in \mathbb{R}_+^E$ is an outflow equilibrium if for any player $i \in V$:*

$$\mathbf{q}^i \in \arg \max_{\overline{\mathbf{q}}^i \in \mathbb{R}_+^{V(i)}} w_i(\overline{\mathbf{q}}^i, \mathbf{q}^{-i}) \quad s.t. \quad q_i \geq 0$$

2.3 A Four Player Example

Before discussing how outflow competition in a networked market affects the allocation of goods and prices, consider a four player example $V = \{a, b, c, d\}$. Utility from consumption is a constant relative risk aversion map for any player $i \in V$:

$$u_i(q) = q^{1/2}$$

Consumption good endowments are $Q_a = 3$, $Q_b = 1$ and $Q_c = Q_d = 0$. For any connected trade network constrained efficiency requires that all consumers split the consumption good equally. The social welfare at such allocation is maximal and equal to 4. Equal sharing of consumption however is not an equilibrium outcome when individuals compete on outflows, even if all trades are feasible.

Consider the fully connected economy. Because players c and d are identical and in a symmetric position, no trade occurs amongst them and they are supplied in equal amounts by a and b. Since no trade occurs amongst them, removing the cd edge from the relationship network would not affect equilibrium flows. In the outflow equilibrium of this economy the two well endowed individuals supply to both consumers with no endowment, but in different amounts. Additionally player b

buys some consumption good from player a at a lower price in order to resell most of it to the other two at a markup. For player a it is revenue-maximizing to supply such goods even though they are used to compete against him for the supply of the remaining players. Such phenomenon is common in a networked market, because if an individual was not sell to players that compete against him, he would face a tougher competition in the supply of the remaining player. The equilibrium flows of this economy are reported in table 1 and are depicted in figure 1.

B\S	a	b	c	d	Tot
a	-	0	0	0	0
b	0.397	-	0	0	0.397
c	0.479	0.144	-	0	0.623
d	0.479	0.144	0	-	0.623
Tot	1.355	0.288	0	0	

Var	p	q	u
a	-	1.645	2.078
b	0.475	1.109	1.047
c	0.633	0.623	0.395
d	0.633	0.623	0.395
Tot	-	4.000	3.914

TABLE 1: On the left the flow matrix, in columns sellers and in rows buyers. On the right equilibrium prices paid, consumption and welfare.

Well endowed consumers restrict supply to the other players in the game in order to maximize their gains from trade. Consumers c and d are under-provided in equilibrium. Thus the allocation is inefficient. Social welfare is equal to 3.91. The price paid by consumers c and d for each unit of consumption bought is 0.63. Such price is higher than the price charged by consumer a to b on the units he purchases from him, 0.47. Player b imposes a 34% markup on all the units he resells. The example shows how outflow competition can lead to equilibrium resale of goods with positive profit margins even when all consumers know each other.

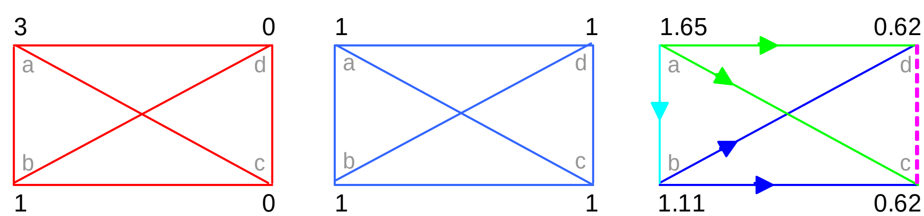

FIGURE 1: On the left the economy, at the center efficiency, on the right Cournot outcomes. Edges on which no flow transits are dashed.

If consumers a and b did not know each other the previous allocation could not obtain in equilibrium. Severing such a link from the network increases consumption of every player, but for player b. The equilibrium flows of this economy are reported in table 2 and depicted in figure 2.

B\S	a	b	c	d	Tot
a	-	-	0	0	0
b	-	-	0	0	0
c	0.543	0.110	-	0	0.653
d	0.543	0.110	0	-	0.653
Tot	1.086	0.220	0	0	

Var	p	q	u
a	-	1.914	2.055
b	-	0.780	1.019
c	0.619	0.653	0.404
d	0.619	0.653	0.404
Tot	-	4.000	3.883

Table 2: On the left the flow matrix, in columns sellers and in rows buyers. On the right equilibrium prices paid, consumption and welfare.

Players c and d receive more consumption good at a lower price and are thus better off. But consumers a and b are worse off, because the direct competition between them reduces their rents. The equilibrium is still inefficient. Social welfare decreases further to 3.88. A unique price is paid by consumers c and d for each unit bought in this economy, namely 0.62. This price coincides with the Cournot equilibrium price for the economy without network.

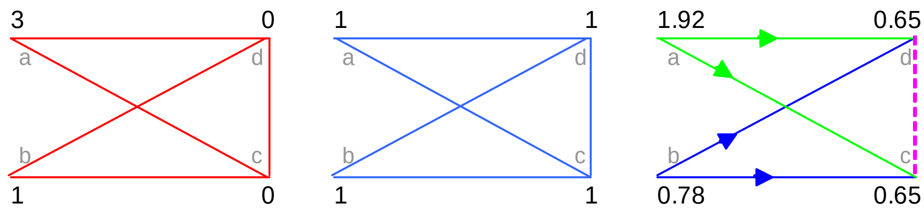

Figure 2: On the left the economy, at the center efficiency, on the right Cournot outcomes. Edges on which no flow transits are dashed.

If also link between player b and player d is removed from the relationship network, consumer b remains with a unique neighbor. Players a and b remain with two relationships each, while c is still connected to all. In this economy player a still sells to both consumers c and d and player b only sells to c. However, even though consumer c ends up with significantly more consumption than d, he prefers not to sell anything to d in equilibrium. Indeed because selling to player d would raise the price of all inflows purchased by c, player c prefers to forgo the revenues he could make from a sale. In these markets it is quite common for connected player with different marginal rates of substitution to prefer not to trade, because a commitment not to resell may significantly reduce the price of the goods purchased. Equilibrium flows and prices are reported in table 3 and depicted in figure 3.

B\S	a	b	c	d	Tot
a	-	-	0	0	0
b	-	-	0	-	0
c	0.561	0.138	-	0	0.699
d	0.488	-	0	-	0.488
Tot	1.049	0.138	0	0	

Var	p	q	u
a	-	1.951	2.081
b	-	0.862	1.011
c	0.598	0.699	0.418
d	0.716	0.488	0.349
Tot	-	4.000	3.860

Table 3: On the left the flow matrix, in columns sellers and in rows buyers. On the right equilibrium prices paid, consumption and welfare.

Since player c has two suppliers, while player d has only one that is active, the price c pays for each unit bought is lower than the price paid by d. Competition amongst suppliers reduces prices and increases the quantity supplied. Player a supplies more consumption good to the competitive market than to the one in which he is a monopolist. Social welfare in this economy is lower than in the previous two examples at 3.86.

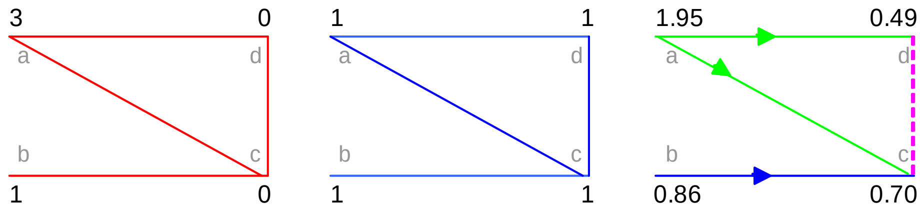

FIGURE 3: On the left the economy, at the center efficiency, on the right Cournot outcomes. Edges on which no flow transits are dashed.

In the final example the network structure is such that all individuals know player c and no other relationship belongs to the market. In such an economy players a and b sell to c, who with the units bought supplies d. The equilibrium flows of this economy are reported in table 4 and depicted in figure 4.

B\S	a	b	c	d	Tot
a	-	-	0	-	0
b	-	-	0	-	0
c	0.639	0.137	-	0	0.776
d	-	-	0.073	-	0.073
Tot	0.639	0.137	0.073	0	

Var	p	q	u
a	-	2.361	1.918
b	-	0.863	1.011
c	0.596	0.703	0.511
d	1.850	0.073	0.135
Tot	-	4.000	3.574

TABLE 4: On the left the flow matrix, in columns sellers and in rows buyers. On the right equilibrium prices paid, consumption and welfare.

Player c's markup on the units the resells to d is of 210%. Resale occurs despite such markup, because c has access to all suppliers. The amount of resale is constrained by the effects that such a trade bears on the prices paid by player c to his suppliers. Social welfare drops significantly to 3.57. Consumers a and d are negatively affected by the change from the previous environment. Consumer c gains from the previous situation, because he gets monopoly on four and because all the supply in the market directed towards him.

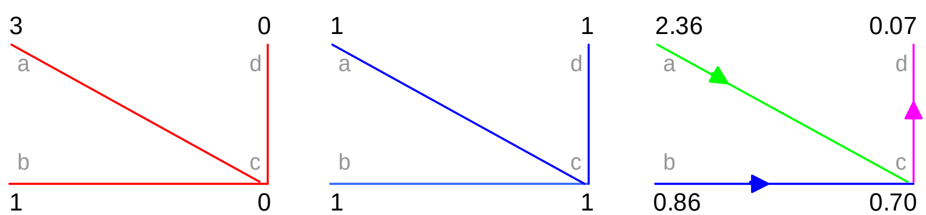

FIGURE 4: On the left the economy, at the center efficiency, on the right Cournot outcomes.

2.4 Outflow Equilibrium Existence

In this section sufficient conditions for the existence of an outflow equilibrium are introduced. It is shown that many commonly used families of utility functions grant equilibrium existence. Conditions for uniqueness of the outflow equilibrium are discussed. The section proceeds as follows first the notion of complementarity problem is defined and conditions for the existence and uniqueness of a solution to such problem are reviewed. Then it shown that the necessary first order conditions of the outflow equilibrium define a complementarity problem. Sufficient conditions on the primitives of the problem are provided to guarantee that each solution to that complementarity problem is an outflow equilibrium. Finally conditions for existence and uniqueness are derived exploiting the specific nature of the problem at hand. Section 6.1 of the appendix discusses further results on existence.

The proof of existence of an outflow equilibrium provided here is closely related to a powerful result about the solutions of complementarity problems which needs to be introduced. A general complementarity problem is defined as follows:

Complementarity Problem *Find* $\mathbf{y} \in \mathbb{R}^N$ *such that for any* $n \in N$:

$$y_n \geq 0 \quad f_n(\mathbf{y}) \geq 0 \quad y_n f_n(\mathbf{y}) = 0$$

Sufficient conditions for existence of the solution to a complementarity problem were found by Karamardian in [17]. Necessary and sufficient conditions for the uniqueness of a solution were provided by Kolstad and Mathiensen in [20]. To guarantee existence of the solution to a complementarity problem, a relevant boundary condition needs to be satisfied. It states that:

Boundary Condition BC *There exists a non-empty compact set* $C \subset \mathbb{R}_+^N$ such that for any $\mathbf{x} \in \mathbb{R}_+^N \backslash C$ there exists $\mathbf{y} \in C$ such that:

$$\sum_{i \in N} (y_n - x_n) f_n(\mathbf{x}) < 0$$

Define $T(\mathbf{x}) = \{n \in N | x_n > 0\}$ to be the set of active indices in the problem and let $J_T(\mathbf{x})$ denote the principal minor of the Jacobian of f associated to the indices in $T(\mathbf{x})$. Further suppose that if $T(\mathbf{x}) = \emptyset$ then $\det J_T(\mathbf{x}) = 1$. Kolstad and Mathiensen prove the following theorem:

Theorem 1 *If* $f : \mathbb{R}_+^N \to \mathbb{R}^N$ *is continuously differentiable and satisfies* ***BC*** *then:*
(1) the complementarity problem has a solution
If at each solution $\mathbf{y}^*$ *of the complementarity problem* $y_n^* = 0$ *implies* $f_n(\mathbf{y}^*) > 0$*, then:*
(2) if at any solution $\det J_T(\mathbf{x}^*) > 0$ *there if only one solution*
(3) if there is only one solution then $\det J_T(\mathbf{x}^*) \geq 0$

There is a close connection between the outflow equilibria and the solution of a particular complementarity problem. Indeed, let $\mu_i \geq 0$ denote the multiplier on the nonnegativity constraint for consumption of player $i \in V$ and define the function f as follows:

$$f_j^i(\mathbf{q}, \boldsymbol{\mu}) = -\partial w_i(\mathbf{q})/\partial q_j^i + \mu_i \quad \text{and} \quad f_i(\mathbf{q}, \boldsymbol{\mu}) = q_i$$

Then the system KKT first order conditions characterizing the outflow equilibria defines following complementarity problem:[3]

[3]Karush Kuhn Tucker first order conditions.

Complementarity Problem CP *Find* $(\mathbf{q}, \boldsymbol{\mu}) \in \mathbb{R}^{E \cup V}$ *such that for any* $i \in V$ *and* $j \in V(i)$:

$$\begin{cases} q_j^i \geq 0 & f_j^i(\mathbf{q}, \boldsymbol{\mu}) \geq 0 & q_j^i f_j^i(\mathbf{q}, \boldsymbol{\mu}) = 0 \\ \mu_i \geq 0 & f_i(\mathbf{q}, \boldsymbol{\mu}) \geq 0 & \mu_i f_i(\mathbf{q}, \boldsymbol{\mu}) = 0 \end{cases}$$

Since such conditions are necessary for optimality, any outflow equilibrium satisfies them. Also let $\mathbf{CP}(i)$ define the problem of finding $\mathbf{q}^i \in \mathbb{R}^{V(i)}$ such that for any $j \in V(i)$ the aforementioned conditions hold. This reduced problem is closely related to the best reply correspondence of player $i \in V$.

In order to guarantee that any solution to the **CP** problem is an outflow equilibrium further conditions on preferences need to be invoked. Pseudo-concavity of individual welfare with respect to outflows suffices. Such assumption however is not easy to reconcile with conditions on preferences. Section 6.1 explores this case. Different assumptions on the cost to supply outflows and on the revenues made by suppliers in each market are explored here. Such assumption have direct implications on the shape of preferences. In the outflow model the *marginal cost* to individual $i \in V$ of suppling outflows if he has q_i units and has bought q_i° units, is given by:

$$u_i'(q_i) - u_i''(q_i) q_i^\circ$$

The *revenue* to individual $i \in V$ from suppling q_j^i units to market $j \in V(i)$ when j has already $q_j$units to consume, is given by:

$$u_j'(q_j + q_j^i) q_j^i$$

The connection between an outflow equilibrium and solution of the **CP** problem is made explicit by the following lemma:

Lemma 2 *If the costs of suppling outflows are convex and if revenues are pseudo-concave and concave if increasing, an allocation is an outflow equilibrium if and only if solves **CP**.*

Pseudo-concavity requires profits to be quasi-concave and to have no inflection points with horizontal tangents.[4] Both the assumption on the costs and that on the revenues have implications on the shape of the utility map. Sufficient conditions on the utility function to satisfy the cost and revenue assumption are provided in the next results. The first proposition requires marginal utility of consumption to be bounded on the interval $[-Q, Q]$. In this case if the marginal utility is convex and if the absolute risk aversion does not decrease too much, the desired conditions on costs and revenues hold.

Lemma 3 *If assumption A1 holds,* $u_i''' \geq 0$ *and the coefficient of absolute risk aversion of player satisfies for any player* $i \in V$*:*

$$\frac{\partial}{\partial q}\left(-\frac{u_i''(q)}{u_i'(q)}\right) \geq -\left(\frac{u_i''(q)}{u_i'(q)}\right)^2 \quad \text{for any } q \in [-Q, Q]$$

[4]The differentiable function $f : \Gamma \to \mathbb{R}^n$ is said to be pseudo-concave $\nabla f(x)(y - x) \leq 0$ implies $f(y) \leq f(x)$ for any $x, y \in \Gamma$.

Then the costs are convex, the revenues are pseudo-concave and concave if increasing.

These conditions are sufficient, but not necessary. The assumptions on the utility map imply that any individual is willing to pay some money for the units purchased even if he cannot meet the consumption budget $q_i \geq 0$. Having units players that value units also in short positions complicates the proof of existence for sake of generality. Section V of the appendix shows that proposition 3 holds even if all the conditions on the preferences hold strictly, but only on $[0, Q]$ and instead one assumes that:

$$w_i(\mathbf{q}) = 0 \quad \text{if} \quad q_i < 0$$

The class of maps satisfying these assumptions includes all constant absolute risk aversion maps. All increasing absolute risk aversion maps for which $u_i''' \geq 0$ satisfy the assumptions as well. Some decreasing absolute risk aversion maps also satisfy the desired assumptions. For instance the log utility function satisfies the condition on the absolute risk aversion at equality on $(0, \infty)$ and provides a bound on the set of admissible utility maps. Such map however satisfies assumption A1 only if on the positive orthant. Therefore only shifting the map can insure that a pure strategy equilibrium exists.

The second proposition instead addresses functional forms for which assumption A1 cannot hold since the utility map is not defined on the negative orthant. To such class belong most CRRA functionals. With such preferences shifting the map so that the utility is defined on the entire choice domain may suffice to guarantee existence. The next claim states that any shifted weakly increasing relative risk aversion map meets the assumptions on cost and revenues.

Proposition 4 *For any player $i \in V$ consider a map $\overline{u}_i$ that satisfies assumption A1 on $\mathbb{R}_+$ alone, with $\overline{u}_i''' \geq 0$ and with coefficient of relative risk aversion that satisfies:*

$$\frac{\partial}{\partial q}\left(-\frac{q\overline{u}_i''(q)}{\overline{u}_i'(q)}\right) \geq 0 \quad \text{for any } q \in [0, Q]$$

Let $u_i(q_i) = \overline{u}_i(c_i + q_i) - \overline{u}_i(c_i)$ denote utility of player i for some constant $c_i \geq Q$. Then the costs of suppling outflows are convex, the revenues are pseudo-concave and concave whenever increasing.

The shift is necessary to insure that the maps are well defined for any possible profile of flows. Shifted constant relative risk aversion maps always meet this condition.

The problem with utility being defined only on the positive orthant is that suppliers cannot assess the consequences of leaving any one of their buyers with a negative amount of consumption good, even though such event will never occur in equilibrium. But, such consequences are important to determine their course of action. Moreover if the marginal utility was infinite at zero consumption, any buyer could be exposed to an infinite loss. Such scenarios are irrelevant for allocations that satisfy the following constraint:

$$q_i - \max_{k \in V(i)} \{q_i^k\} \geq 0$$

Such constraint guarantees that no player ever holds a short position if any one of his suppliers deviates and chooses not to supply. Thus it guarantees that payoffs of all players are well defined. If the such constraint holds, proposition 4 holds without any shift to the utility maps. If such constraint is imposed on individual choices existence can be guaranteed for any weakly increasing relative risk aversion map, as pointed out in proposition 24 in appendix. If neither the constraint nor the conditions on preferences are imposed it is possible all equilibria of the Cournot game are in mixed strategies and that no pure strategy equilibrium exists.

In order to state the existence result, define outflows to be *bounded* if there exists a non-empty compact set $C \subset \mathbb{R}_+^E$ such that for any $\mathbf{q} \in \mathbb{R}_+^E \backslash C$:

$$\partial w_i(\mathbf{q})/\partial q_j^i < 0 \text{ for any } ij \in E$$

Notice that if such condition is met the boundary condition of the **CP** problem will be satisfied. In fact the further requirements on the multipliers are automatically satisfied when this condition holds. Also define an outflow equilibrium $\mathbf{q} \in \mathbb{R}_+^E$ to be *non-degenerate* if:

$$q_j^i = 0 \text{ implies } \partial w_i(\mathbf{q})/\partial q_j^i < 0$$

Given these definitions and the previous observations all that remains to be done is to apply Kolstad and Mathiensen's result to this setup. It follows that:

Theorem 5 *If A1 holds and if:*
(1) marginal utility satisfies $\lim_{q_i \to -\infty} u_i'(q_i) = \infty$ *and endowments are positive*
(2) costs convex, revenues are pseudo-concave and concave if increasing
Then best reply functions are single valued and an outflow equilibrium exists.
If all equilibria are non-degenerate:
Then there is a unique equilibrium if and only if $\det J_T(\mathbf{q}, \boldsymbol{\mu}) > 0$ *at any equilibrium*

Where $J_T(\mathbf{q}, \boldsymbol{\mu})$ is the leading minor of the Jacobian of the system associated to the active variables, flows and multipliers. The theorem provides sufficient conditions for the existence of a pure strategy equilibrium of the outflow model. The assumption that endowments are positive is required only to insure that the problem is well defined when short selling constraints bind. If players can hold short positions such assumption can be dispensed. The condition for equilibrium uniqueness provides a test to verify whether an equilibrium is unique. It it is conjectured that assumption (2) implies that $\det J_T(\mathbf{q}, \boldsymbol{\mu}) > 0$ at any equilibrium. Such result however has been proven just for economies with two or three players for the moment.

Since conditions on preferences that guarantee that any solution to the **CP** problem is an outflow equilibrium meet the boundary condition in (1), they suffice to prove existence. Therefore any economy in which the utility maps of all players satisfy the assumptions in lemmata 3 and 4 have an outflow equilibrium and provide a test for uniqueness. In fact, it must be that:

Corollary 6 *If the assumptions of either lemma 3 or lemma 4 hold and if endowments are positive then an outflow equilibrium exists.*

The requirement that endowments be positive guarantees that the problem is well defined and bounded, but is still in general stronger than necessary. This concludes the discussion of outflow equilibrium existence.

2.5 Basic Properties of Outflow Equilibria

This section presents results about equilibrium flow pattern and markups in economies with a finite number of players. An immediate implication of outflow competition model is that the price effects cause flows to move in only one direction on each link. If in an outflow equilibrium consumer i supplies a positive amount of consumption good to j, then j does not supply any to i. Indeed player i supplies j only if:

$$u_j'(q_j) = p_j(\mathbf{q}) > u_i'(q_i)$$

Thus j cannot supply i since reverse inequality would have to hold as well. At most $|E|/2$ flow are positive in equilibrium. Without price distortions any two linked consumers with different marginal rate of substitution would have an incentive to trade. Even if the price distortions were to vanish for arbitrarily low outflows, a trade would always occur between any two linked consumers with different marginal rates of substitution, as in the Cournot model.

In the outflow competition model however distortions on inflow prices do not vanish as outflows decrease. Thus any individual purchasing goods will sell to a neighbor only if the gains from trade can compensate for the monopsony price distortion on inflows. Indeed equilibrium markups remain positive even as $q_j^i \to 0$ for any individual reselling a positive amount of consumption good:

$$p_j(q_j) - p_i(q_i) = -u_j''(q_j)q_j^i - u_i''(q_i)\sum_{k\in V(i)} q_i^k \geq -u_i''(q_i)\sum_{k\in V(i)} q_i^k > 0$$

Resale of consumption good is pervasive in these economies. There are two motives for resale. The more immediate motive is that the trading network has a limited number of links that can be used to transfer goods. The second motive is that each seller has an incentive to price discriminate buyers by selling to the highest number of individuals available at different prices, provided that he makes gains on each trade. Which explains why even a fully connected economy can display equilibrium resale (as argued in section 2.7).

The observation that resale markups are always positive has two immediate implications. The first is that individuals that are linked and have different marginal rates of substitution do not necessarily trade in equilibrium. In fact if the gains from retail may too low to overcome the monopsonistic distortion, no trade will occur. Buyers with similar marginal rates of substitution usually refrain from trading even if connected. A simple economy in which this phenomenon occurs was reported in the third example of section 2.3. The other straightforward consequence of goods being resold at strictly positive markups is that flows of goods never cycle on the network. In fact because the marginal utility of consumption grows along the supply chain, it can never be that an individual buys some of the units he previously sold:

Remark 7 *The set of active trading links* $T(\mathbf{q}) = \left\{ij \in E | q_j^i > 0\right\}$ *contains no cycle in any outflow equilibrium* $\mathbf{q} \in \mathbb{R}_+^E$.

Since goods do not cycle in such markets the flows of goods start at some node (*sources*) to end at some different node (*sinks*). The flow pattern however can have more than one source and/or sink in equilibrium. It can be shown that any individual having lower marginal utility than all his neighbors is a source and that any individual with higher marginal utility than all his neighbors is a sink. Specifically:

Remark 8 *In any outflow equilibrium an individual is a source (sink) if his marginal utility of consumption is not higher (lower) than that of any of his neighbors.*

Individual with lower marginal utility than all their neighbors can never buy, because only players with lower marginal utility would sell to him. Similarly individuals with higher marginal utility than all their neighbors can never sell. Thus, in any outflow equilibrium the players with the lowest marginal utility for consumption are sources. While the players with the highest marginal utility are sinks.

Sources in this model sell to every neighbor with higher marginal utility, because for any source having a lower marginal utility is not only necessary, but also sufficient for a trade to occur. Since a source has no inflows, all that matters are outflow price distortions and such distortions vanish as outflows decrease.

Remark 9 *In any outflow equilibrium sources sell to all their neighbors with strictly higher marginal utility.*

Another implication of the outflow model is that if two players have a neighbor in common, that neighbor sells to the low marginal utility player only if he sells to the high marginal utility player:

Remark 10 *In any outflow equilibrium* $\mathbf{q} \in \mathbb{R}^E_+$ *if* $i, j \in V(k)$ *and* $u'_i(q_i) < u'_j(q_j)$*, then* i *buys from* k *only if* j *buys from* k*.*

Intuitively because the lower marginal utility player would always pay a lower price, he would be supplied only if the high marginal utility consumer is supplied first.

Whenever the allocation of endowments is inefficient, equilibrium consumption in any economy populated by a finite number of players is inefficient. In general poorly endowed consumers tend to consume less than what would be efficient. Similarly well endowed players tend overconsume. Exceptions to this rule of thumb are possible for specific network and endowment configurations, but never lead to equilibrium efficiency. Occasionally well connected but poorly endowed players can make large gains both in consumption and money through retrade.

The last result presented in this section consider what happens to this economy if players have several instances to trade quantities on the network. In such environment the endowment at each trading round is the final allocation of the previous trading round. The current version of the result makes use of three strong assumptions: agents do not discount, good do not perish, agents act myopically at each round. If such assumptions hold, it can be shown that as the number of instances in which to trade on the network grows, the outflow equilibrium outcome becomes efficient. Therefore if players have arbitrarily many instances to trade the limiting allocation of consumption will be efficient and the limiting prices will be competitive in each component of the network. However since the price paid for a specific flow differs along the sequence of trades the distribution of individual welfare will not correspond to that of a Walrasian economy.

Lemma 11 *If individuals have arbitrarily many instances to trade, do not discount and do not account for future actions the sequence of outflow equilibrium trades converges to the constrained efficient outcome whenever such outcome is interior.*

The result hinges on two observation. The first is that all individuals with the lowest equilibrium marginal utility for consumption at round t always have an incentive to sell to all their neighbors at round $t+1$. In turn this implies that the sequence of outflow equilibrium allocations converges. The second observation is that such sequence cannot converge to an allocation that is not constrained efficient, since at the limiting allocation individuals that benefit from a deviation would exist. Requiring the efficient allocation to be interior guarantees that goods do not remain stuck with suppliers which value them little and that can only sell them to individuals which value them less. Since individuals are myopic they are willing to pay different prices for the same flow at different trading instances. Therefore the distribution of welfare will differ form the Walrasian outcome of each component of the network even though the limiting prices are competitive. The generalization of the lemma to forward looking players without discounting should account for the fact that individuals do not want to pay different prices along the sequence of equilibria. Such result is still under investigation and shall be added as soon as time permits.

Comments on Prices, Welfare and Market Power

In the outflow competition model each individual on the network can be interpreted as a separate local market. Oligopolistic competitors use their access to different local markets to price discriminate their customers. Since quantity competition prevents price discrimination within each market, discriminating across markets is welfare maximizing for each supplier. Neighbors' endowments and access to markets will determine buying prices in equilibrium. High equilibrium marginal utility neighbors pay more than low marginal utility neighbors. Goods are occasionally traded below the competitive equilibrium price. As was the case for player b's inflow price in the first example of section 2.3.

Price discrimination provides incentives for suppliers to sell to smaller competitors. Suppliers sell some goods to competitors at a discount, even though such goods are used to compete against them to supply consumers at a higher price. Such instances occur when the sales to competitors bring sufficiently many revenues to overcome the negative effects of increased competition in the high value markets. Resale is pervasive in such economies. Even a fully connected economy can display equilibrium resale. It is driven by the profit opportunities that the different prices in the economy present to players. The monopsony wedge is the main force limiting resale, because it increases the cost of supplying goods at each step. The section about large economies shows that resale vanishes in any economy that becomes competitive as it grows large.

In a flow competition model whenever the economy's endowment is efficient, no trade occurs and a unique price for consumption reigns on all links. Namely, the competitive equilibrium price.

Thus, there always exists transfers of consumption good amongst players that bring economy to the competitive outcome. But, no economy of finite size and with an inefficient endowment profile, can ever become efficient as a result of trades in such markets. In fact, even though goods flow to under-endowed individuals the distortions caused by the price effects cannot vanish in any finite size economy.

Individual welfare in a networked market is determined not only by the endowment, but also by the position held in the market. Since oligopolistic rents are higher in markets with fewer competition, having access to more of such markets benefits a seller. Buyers benefit from having access to more sellers since competition decreases the markup that supplier are able to charge to them. Poorly endowed but well connected individuals thrive in these economies by selling at a markup most of the units bought to low competition markets. Rents from trades are distributed along the supply chain. Players reselling goods may opt not to supply neighboring high competition markets, because of the additional markup caused by the increase in inflow prices.

2.6 Adding Links and Welfare

This section discusses the effects on individual and social welfare of adding links to the network. Two prototypical examples are introduced. The first shows that adding a connection can reduce social welfare, an instance of Braess's Paradox. The second instead shows that adding a link may reduce the welfare of one of the two players on the newly crated trading relationship. Such examples can motivate the study of networks that are not complete, since instances in which an individual prefers not to be able to trade with certain players, can be found.

Braess's Paradox

In the economy presented increasing the set of trading relationships reduces social welfare. Consider a market with three consumers $\{a, b, c\}$. Individual a is endowed with two units of consumption good, b with one unit and the c with none. Preferences of all players satisfy $u(q) = q^{1/2}$. If only player a and c can trade, then a sells 0.4 units to b at a price of 0.8 and social welfare is 3.9. Equilibrium prices and allocations for this economy are reported in table 5 left and depicted in figure 5A.

If also consumers a and b can trade social welfare in the outflow equilibrium drops. Indeed in such an economy player a supplies both his neighbors. Player b is supplied with 0.2 units and player c with 0.36 units at different prices. Equilibrium allocations and prices are reported in table 5 right and figure 5B.

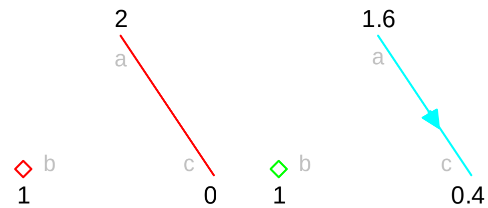

FIGURE 5A: Endowments left and equilibrium outcomes right

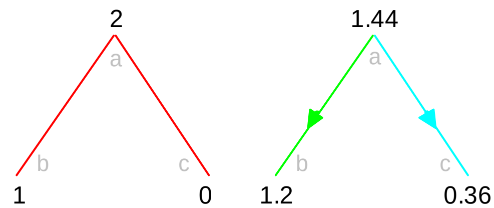

FIGURE 5A: Endowments left and equilibrium outcomes right

Social welfare in this economy decreases to 3.89. Such drop is a consequence of individual a's price discrimination of consumers b and c. Indeed player a prefers to curtail his supply to c and extract a higher per unit rent, because he can recoup that loss in revenue by selling to b.

In such markets arbitrarily increasing the set of trading relationships may decrease social welfare. Despite such negative result, it is in general true that a trading relationship exists that if added to the economy increases social welfare. In the example presented adding link bc improves social welfare and the complete network is the welfare maximizing market structure. The flows and prices for such market structure can be found in the examples section of the appendix.

Var	p	q	u
a	-	1.6	1.581
b	-	1.0	1.000
c	0.791	0.4	0.316
Tot	-	3.0	2.897

Var	p	q	u
a	-	1.437	1.591
b	0.456	1.204	1.004
c	0.834	0.359	0.300
Tot	-	3.000	2.895

TABLE 5: Prices paid, consumption and welfare. Left: $E = \{ac\}$ Right: $E = \{ac, ab\}$

Individuals Prefer not to be Linked

It may appear that adding a link to the networked market always improves the individual welfare of the two players being linked. However, this is not the case in all economies. General equilibrium effects may decrease the welfare of either member in the new trading relationship. The next example presents an economy in which adding a link reduces the welfare of the supplier on the new link.

Consider a slight perturbation of the leading example discussed in section 2.3. The endowment profile differs slightly and is given by $\mathbf{Q} = [2.975, 1, 0, 0.025]$. Consider a market in which player a can trade with d and player b can trade with c. In the outflow equilibrium of this economy players a and b supply their respective customers as monopolies. Allocations and prices for this economy are reported in table 6 left and depicted in figure 6A. Then consider how the individual welfare of all players is affected if individuals c and d are linked. If such connection is added player d competes with player b to supply player c with consumption good. Allocations and prices for this economy are depicted in figure 6B and reported in table 6 right.

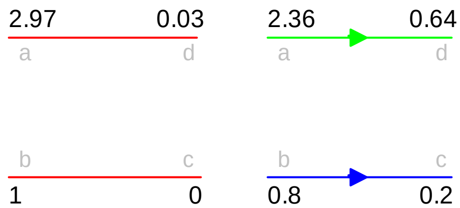

FIGURE 6A: Endowments left and equilibrium outcomes right.

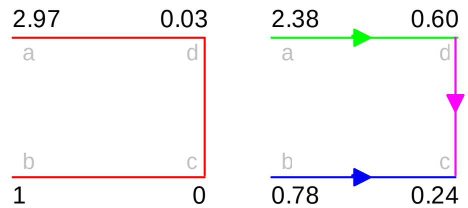

FIGURE 6B: Endowments left and equilibrium outcomes right.

In the outflow equilibrium of this economy consumer d is worse off than when he cannot sell to c. In fact when such a trading relationship is added to the network, the payoff of player four decreases form 0.42 to 0.41. Even though player d chooses to supply c, having the option to sell affects the quantity sold to him from a and thus reduces his welfare. When the link is added, all gains from trade on the newly created link are either kept by c or transferred to a. Player a being the monopoly supplier of d is able to extract more rents, because of the steeper demand schedule he faces when d resells.

Var	p	q	u
a	-	2.362	1.921
b	-	0.800	1.118
c	1.118	0.200	0.224
d	0.626	0.638	0.415
Tot	-	4.000	3.677

Var	p	q	u
a	-	2.38	1.928
b	-	0.786	1.106
c	1.023	0.239	0.244
d	0.648	0.596	0.411
Tot	-	4.000	3.690

TABLE 6: Prices paid, consumption and welfare. Left: $E = \{ad, bc\}$ Right: $E = \{ad, dc, bc\}$

The two examples presented showed that adding a link may end up hurting individuals and society if individuals compete on outflows. It would be interesting to argue that despite such negative scenarios a link that, if added, does not decrease social welfare always exists. If such conjecture was to hold, the complete network would be welfare maximizing. But even though the claim holds true for all simulations carried out, the proof of such result remains an open question. Intuitively, a newly added link reduces social welfare only when price discrimination leads to a further misallocation of resources. But since resources are misallocated in the outflow equilibrium, it is always possible to link the two unconnected players with the highest marginal utility differential and expect society to gain from such change.

In the section about large economies it is shown that connecting replicas of the original economy always increases social welfare. Moreover conditions on the network structure which guarantee that social welfare becomes efficient as the economy grows large are provided.

2.7 The Complete Network Economy

This section provides a closer characterization of the outflow equilibrium trades for economies in which all the individuals are connected. Since in such economies the player's positions in the market are symmetric, all differences in consumption and welfare are driven by endowments and preferences. The first claim shows that of two players the one with low equilibrium marginal utility for consumption sells more to all his neighbors than the one with high marginal utility.

Proposition 12 *If assumption A1 holds in any outflow equilibrium of the complete networked economy if $u_i'(q_i) < u_j'(q_j)$ then:*

(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ imply $q_k^i > q_k^j$
(2) $q_j^i \geq q_i^j$ and $q_j^i > 0$ imply $q_i^j = 0$

Even though such claim is intuitive it still requires some discipline on inflows. Because the cost of suppling units consists not only of forgone consumption, but also of increased expenditure on inflows, such claim requires the ranking of marginal utilities to impose some discipline on the ranking of supply costs. The complete network structure guarantees that such discipline is maintained and requires low marginal utility players to sell more. However without further assumptions it is impossible to guarantee that players with low marginal utilities also buy less from their neighbors.

If all players have a common utility function u and if such utility function satisfies assumptions that are sufficient for equilibrium existence, stronger results can be derived. Specifically assume that:

Assumption A2 *For any player $i \in V$ assume that $u_i = u$, $u''' > 0$ and the coefficient of absolute risk aversion is weakly increasing:*

$$\frac{\partial}{\partial q}\left(-\frac{u''(q)}{u'(q)}\right) \geq 0$$

Assumption A2 is satisfied by any CARA utility function. Such assumption puts a bound on the third derivative since it requires that $u'''(q) \leq u''(q)^2/u'(q)$. When A2 holds the completeness of the network guarantees that inflows can be ranked across players.

Proposition 13 *If assumptions A1 and A2 hold in any outflow equilibrium of the complete networked economy $q_i > q_j$ if and only if:*

(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ imply $q_k^i > q_k^j$
(2) $q_i^k \leq q_j^k$ and $q_j^k > 0$ imply $q_i^k < q_j^k$
(3) $q_j^i \geq q_i^j$ and $q_j^i > 0$ imply $q_i^j = 0$

Whenever the assumptions A1 and A2 hold true, individuals consuming more goods buy less from their neighbors and sell more to their neighbors when compared to individuals consuming less. The additional assumptions on the utility function were required to motivate any individual to sell more goods to those local markets in which the demand is steeper.

Since those consuming more buy less and sell more, it must be that they started with more endowment. Thus if assumption A2 holds in any outflow equilibrium of the complete network economy an individual consumes more goods if and only if he starts with more goods:

Proposition 14 *If assumptions A1 and A2 hold in any outflow equilibrium of the complete networked economy $Q_i > Q_j$ if and only if $q_i > q_j$.*

If A1 and A2 hold, the flow pattern of the complete network economy is completely pinned down by the endowments alone. Moreover by remark 10 it must be that the individuals with biggest endowment never buy, while individuals with the smallest endowment never sell.

A question that this analysis aims to address and that is supported by simulations, but that remains open is: whether the complete graph is the network structure that maximizes social welfare. The proof of such claim is intimately related to the problem of finding a link that if added increases social welfare. The next section shows that as an economy grows large the complete network becomes welfare maximizing.

3 Large Economies and the Competitive Equilibrium

This section presents necessary and sufficient conditions on the network structure for the solution of the outflow competition model to converge to the competitive equilibrium as the economy grows large. It is shown that resale must vanish if an economy ever becomes competitive and that social welfare increases as an economy grows large. The complete characterization of flows in the limiting economy is also presented. Sufficient conditions for equilibrium existence become weaker as an economy grows large. For sake of clarity all results are presented in the context of replica economies. Section 6.3 in appendix shows how results extend to arbitrary sequences of networked markets. The main goal of this section is to explain when a non-anonymous networked market converges to the competitive equilibrium of the corresponding anonymous economy

Definition 2 *The competitive equilibrium of the economy $\{V, Q, u\}$ consists of a price $p^* \in \mathbb{R}_+$ for consumption good and of an allocation $\mathbf{q}^* \in \mathbb{R}_+^V$ such that:*

(1) each player's allocation is optimal given the price

(2) the market of consumption clears

By assumption A1 it must be the competitive equilibrium exists and is efficient. For convenience let B denote the set of individuals buying goods in the competitive equilibrium and let S denote the set of players selling goods in the competitive equilibrium. Specifically:

$$\begin{aligned} B &= \{i \in V | Q_i < q_i^*\} \\ S &= \{i \in V | Q_i > q_i^*\} \end{aligned}$$

Because the graph describing the networked economy is undirected notice that $V(i) \supseteq S$ for any $i \in B$ if and only if $V(i) \supseteq B$ for any $i \in S$. Players in B are called *competitive buyers* and players in S are *competitive sellers*.

Define a networked economy $\{G^r, Q^r, u^r\}$ to be an r-replica of the economy $\{G, Q, u\}$ for any $r \in \mathbb{N}_+$ if:

$$\begin{aligned} &(1) \quad V^r = \{i(s) | i \in V \cap s \in \{1, ..., r\}\} \\ &(2) \quad E^r = \{i(s)j(t) | ij \in E\} \\ &(3) \quad \{Q^r_{i(s)}, u^r_{i(s)}\} = \{Q_i, u_i\} \quad \text{ for } \forall i(s) \in V^r \end{aligned}$$

The first condition states that the players belonging the r-replica are r copies of each players in the original economy. The second condition requires that all copies of two players which were linked in the original economy, be linked amongst themselves. While the third condition states that all copies of a player have the same endowment and utility that he has. The competitive equilibrium of the economy does not change as the economy gets replicated. Efficient per-capita social welfare remains constant along any sequence of replica economy. Therefore any copy of a competitive buyer of the original economy remains a competitive buyer in any replica and similarly for sellers.

Consider a sequence of replica economies $\{G^r, Q^r, u^r\}_{r=1}^{\infty}$. The next proposition provides conditions on the market structure to guarantee that outflow equilibrium allocations and prices of the sequence of economies converges to competitive equilibrium as the number of players grows large.

Proposition 15 *Consider a sequence of replica economies $\{G^r, Q^r, u^r\}_{r=1}^{\infty}$, if $V(i) \supseteq S$ for any $i \in B$ then any symmetric outflow equilibrium converges to the competitive equilibrium as the economy grows large.*

The proposition states that as networked economies in which all competitive buyers can trade with all competitive sellers grow large, equilibrium outcomes become competitive. Such claim views the anonymous Walrasian market place as an approximation of a non-anonymous large market in which a large number of buyers and sellers can trade. If such condition on the market is satisfied, all markups that sellers can impose to the buyers vanish as the economy grows large. The outflow equilibrium converges to the efficient allocation and a unique price reigns throughout the limiting economy. Any complete network economy trivially satisfies the aforementioned condition and therefore when replicated converges to a competitive outcome. Therefore such network structure must maximize social welfare as an economy grows large. The definition of replica economy provided imposes more discipline on the sequence of economies than is required. Section 6.3 shows that such claim hold true even for arbitrary sequences of growing economies. The condition on the network structure then requires that all competitive sellers get to know all competitive buyers as the economy grows large.

The condition on the network structure proposed in the last proposition depends on the definitions of B and S, which in turn hinges on the definition of competitive equilibrium. Such condition, requires the knowledge of endowments and preferences in order to establish whether the market structure favors competition, unless the complete network economy is considered. The next proposition shows why the requirement on buyers and sellers is in some sense minimal and why economies that do not satisfy it will not in general converge to competitive equilibrium.

Specifically consider any economy whose sequence of outflow equilibria converges to competitive equilibrium if replicated arbitrarily many times. The next result states that the fraction of goods sold directly from buyers to sellers in an outflow equilibrium of such economy converges to one as the market grows large. Indeed if intermediaries are necessary to distribute goods in an economy they command a rent independently of the size and structure of the market. The result can be stead as follows:

Proposition 16 *If the outflow equilibria of sequence of replica economies converge to the competitive equilibrium, then amount of consumption good resold by any individual vanishes as the economy grows large.*

The proof hinges on two observations: (1) all trades occur at one price in a competitive economy, and (2) no individual ever resells any positive quantity of goods at a zero markup. In the outflow model increasing the size of the economy causes individuals to become price takers as sellers, since each local market becomes more competitive. But individuals never become price takers as buyers since they maintain their monopsony power when purchasing goods at their local market. Thus the wedge on inflow prices cannot disappear if resale persists. Hence nobody ever resells in a competitive economy. The result does not hinge on the proposed definition of replica, but on the power that each intermediary commands in his local market.

Proposition 16 implies that unless flows can be found that support the competitive allocation and are without resale, the outflow equilibrium sequence never converges to the competitive equilibrium. Proposition 16 also motivates why no assumption on the market structure weaker than that offered in proposition 15, can guarantee that an economy becomes competitive. Particular preferences and endowments for which fewer links suffice to get a competitive economy can be found (e.g. if players' endowments are efficient). The result bears a highly negative view of behavior of middlemen. In such simple economies middlemen serve no purpose if the economy is sufficiently well connected and thus have no role in a competitive economy. Certainly different assumptions can be envisioned for middlemen to persist in a competitive market. However if players compete on outflows and if their marginal benefits from consumption are decreasing, any complication of the model will still have to account for a negative force trying to eradicate the presence of middlemen in any limiting competitive economy.

As in most quantity competition models existence is easier to proof as an economy grows large. For the moment restrict attention to symmetric equilibria. Let $\overline{q}^i_j = \lim_{r\to\infty} rq^{i(t)}_{j(s)}(r)$ denote the amount of goods sold in the limiting economy from an individual of type i to all individuals of type j. The symmetric equilibrium flows of the limiting economy can be shown to solve the **CP** complementarity problem for the following optimality conditions:

$$f^i_j(\overline{\mathbf{q}}, \boldsymbol{\mu}) = u'_i(\overline{q}_i) - u'_j(\overline{q}_j) - u''_i(\overline{q}_i)\sum_{k\in V(i)}^k \overline{q}_i + \mu_i \geq 0$$

Such conditions differ from the previous ones only because the price effect on outflows has vanished. Such wedge vanishes because in the limit economy infinitely many individuals compete to supply each local market. The price effect on inflows instead always remains positive for those buying and selling goods in the limit economy. But, since the outflow wedge was the complicating factor in the proof of existence a stronger result can be stated for the limit economy.

Proposition 17 *If assumption A1 holds, $u'''_i > 0$ and endowments are positive, then the limit economy always possesses a symmetric outflow equilibrium.*

In fact revenues at each local market are concave in the limiting economy. What remains to be shown is that the cost of supplying outflows is convex. The assumption on the third derivative ensures that such condition always holds. Thus lemma 5 applies.

Before stating a final result about social welfare, consider two examples of replica economies to discuss the dynamics of flows as the economy grows large. The two economies differ in the market structure, but have common preferences and endowments. In both there are three types of players: one type of player is endowed with two units, one with one unit and the last with none. Call them producers, intermediaries and consumers. All three players have a constant relative risk aversion utility function with coefficient 1/2. In the first economy all players are connected. Such economy if replicated arbitrarily many times converges to the competitive equilibrium, since producers can directly supply all consumers. Equilibrium consumption of all three types of players converges to one. Consumption of producers decreases monotonically, consumption of consumers increases monotonically. Intermediaries' consumption first increases and then drops. The price paid by those players thus first declines and then converges from below to 1/2. The price paid by consumers instead monotonically decreases to the competitive equilibrium price. Equilibrium resale vanishes in the limiting economy and intermediaries do not trade in the limit. Per capita social welfare increases monotonically as the equilibrium converges to the competitive outcome. Figures 9A and 9B depict the sequences of consumption and prices of the replicas.

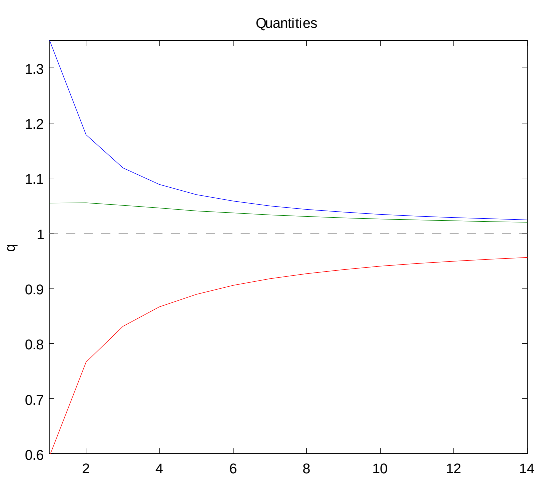

FIGURE 9A: On the vertical axis consumption on the horizontal axis the replica

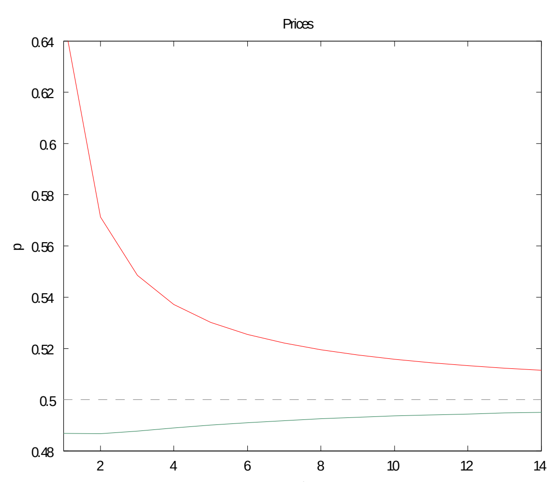

FIGURE 9B: On the vertical axis prices on the horizontal axis the replica

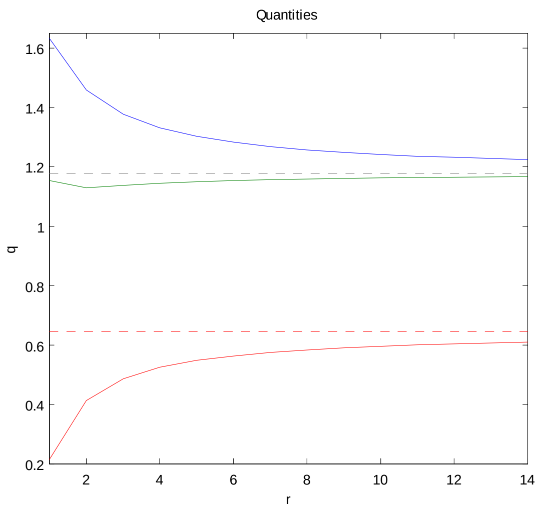

FIGURE 10A: On the vertical axis consumption on the horizontal axis the replica

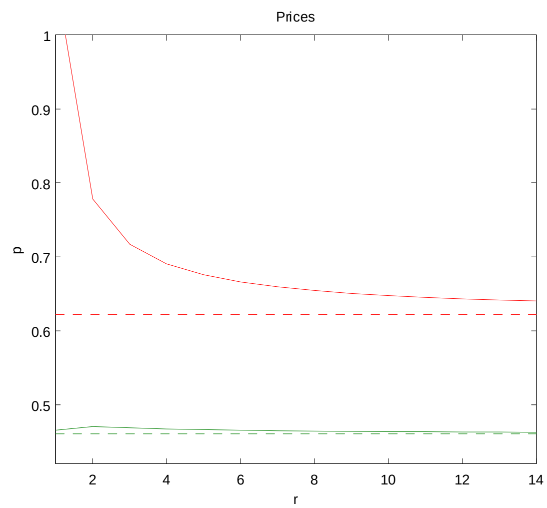

FIGURE 10B: On the vertical axis prices on the horizontal axis the replica

Now consider the economy in which the links between producers and consumers are removed. In such market intermediaries unit act as middlemen buying from producers and selling to consumers. Such economy even if replicated arbitrarily many times cannot converge to the competitive equilibrium, since producers cannot directly supply all individuals in need of consumption good. Equilibrium consumption of all three types of players does not converge. In the limiting economy consumption of producers converges to that of their intermediaries. Again consumption of producers decreases monotonically and consumption of consumers increases monotonically. Consumption of the middleman first decreases and then grows to equalize to that of their suppliers. The price paid by intermediaries first grows and then declines converging to a value lower than the competitive price. The price paid by consumers instead monotonically decreases but remains higher than the competitive equilibrium price. The limiting markup made by middlemen converges to approximately 30%. Per-capita social welfare increases monotonically, but remains inefficient in the limiting economy. Figures 10A and 10B depict the sequences of consumption and prices of the replicas of such economy. The outflow model recognizes that the second economy cannot mimic an anonymous Walrasian market because intermediaries are required to exchange goods. Such example determines the relevant prices for the two anonymous market squares that arise in the limit. Intermediaries in such interpretation are the only players allowed to enter both market squares and collect a rent by transferring goods through the two markets.

The last result presented in this section provides conditions under which per-capita social welfare increases, when an economy gets replicated. Because the definition of replica requires each local market to become more competitive as the economy gets replicated it is reasonable to expect such result to hold with some generality.

Proposition 18 *If sufficient conditions for existence are met and if at any replica r the economy possesses a unique symmetric outflow equilibrium, then in such equilibrium per-capita social welfare* *increases every time the economy is replicated.*

Thus if a unique symmetric equilibrium exists along the sequence of replica economies, per-capita social welfare increases. It implies that even economies in which the retail sector does not vanish become more competitive, though not perfectly competitive, as they grow large. This proposition establishes a formal link between social welfare and network structure exploiting the nature of the Jacobian matrix of the complementarity problem.

4 Inflow Competition in Networked Markets

This section presents the inflow competition model. First the model is introduced. Then a brief summary of results for this model is presented. An example of an inflow competition economy is provided in the examples section of the appendix.

4.1 Inflow Competition

Competition in the inflow proceeds as follows. Initially each individual decides how much consumption good to buy from each neighbor. Then given the chosen flows market prices are determined at each node so that each local market clears. In this setup a seller is paid equally for all of his outflows at the marginal value of the last unit purchased. The inverse supply function at the node $j \in V$ satisfies for any $i \in V(j)$:

$$p_i^j(\mathbf{q}) = p^j(q_j) = u_j'(q_j)$$

Again an argument à la Kreps and Scheinkman shows that if individuals can commit to their choices of inflows, price competition amongst buyers leads to such price arising in each local market. Any individual offering a lower price would be made worse off because part of his demand would remain unfulfilled. Moreover no buyer would have an incentive to offer a higher price since his demand is fulfilled at the lower price.

The price consumer j receives for the units he sells not only decreases when his inflows increase, but also increases when his outflows increase. The concavity of the utility map implies that $\partial p^j(q_j)/\partial q_i^j \geq 0$ and $\partial p^j(q_j)/\partial q_j^i \leq 0$ for any $i \in V(j)$. Changes in other flows in the network do not affect the price paid by i as long as his inflows and outflows remain unchanged.

Buyers take into account the effects that their inflow choice has both on the prices they get for each unit sold and on the prices they pay for each units bought. The problem of buyer $i \in V$ is to choose which quantities to buy from each of his neighbors given price effects:

$$\max_{q_i^j \geq \mathbf{0} \text{ for } j \in V(i)} u_i(q_i) + \textstyle\sum_{k \in V(i)} [p^i(q_i) q_k^i - p^k(q_k) q_i^k] \quad \text{s.t.} \quad q_i \geq 0$$

In the interior of the domain the first order condition for q_i^j in the knowledge constrained economy requires that:

$$u_i'(q_i) - p^j(q_j) - \frac{\partial p^j(q_j)}{\partial q_i^j} q_i^j + \frac{\partial p^i(q_i)}{\partial q_i^j} \textstyle\sum_{k \in V(i)} q_k^i \leq 0$$

Such condition holds with equality whenever a positive quantity is supplied $q_i^j > 0$. The first order condition states that the marginal benefit of buying more consumption must cover the price paid, monopoly price distortion on all units sold and the Cournot distortion on the units purchased by seller j. This equation differs from the outflow equation because a different set of price distortions is considered.

In the outflow model the suppliers can commit outflows, while in the inflow model the buyers have the option to commit to inflows. The group having such power benefits in equilibrium by appropriating all the gains from trade. Since rents in the inflow model go to buyers, more goods flow to them. An inflow economy is in general be more efficient than an outflow economy, since all that matters is the final allocation of goods. The first order condition for the inflow model written only in terms of utility functions states that:

$$u_i'(q_i) - u_j'(q_j) + u_j''(q_j)q_i^j + u_i''(q_i)\sum_{k\in V(i)} q_k^i \leq 0$$

In what follows the expression *inflow equilibrium* will be used to refer to a pure strategy Nash equilibria of the inflow competition model.

4.2 Results and Discussion

The proof existence of an inflow equilibrium requires stronger conditions on preferences than in the outflow model. The following conditions suffice to prove the existence of a pure strategy equilibrium in this setup:

Proposition 19 *If A1 holds, endowments are positive, $u_i''' \geq 0$ and for any player $i \in V$:*

$$\frac{\partial}{\partial q}\left(-\frac{u_i''(q)}{u_i'(q)}\right) \geq 0 \quad \text{for any } q \in [-Q, Q]$$

then an inflow equilibrium exists.

A more detailed analysis of equilibrium existence can be developed by applying the complementarity approach of section 2.4. Again the assumption that endowments are positive is seldom required to insure that an inflow equilibrium exists.

In the inflow model each seller supplies all his customers at a single price. Buyers however purchase the same good from different suppliers at different prices. Again it is in their best interest to do so given the choices of others, because price distortion would increase their expenses if they were to concentrate their demand on a single neighboring market. As in the outflow model resale is pervasive, but connected individuals do not necessarily trade because of the non vanishing price distortions. In this setup the outflow price distortion does not disappear as inflows vanish.Examples reported in the appendix show that adding links can still reduce social welfare or the welfare on one of the individuals being connected. In the inflow model markups are generally smaller, because they depend on the concavity of the most endowed player. This result holds always in two-sided markets. But for more general market structures it does not hold in full generality. The inflow model usually allocates good more efficiently since rents are given to the weaker side of the market.

The results for large economies are substantially unchanged. Again any economy in which the retail sector does not vanish cannot become competitive as it grows large. If all local markets become more competitive as the economy grows large, social welfare still increases as an economy grows large. Moreover as in the outflow model, conditions for equilibrium existence simplify in the limiting economy. For replica economies the system of equations characterizing the limiting economy can be obtained by discarding the inflow price distortions from the equations which define the inflow equilibrium of the original economy.

5 Conclusions

The analysis discussed how networked oligopolistic markets operate. A general equilibrium model was introduced to study flow patterns and pricing in networked economies. Sufficient conditions for pure strategy equilibrium existence were presented.

The study showed that trade patterns in such economies are dictated not only by the network structure, but also by welfare-maximizing properties of price discrimination across local markets. Even well connected economies were shown to display significant price dispersion and non-trivial trade patterns across local markets. Moreover since goods were resold at non-vanishing markups, not all neighboring players with different marginal rates of substitution would necessarily trade in a flow equilibrium. Price setting behavior of individuals implied that changes in the network structure could have non trivial effects on individual and social welfare. Indeed adding trading relationship could negatively affect both social welfare and the individual welfare of one of the players being connected. Such negative scenarios however were less likely in large economies.

Sufficient conditions on the network structure were presented to insure that non-anonymous networked markets become competitive as the number of players grows large. It was shown that in this setup no economy that becomes competitive could have a non-vanishing retail sector, since retail would only occur at positive markups. Moreover it was shown that replicating the economy has positive effects on social welfare.

References

[1] Ahuja R., Magnanti T., Orlin J., “Network Flows”, Prentice Hall, 1993.

[2] Acemouglu D., Ozdaglar A., “Competition in Parallel Serial-Networks”, IEEE Journal on Selected Areas in Communication, 25/6, 2007.

[3] Blume L., Easley D., Kleinberg J., Tardos E., “Trading Networks with Price-Setting Agents”, Electronic Commerce, Proceedings of the 8th ACM conference, 2007.

[4] Bollobas B., “Modern Graph Theory”, Springer, 1998.

[5] Cournot A., “Recherches sur les Principes Mathematiques de la Théorie des Richesses”, 1838, Translation Macmillan, 1897.

[6] Corominas-Bosh M., "Two Sided Networked Markets", Journal of Economic Theory 115/1, 2004.

[7] Cook K., Emerson R., Gillmore M., Yamagishi T., "The Distribution of Power in Exchange Networks: Theory and Experimental Results", American Journal of Sociology 89, 1983.

[8] Chawla S., Roughgarden T., "Bertrand Competition in Networks", Working Paper, 2007.

[9] Collie D., "International Trade and Cournot Equilibrium", Bulletin of Economic Research 44/1, 1992.

[10] Cook K., Whitmeyer J., "Two Approaches to the Social Structure: Exchange Theory and Network Analysis", Annual Review of Sociology 18, 1992.

[11] Goyal S., Joshi S., "Networks of Collaboration in Oligopoly", Games and Economic Behavior 43, 2003.

[12] Greenhut M., Ohta H., "Vertical Integration of Successive Oligopolists", American Economic Review 69/1, 1979.

[13] Dasgupta P., Maskin E., "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory", Review of Economic Studies 53/1, 1986.

[14] Jackson M., "Social and Economic Networks", Princeton University Press, 2008.

[15] Jackson M., Watts A., "The Evolution of Social and Economic Networks", Journal of Economic Theory 106/2, 1998.

[16] Jackson M., Wolinsky A., "A strategic model of social and economic networks", Journal of Economic Theory 71/1, 1996.

[17] Karamardian D., "The Complementarity Problem", Mathematical Programming 2, 1972.

[18] Kolstad C., Burris A., "Imperfectly Competitive Equilibria in International Commodity Markets", Journal of Agricultural Economics 68/1, 1986.

[19] Kakade S., Kearns M., Ortiz L., "Graphical Economics", Lecture Notes in Computer Science 3120, 2004.

[20] Kolstad C., Mathiesen L., "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium", Review of Economic Studies 54/4, 1987.

[21] Kranton R., Minehart D., "Competition in Buyer-Seller Networks", American Economic Review 91/3, 2001.

[22] Kreps D., Sheinkman J., "Quantity Pre-commitment and Bertrand Competition Yield Cournot Outcomes", Bell Journal of Economics 14/2, 1983.

[23] Lerner A., "The Concept of Monopoly and the Measurement of Monopoly Power", Review of Economic Studies 1/3, 1934.

6 Appendix

6.1 More on Existence

This section presents further results on equilibrium existence. In particular it discusses: (1) the case of concave individual welfare, (2) the pseudo-concave case, (3) the case in which short positions lead to default and (4) the case in which short sales are limited.

(1) Since by assumption A1 the individual welfare $w_i(\mathbf{q})$ is continuous in all flows, a sufficient condition for existence is the concavity of individual i's welfare in his outflows $\mathbf{q}^i$. This condition requires all leading minors of the Hessian of $-w_i(\mathbf{q})$ to be positive definite. Such Hessian is defined by:

$$J_{V(i)}(\mathbf{q}) = \left[-\frac{\partial^2 w_i(\mathbf{q})}{\partial q_j^i \partial q_k^i} \right]_{k,j \in V(i)} > 0$$

Since all outflows are perfect substitutes for a supplier the cross partials of the Hessian are identical and do not depend on the outflows:

$$\frac{\partial^2 w_i(\mathbf{q})}{\partial q_j^i \partial q_k^i} = \begin{cases} 2u_j''(q_j) + u_j'''(q_j)q_j^i + u_i''(q_i) - u_i'''(q_i)\sum_{k \in V(i)} q_i^k & \text{if} \quad j = k \\ u_i''(q_i) - u_i'''(q_i)\sum_{k \in V(i)} q_i^k & \text{if} \quad j \neq k \end{cases}$$

Therefore, one gets that:

Lemma 20 *If assumption A1 and one of the following holds:*
(1) $u_i''' \geq 0$ *and* $2u_i''(q) + u_i'''(q)q' < 0$ *for any* $q \in [0, Q]$, *any* $q' \in [0, Q - q]$
(2) $u_i''' \leq 0$ *and* $u_i''(q) - u_i'''(q)q' < 0$ *for any* $q \in [0, Q]$, *any* $q' \in [0, q]$
Then player i's welfare is concave.

Such conditions if met guarantee existence of an outflow equilibrium. But, they are not general enough to guarantee existence for many commonly used utility functions.

(2) A connection between outflow equilibria and solutions of the **CP** problem can be made even when individual welfare is pseudo-concave. In particular its is possible to prove that:

Lemma 21 *Any outflow equilibrium solves **CP**. Further if individual welfare is pseudo-concave with respect to own outflows an allocation is an outflow equilibrium if and only if solves **CP**.*

Theorem 22 *If utility functions are* C^3 *and if:*
(1)outflows are bounded and endowments are positive
(2) individual welfare is pseudo-concave with respect to own outflows
Then an outflow equilibrium exists. Moreover
If all outflow equilibria are non-degenerate, $\det J_T(\mathbf{q}, \boldsymbol{\mu}) > 0$ *at any equilibrium if and only if there is a unique outflow equilibrium*

(3) Finally consider the case in which short consumption positions lead to bankruptcy. In particular assume: $q_i < 0$ implies $w_i(\mathbf{q}) = 0$ and thus $p_i(q_i) = 0$.

Lemma 23 *If for any player $i \in V$ A1 holds, $w_i(\mathbf{q}) = 0$ when $q_i < 0$, $u_i''' \geq 0$ and:*

$$\frac{\partial}{\partial q}\left(-\frac{u_i''(q)}{u_i'(q)}\right) > -\left(\frac{u_i''(q)}{u_i'(q)}\right)^2 \quad \textit{for any } q \in [0, Q]$$

Then the costs are convex, the revenues are pseudo-concave and concave if strictly increasing.

Indeed since leaving a buyer short on consumption is weakly dominated by not selling such strategy is never adopted. Moreover the appropriate modification of lemma 2 holds, given tat leaving buyers short is dominated.

(4) Limiting the extent of short sales can generalize the conditions for existence without shifting preferences.

Proposition 24 *If assumption A1 holds, $q_i - \max_{k \in V(i)}\{q_i^k\} \geq 0$, $u_i''' \geq 0$ and the coefficient of relative risk aversion of player $i \in V$ satisfies:*

$$\frac{\partial}{\partial q}\left(-\frac{q u_i''(q)}{u_i'(q)}\right) \geq 0 \textit{ for any } q \in [0, Q]$$

Then the costs are convex, the revenues are pseudo-concave and concave if increasing.

6.2 Relevant Examples

Braess's Paradox Example Continued

This completes the analysis economy discussed in section 2.6. Recall that in such market the first individual is endowed with two units of consumption, the second with one unit and the third with none. All three players have a common CRRA utility for consumption with coefficient of 1/2.

B\S	1	2	3	Tot
1	-	0	0	0
2	0.247	-	0	0.247
3	0.401	0.192	-	0.593
Tot	0.648	0.192	0	

Var	p	q	u
1	-	1.352	1.543
2	0.487	1.054	1.032
3	0.649	0.539	0.385
Tot	-	3.000	2.960

TABLE 9: On the left the flow matrix. On the right prices, consumption and welfare.

In the unique outflow equilibrium of this market consumer one sell to two and three, while two sells to three. Player one is willing to supply units to two that will then be resold to three at a higher price. He does so only because it is the best way to mitigate the competition from player two when supplying the third individual. The social welfare in this economy is of 2.96 and is higher than with any other market structure.

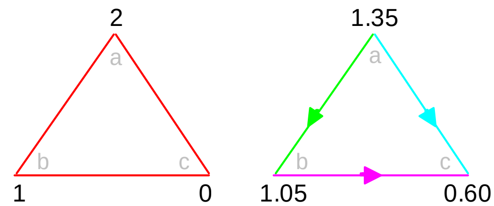

FIGURE 11: On the left endowments, on the right Cournot outcomes.

Inflow Competition Example

Again consider the economy discussed in the Braess type example. Recall that the economy has three players. One endowed with two units, another endowed with one and the last with none. All consumers have a CRRA utility map with coefficient of 1/2. If only the most and the least endowed individuals can trade, 0.75 units are exchanged between them at a price of 0.45. Thus more goods are traded at a lower price in the inflow equilibrium of the economy when compared to the outflow outcome discussed in section 2.6. Thus social welfare in the inflow model exceeds that of the outflow outcome. The precise equilibrium quantities are reported in table 10.

B\S	1	2	3	Tot
1	-	-	0	0
2	-	-	-	0
3	0.746	-	-	0.746
Tot	0.746	-	0	

Var	p	q	u
1	0.446	1.254	1.453
2	-	1.000	1.000
3	-	0.746	0.531
Tot	-	3.000	2.984

TABLE 10: On the left the flow matrix. On the right prices, consumption and welfare.

If the link between the two well endowed agent is added, the inflow equilibrium social welfare drops. As was the case for the outflow model, in such economy too many goods would flow to the efficiently endowed individual. Indeed in such economy the player endowed with two units sells 0.71 and 0.1 units respectively to his low and high endowment neighbor at a price per unit of 0.46. Such change in flows negatively affects the player with the highest demand for consumption and therefore reduces equilibrium social welfare. Again the inflow economy outperforms the outflow outcome, since rents flow to consumers more in need. Equilibrium prices and flows for this economy are reported in table 11.

B\S	1	2	3	Tot
1	-	0	0	0
2	0.099	-	-	0.099
3	0.709	-	-	0.709
Tot	0.808	0	0	

Var	p	q	u
1	0.458	1.192	1.462
2	-	1.099	1.003
3	-	0.709	0.517
Tot	-	3.000	2.982

TABLE 11: On the left the flow matrix. On the right prices, consumption and welfare.

Finally consider the fully connected networked market. In the inflow equilibrium of this economy two prices reign. The well endowed individual still sell goods at a price of 0.45 per unit. But the individual endowed with a single unit resells more units than he bought at a price of 0.51. All trades take place in equilibrium and it is in the best interest of the unendowed player to buy from both supplier at different prices. Though equilibrium flows are significant, social welfare in the economy is still inefficient. The inflow economy still outperforms the outflow outcome in terms of efficiency. Table 12 reports inflow equilibrium prices and flows for this market.

B\S	1	2	3	Tot
1	-	0	0	0
2	0.117	-	0	0.117
3	0.630	0.169	-	0.799
Tot	0.747	0.169	0	

Var	p	q	u
1	0.447	1.253	1.453
2	0.513	0.948	1.008
3	-	0.799	0.526
Tot	-	3.000	2.987

TABLE 12: On the left the flow matrix. On the right prices, consumption and welfare.

6.3 Large Markets Without Replica

A sequence of networked economies $\{G^r, Q^r, u^r\}_{r\in\mathbb{N}}$ is said to increase if for any $r \in \mathbb{N}$:

$$\begin{array}{ll} (1) & V^r \subset V^{r+1} \ \ \& \ \ E^r \subset E^{r+1} \\ (2) & \{Q_i^r, u_i^r\} = \{Q_i, u_i\} \text{ for } \forall i \in V^r \end{array}$$

The first conditions states that the number of players and connections grows. The second states that player's tastes and endowments do not depend on the market structure. Let B^r denote the set of individuals buying goods in the competitive equilibrium $\mathbf{q}^*(r) \in \mathbb{R}_+^{V^r}$ of the r-th economy and let S^r denote the set of players selling goods:

$$\begin{aligned} B^r &= \{i \in V | Q_i < q_i^*(r)\} \\ S^r &= \{i \in V | Q_i > q_i^*(r)\} \end{aligned}$$

Definition 3 *An networked economy becomes competitive, if a selection of the outflow equilibrium correspondence converges to the competitive equilibrium.*

Given such definitions it is possible to state the following two results about competition in large markets:

Proposition 25 *Consider an increasing sequence $\{G^r, Q^r, u^r\}_{r\in\mathbb{N}}$, if the economy becomes competitive, then the amount of goods resold by any individual vanishes.*

Proposition 26 *Consider an increasing sequence $\{G^r, Q^r, u^r\}_{r\in\mathbb{N}}$, if $\exists \overline{r} \in \mathbb{N}$ such that $V^r(i) \supseteq S^r$ for any $i \in B^r$ and $r > \overline{r}$ and if $\lim_{r\to\infty} |B^r| = \infty$, then any outflow equilibrium becomes competitive.*

As for replica economies no goods can be resold in a competitive economy, since resale occurs only at positive markups. As the number of competitive buyers and sellers grows large and if all buyers meet all sellers the networked economy becomes competitive. In fact if all direct trades are available, retailers get squeezed out of the market, since the rents from selling become arbitrarily small.

6.4 Proofs

Outflow Equilibrium Existence

Lemma 2 *If the costs of suppling outflows are convex, if revenues are pseudo-concave and concave if increasing, an allocation is an outflow equilibrium if and only if solves* ***CP****.*

Proof. Suppose that $\mathbf{q} \in \mathbb{R}^E$ solves **CP**. But suppose it's not an outflow equilibrium. If so there exists a player $i \in V$ and a profile of outflows $\overline{\mathbf{q}}^i \in \mathbb{R}^{V(i)}$ such that $w_i(\overline{\mathbf{q}}^i, \mathbf{q}^{-i}) > w_i(\mathbf{q})$. Pick $\overline{\mathbf{q}}^i$ in the best reply so that:

$$\overline{\mathbf{q}}^i \in \arg \max_{\widehat{\mathbf{q}}^i \in \mathbb{R}_+^{V(i)}} w_i(\widehat{\mathbf{q}}^i, \mathbf{q}^{-i}) \;\; \text{s.t.} \;\; \widehat{q}_i \geq 0$$

Because conditions in $\mathbf{CP}(i)$ are necessary for player i's optimality, they are met at $(\overline{\mathbf{q}}^i, \mathbf{q}^{-i})$. Consider three separate scenarios. In the first scenario $\sum_{k \in V(i)} q_k^i = \sum_{k \in V(i)} \overline{q}_k^i$. If so:

$$\begin{array}{rcl} u_i'(q_i) - u_i''(q_i) \sum_{k \in V(i)} q_i^k & = & u_i'(\overline{q}_i) - u_i''(\overline{q}_i) \sum_{k \in V(i)} \overline{q}_i^k \\ \Rightarrow \quad u_j'(q_j) + u_j''(q_j) q_j^i - \mu_i & = & u_j'(\overline{q}_j) + u_j''(\overline{q}_j) \overline{q}_j^i - \overline{\mu}_i \\ \Rightarrow \quad \dfrac{u_j'(q_j) u_k''(q_k)}{u_j''(q_j) u_k'(q_k)} = \dfrac{q_j^i}{q_k^i} & = & \dfrac{\overline{q}_j^i}{\overline{q}_k^i} = \dfrac{u_j'(\overline{q}_j) u_k''(\overline{q}_k)}{u_j''(\overline{q}_j) u_k'(\overline{q}_k)} \end{array}$$

For any two flows that occur in equilibrium. In fact because $u_j'(q_j) q_j^i$ is pseudo-concave in q_j^i, the region on which revenues increase is convex. Such region is always non-empty since $u_j' > 0$. Thus because $\mathbf{CP}(i)$ and $q_j^i > 0$ requires $u_j'(q_j) + u_j''(q_j) q_j^i > 0$ and $\overline{q}_j^i > 0$, it must be that $q_j^i = \overline{q}_j^i$ and that $\mu_i = \overline{\mu}_i$. Thus no profitable deviation of this type exists.

In the second scenario $\sum_{k \in V(i)} q_k^i > \sum_{k \in V(i)} \overline{q}_k^i$. Thus it must be that $\overline{\mu}_i = 0$ because $0 \leq q_i < \overline{q}_i$. If so by assumption A1 and because marginal costs are increasing:

$$\begin{array}{rcl} u_i'(q_i) - u_i''(q_i) \sum_{k \in V(i)} q_i^k & > & u_i'(\overline{q}_i) - u_i''(\overline{q}_i) \sum_{k \in V(i)} \overline{q}_i^k \\ \Rightarrow \quad u_j'(q_j) + u_j''(q_j) q_j^i & \geq & u_j'(q_j) + u_j''(q_j) q_j^i - \mu_i > u_j'(\overline{q}_j) + u_j''(\overline{q}_j) \overline{q}_j^i \end{array}$$

The region on which revenues increase is still convex and non-empty since $u_j' > 0$. Also because $\mathbf{CP}(i)$ and $q_j^i > 0$ requires $u_j'(q_j) + u_j''(q_j) q_j^i > 0$ and $\overline{q}_j^i > 0$, it must be that the marginal revenue are decreasing by concavity. Therefore $q_j^i < \overline{q}_j^i$ for any $j \in V(i)$. But this cannot be since $\sum_{k \in V(i)} q_k^i > \sum_{k \in V(i)} \overline{q}_k^i$ and not profitable deviation of this type exists.

In the last scenario $\sum_{k \in V(i)} q_k^i < \sum_{k \in V(i)} \overline{q}_k^i$. Thus it must be that $\mu_i = 0$ because $0 \leq \overline{q}_i < q_i$. If so by assumption A1 and because marginal costs are increasing:

$$\begin{array}{rcl} u_i'(q_i) - u_i''(q_i) \sum_{k \in V(i)} q_i^k & < & u_i'(\overline{q}_i) - u_i''(\overline{q}_i) \sum_{k \in V(i)} \overline{q}_i^k \\ \Rightarrow \quad u_j'(q_j) + u_j''(q_j) q_j^i & \leq & u_j'(\overline{q}_j) + u_j''(\overline{q}_j) \overline{q}_j^i - \mu_i < u_j'(\overline{q}_j) + u_j''(\overline{q}_j) \overline{q}_j^i \end{array}$$

The region on which revenues increasing is still convex and non-empty whenever $u_j'(q_j - q_j^i) > 0$. Also because $\mathbf{CP}(i)$ and $q_j^i > 0$ requires $u_j'(q_j) + u_j''(q_j) q_j^i > 0$ and $\overline{q}_j^i > 0$, it must be that the marginal revenue are decreasing by concavity. Therefore $q_j^i > \overline{q}_j^i$ for any $j \in V(i)$. But this cannot be since $\sum_{k \in V(i)} q_k^i < \sum_{k \in V(i)} \overline{q}_k^i$ and not profitable deviation of this type exists. ■

Proposition 3 *If for any player $i \in V$ assumption A1 holds, $u_i''' \geq 0$ and :*

$$\frac{\partial}{\partial q}\left(-\frac{u_i''(q)}{u_i'(q)}\right) \geq -\left(\frac{u_i''(q)}{u_i'(q)}\right)^2 \text{ for any } q \in [-Q, Q]$$

Then the costs are convex, the revenues are pseudo-concave and concave if increasing.

Proof. Because A1 holds and since $u_i''' \geq 0$ total costs are convex:

$$-u_i''(q_i) + u_i'''(q_i) \textstyle\sum_{k \in V(i)} q_i^k > 0$$

Now suppose that revenues from selling q_j^i are increasing at q_j. If so it must be that:

$$-\frac{u_j''(q_j)}{u_j'(q_j)} \leq \frac{1}{q_j^i}$$

But the assumption on the absolute risk aversion coefficient implies that:

$$-\frac{u_j'''(q_j)}{u_j''(q_j)} \leq -2\frac{u_j''(q_j)}{u_j'(q_j)} \leq \frac{2}{q_j^i}$$

Which indeed implies that revenues on the sale are concave:

$$2u_j''(q_j) + u_j'''(q_j)q_j^i \leq 0$$

It remains to be shown that revenues $\rho(q_j^i) = u_j'(q_j)q_j^i$ are pseudo-concave. Suppose that they are not. If so the exist $x, y \in \mathbb{R}_+$ such that $\rho'(x)(y - x) \leq 0$ and $\rho(y) > \rho(x)$. Also suppose that $\rho'(x) > 0$. But, because $y \leq x$ and $\rho \in C^3$, there exists $z \in [x, y]$ such that $\rho'(z) = 0$ and $\rho''(z) > 0$, which violates the condition that revenues be concave if increasing. Similarly if $\rho'(x) < 0$. ■

Proposition 4 *For any player $i \in V$ consider a map $\overline{u}_i$ that satisfies assumption A1 on R_+ alone, with $\overline{u}_i''' \geq 0$ and with coefficient of relative risk aversion that satisfies:*

$$\frac{\partial}{\partial q}\left(-\frac{q\overline{u}_i''(q)}{\overline{u}_i'(q)}\right) \geq 0 \text{ for any } q \in [0, Q]$$

Let $u_i(q_i) = \overline{u}_i(c_i + q_i) - \overline{u}_i(c_i)$ denote utility of player i for some constant $c_i \geq Q$. Then sales' costs are convex, the revenues are pseudo-concave and concave if increasing.

Proof. Because A1 holds and since $u_i''' \geq 0$ total costs are convex. Suppose that revenues from selling q_j^i are increasing at q_j. If so it must be that:

$$-\frac{u_j''(q_j)}{u_j'(q_j)} \leq \frac{1}{q_j^i}$$

But the assumption on the relative risk aversion coefficient and $c_j \geq Q \geq q_j^i$ imply that:

$$-\frac{u_j'''(q_j)}{u_j''(q_i)} \leq -\frac{u_j''(q_j)}{u_j'(q_j)} + \frac{1}{c_j + q_j} \leq \frac{1}{q_j^i} + \frac{1}{c_j + q_j} \leq \frac{2}{q_j^i}$$

Revenues on the sale are concave if increasing since $2u_j''(q_j) + u_j'''(q_j)q_j^i \leq 0$.

It remains to be shown that revenues $\rho(q_j^i) = u_j'(q_j)q_j^i$ are pseudo-concave. Suppose that they are not. If so the exist $x, y \in \mathbb{R}_+$ such that $\rho'(x)(y-x) \leq 0$ and $\rho(y) > \rho(x)$. Also suppose that $\rho'(x) > 0$. But, because $y \leq x$ and $\rho \in C^3$, there exists $z \in [x, y]$ such that $\rho'(z) = 0$ and $\rho''(z) > 0$, which violates the condition that revenues be concave if increasing. Similarly if $\rho'(x) < 0$. ■

Theorem 5 *If A1 holds and if:*

(1) marginal utility satisfies $\lim_{q_i \to -\infty} u_i'(q_i) = \infty$ *and endowments are positive*

(2) costs convex, revenues are pseudo-concave and concave if increasing

Then best reply functions are single valued and an outflow equilibrium exists.

If all equilibria are non-degenerate:

Then there is a unique equilibrium if and only if $\det J_T(q, \mu) > 0$ *at any equilibrium*

Proof. Consider the $\mathbf{CP}(i)$ problem. Suppose that $Q_i + \sum_{k \in V(i)} q_i^k \geq 0$ for the problem not to be trivial. Notice that (1) and (2) imply that condition **BC** applies to this smaller system. In fact fix $\mathbf{q}^{-i} \in \mathbb{R}_+^{E-V(i)}$ and define $\overline{q}_j^i > 0$ by:

$$u_i'(Q_i + \textstyle\sum_{k \in V(i)} q_i^k - \overline{q}_j^i) = u_j'(q_j^{-i})$$

If it exists since the left-hand side increasing while the right-hand side is constant. If instead no such number exists let $\overline{q}_j^i = 0$ since:

$$u_i'(Q_i + \textstyle\sum_{k \in V(i)} q_i^k) > u_j'(q_j^{-i})$$

Assumption (2) rules out the possibility that $\overline{q}_j^i$ is unbounded in any trade. Thus define the following set:

$$C = \left\{ (\mathbf{q}^i, \mu_i) \in \mathbb{R}_+^{V(i)+1} \;\middle|\; \begin{array}{c} \sum_{k \in V(i)} q_k^i \leq \max_{j \in V(i)} \left\{ \overline{q}_j^i \right\} \text{ and} \\ \mu_i \leq \max_{j \in V(i)} \left\{ u_j'(q_j^{-i}) \right\} \frac{\max_{j \in V(i)} \{\overline{q}_j^i\}}{Q_i + \sum_{k \in V(i)} q_i^k} \end{array} \right\}$$

Such set is closed and bounded and therefore compact. Consider $(\mathbf{q}^i, \mu_i) \in \mathbb{R}_+^{V(i)+1} \backslash C$ and suppose that either $\sum_{k \in V(i)} q_k^i > \max_{j \in V(i)} \left\{ \overline{q}_j^i \right\}$. In such scenario for the boundary condition of problem $\mathbf{CP}(i)$ to hold it must be that:

$$\sum_{j \in V(i)} q_j^i f_j^i(\mathbf{q}, \mu_i) + \mu_i q_i = \sum_{j \in V(i)} -\frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} + \mu_i (Q_i + \textstyle\sum_{k \in V(i)} q_i^k) > 0$$

But by assumption marginal utility decreases and because $\sum_{k\in V(i)} q_k^i > \max_{j\in V(i)} \left\{\overline{q}_j^i\right\}$:

$$\begin{aligned} -\frac{\partial w_i(\mathbf{q})}{\partial q_j^i} &\geq -\frac{\partial w_i(\mathbf{q})}{\partial q_j^i} + [u_j'(q_j) - u_j'(q_j^{-i})] + q_j^i u_j''(q_j) + \sum_{k\in V(i)} q_i^k u_i''(q_i) = \\ &= u_i'(q_i) - u_j'(q_j^{-i}) > u_i'(Q_i + \textstyle\sum_{k\in V(i)} q_i^k - \overline{q}_j^i) - u_j'(q_j^{-i}) \geq 0 \end{aligned}$$

Thus the boundary condition is satisfied if the first condition fails. If instead $\sum_{k\in V(i)} q_k^i \leq \max_{j\in V(i)} \left\{\overline{q}_j^i\right\}$, but $\mu_i > \max_{j\in V(i)} \left\{u_j'(q_j^{-i})\right\} \frac{\max_{j\in V(i)}\left\{\overline{q}_j^i\right\}}{Q_i+\sum_{k\in V(i)} q_i^k}$ the boundary condition still holds since:

$$\begin{aligned} &\sum_{j\in V(i)} q_j^i f_j^i(\mathbf{q}, \mu_i) + \mu_i q_i = \\ = &\sum_{j\in V(i)} q_j^i [u_i'(q_i) - u_i''(q_i) \sum_{k\in V(i)} q_i^k - u_j'(q_j) - q_j^i u_j''(q_j)] + \mu_i (Q_i + \sum_{k\in V(i)} q_i^k) > \\ \geq &\sum_{j\in V(i)} q_j^i [-u_j'(q_j)] + \mu_i (Q_i + \sum_{k\in V(i)} q_i^k) \geq \sum_{j\in V(i)} q_j^i [-u_j'(q_j^{-i})] + \mu_i (Q_i + \sum_{k\in V(i)} q_i^k) > \\ > &\min_{j\in V(i)} \left\{-u_j'(q_j^{-i})\right\} \sum_{j\in V(i)} q_j^i + \max_{j\in V(i)} \left\{u_j'(q_j^{-i})\right\} \frac{\max_{j\in V(i)} \left\{\overline{q}_j^i\right\}}{Q_i + \sum_{k\in V(i)} q_i^k} (Q_i + \sum_{k\in V(i)} q_i^k) \geq \\ \geq &\max_{j\in V(i)} \left\{u_j'(q_j^{-i})\right\} \left(\max_{j\in V(i)} \{\overline{q}_j^i\} - \sum_{j\in V(i)} q_j^i\right) \geq 0 \end{aligned}$$

Since all other assumptions for Kolstad and Mathiensen result apply to this reduced system we get that the best reply of each player correspondence exists. Additionally if $J_{T(i)}(\mathbf{q}, \mu_i) < 0$ at all solutions of $\mathbf{CP}(i)$ for any $\mathbf{q}^{-i} \in \mathbb{R}_+^{E-V(i)}$, then player i's best reply is single-valued. But since an outflow to occurs in equilibrium only if marginal revenues in that market are increasing By assumptions they are also concave. Thus the Jacobian of the system is always positive definite at a solution. Indeed the Jacobian of the system is a leading minor of the bordered Hessian of player i's welfare. Thus if $T(i) = \left\{j \in V(i) | q_j^i > 0\right\} \cup \{i | \mu_i > 0\}$ one gets that $\det J_{T(i)}(\mathbf{q}, \mu_i)$ is equal to:

$$\begin{cases} \left[1 + \sum_{j\in T(i)} \frac{u_i''(q_i) - u_i'''(q_i) \sum_{k\in V(i)} q_i^k}{2u_j''(q_j) + u_j'''(q_j) q_j^i}\right] \prod_{j\in T(i)} -\left[2u_j''(q_j) + u_j'''(q_j) q_j^i\right] & \text{if} \quad i \notin T(i) \\ \left[\sum_{j\in T(i)} \frac{1}{2u_j''(q_j) + u_j'''(q_j) q_j^i}\right] \prod_{j\in T(i)} -\left[2u_j''(q_j) + u_j'''(q_j) q_j^i\right] & \text{if} \quad i \in T(i) \end{cases}$$

Which is positive in an outflow equilibrium since by assumption $u_j'(q_j) + u_j''(q_j) q_j^i > 0$ implies $2u_j''(q_j) + u_j'''(q_j) q_j^i < 0$.

Therefore because best reply functions are single-valued, Brower fixed point theorem applies. Thus an equilibrium exists. The uniqueness conditions follow then trivially since if the system is increasing at all fixed points can have at most a single fixed point. ■

Corollary 6 *If the assumptions of either lemma 3 or lemma 24 hold, if all outflow equilibria (if any) are non-degenerate and if endowments are positive then an outflow equilibrium exists.*

Proof. Notice that the aforementioned lemmata provide conditions on the utility function for which conditions (3i) and (3ii) hold. While A1 is implicit in the assumptions of the lemmata. Moreover $u_i''' > 0$ implies $\lim_{q_i \to -\infty} u_i'(q_i) = \infty$. Therefore the previous theorem applies. ■

Basic Properties of the Outflow Model

Remark 7 *In any pure strategy equilibrium* $\mathbf{q} \in \mathbb{R}_+^E$ *of the outflow competition model the set of active trading links* $T(\mathbf{q}) = \left\{ij \in E | q_j^i > 0\right\}$ *contains no cycle.*

Proof. If $ij \in T(\mathbf{q})$ by first order optimality it must be that $u_i'(q_i) < u_j'(q_j)$. If by way of contradiction a cycle $c = \{ij, jk, ..., li\}$ were to belong to $T(\mathbf{q})$, thus one would get that:

$$u_i'(q_i) < u_j'(q_j) < u_k'(q_k) < ... < u_l'(q_l) < u_i'(q_i)$$

and a contradiction. ■

Remark 8 *In any outflow equilibrium an individual is a source (sink) if his marginal utility of consumption is not higher (lower) than that of any of his neighbors.*

Proof. If for player $i \in V$ and any neighbor $j \in V(i)$ equilibrium dictates that $u_i'(q_i) \leq u_j'(q_j)$, then i cannot buy from any neighbor. Indeed the first order conditions for goods flowing to him cannot hold with equality. Therefore his neighbors prefer not to sell to him and $q_i^j = 0$. Similarly whenever $u_i'(q_k) \geq u_j'(q_j)$ for any $j \in V(i)$, player i cannot be selling to any neighbor, since $u_i'(q_i) < u_j'(q_j)$ is necessary for $q_j^i > 0$. ■

Remark 9 *In any outflow equilibrium sources sell to all their neighbors with strictly higher marginal utility*

Proof. By lemma 8 if i is a source $u_i''(q_i) \sum_{k \in V(i)} q_i^k = 0$. Which in turn implies that player i sells to any neighbor $j \in V(i)$ with $u_i'(q_i) < u_j'(q_j)$. Since there always exists $q_j^i > 0$ for which:

$$-u_i'(q_i) + u_j'(q_j) + q_j^i u_j''(q_j) = 0$$

proving the result. ■

Remark 10 *In any outflow equilibrium if* $i, j \in V(k)$ *and* $u_i'(q_i) < u_j'(q_j)$, *then* $q_i^k > 0$ *implies* $q_j^k > 0$.

Proof. Given the stated assumptions optimality of the trade from k to i requires that:

$$u_k'(q_k) - u_k''(q_k) \sum_{l \in V(k)} q_k^l = u_i'(q_i) + q_i^k u_i''(q_i) < u_i'(q_i)$$

Therefore it must be that:

$$u_k'(q_k) - u_k''(q_k) \sum_{l \in V(k)} q_k^l < u_j'(q_j)$$

Which is necessary and sufficient for a trade from k to j to occur whenever a pure strategy equilibrium exists. ■

Lemma 11 *If individuals have arbitrarily many instances to trade, do not discount and do not account for future actions the sequence of outflow equilibrium trades converges to the constrained efficient outcome whenever such outcome is interior.*

Proof. The proof first shows that the lowest marginal utility individuals at any iteration always sell at a later iteration. Then it shows that this requires the sequence of equilibrium allocations to converge. Finally it concludes by showing that such sequence can only converge to the constrained efficient outcome.

It is without loss of generality to restrict attention to connected networks, since separate components of the network will have no influence on each other. Let $q_j^i(t)$ denote the flow of goods from i to j at the t^{th} round of trading, for $t \in \{1, 2...\}$. Consider an inefficient round $t-1$ equilibrium outcome $\mathbf{q}(t-1)$ and an individual $i \in \arg\min_{i \in V} u_i'(q_i(t-1))$. Indeed for any such player i it must be that $q_i(t-1) > 0$ because the efficient outcome being interior implies that:

$$q_i(t-1) > q_i^* > 0$$

Moreover since i is a source at round $t-1$ he sells to all his neighbors with strictly lower marginal utility by lemma 11. But since the graph is connected there exists $i \in \arg\min_{k \in V} u_k'(q_k(t-1))$ and $j \in V(i)$ such that:

$$u_i'(q_i(t-1)) < u_j'(q_j(t-1))$$

or else the outflow equilibrium outcome would be efficient. Therefore $q_j^i(t) > 0$, because i cannot have any inflows at round t and because outflow price distortions vanish. But since the minimal equilibrium marginal utility weakly increases at each iteration and is bounded by $u_i'(q_i^*)$ it must converge. If the upper-bound on the minimal marginal utility holds with equality in the limit the allocation is efficient since marginal utility of all the individuals in the economy converges.

Now by way of contradiction suppose the upper-bound on the minimal marginal utility does not hold in the limit. So that:

$$\lim_{t \to \infty} \min_{k \in V} u_k'(q_k(t-1)) < u_i'(q_i^*)$$

Let $\lim_{t \to \infty} q_i(t-1) = \overline{q}_i$. And notice that $\overline{\mathbf{q}}$ cannot be the limiting outcome, since if individuals had an additional round to trade at $\overline{\mathbf{q}}$ they would have a incentives to do so. Again because there exists $i \in \arg\min_{k \in V} u_k'(\overline{q}_k)$ and $j \in V(i)$ such that $u_i'(\overline{q}_i) < u_j'(\overline{q}_j)$. Thus a contradiction would arise. Hence it must be that:

$$\lim_{t \to \infty} \min_{k \in V} u_k'(q_k(t-1)) = u_i'(q_i^*)$$

which implies that all player end up with the same marginal utility and that the limiting outcome is constrained efficient. ■

The Complete Network Economy

Proposition 12 *In any equilibrium of the complete networked economy if $u_i'(q_i) < u_j'(q_j)$ then:*
(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ implies $q_k^i > q_k^j$
(2) $q_j^i \leq q_i^j$ and $q_j^i > 0$ implies $q_i^j = 0$

Proof. (1) First assume that $i \in \arg\min_{k \in V} u_k'(q_k)$. If so by lemma 8 i sells to all players with marginal utility strictly higher than him and does not buy. If $q_k^i > 0$ for some k, but no other player sells the claim is trivial. Thus suppose that there exists a player j also selling to a k. By assumption it must be that $u_i'(q_i) \leq u_j'(q_j)$. Consider two different scenarios in the first $u_i'(q_i) = u_j'(q_j)$. In which case j does not buy from any player and sells to all players with higher marginal utility by lemma 8. Therefore both i and j sell to the same set of players and in equal amounts $q_j^i = q_j^k$ since optimality requires that:

$$u_i'(q_i) = u_j'(q_j) + u_j''(q_j)q_j^i = u_k'(q_i) = u_j'(q_j) + u_j''(q_j)q_j^k$$

In the second scenario $u_i'(q_i) < u_j'(q_j)$. Therefore by lemma 8 player j buys from i. Consider then any player k buying from both. Optimality for the three trades requires that:

$$\begin{array}{rcl} u_i'(q_i) & = & u_j'(q_j) + q_j^i u_j''(q_j) = u_k'(q_k) + q_k^i u_k''(q_k) \\ u_j'(q_j) & < & u_j'(q_j) - u_j''(q_j) \sum_{l \neq j} q_j^l = u_k'(q_k) + q_k^j u_k''(q_k) \end{array}$$

In turn by manipulation such expressions imply that:

$$q_k^i - q_k^j > q_j^i \frac{u_j''(q_j)}{u_k''(q_k)} \geq 0$$

Which proves that the least marginal utility player sells more to all than any other player to k.

Finally consider the case in which $i, j \notin \arg\min_{k \in V} u_k'(q_k)$. Again assume that $u_i'(q_i) < u_j'(q_j)$ and $q_k^i > 0$ for some $k \in V$. If so and if $q_k^j = 0$ the result holds trivially. Consider the case in which $q_k^j > 0$. Let $S(i) = \left\{ i \in V | q_i^k > 0 \right\}$. Then notice that by lemma 8 and 10 it must be that $\emptyset \neq S(i) \subseteq S(j)$. By the optimality of the trades from any $h \in S(i)$ to i and j and from i and j to k it must be that:

$$\begin{array}{rcl} u_i'(q_i) + u_i''(q_i)q_i^h & = & u_j'(q_j) + u_j''(q_j)q_j^h \\ u_j'(q_j) - u_j''(q_j) \sum_{l \neq j} q_j^l & = & u_k'(q_k) + u_k''(q_k)q_k^j \\ u_i'(q_i) - u_i''(q_i) \sum_{l \neq i} q_i^l & = & u_k'(q_k) + u_k''(q_k)q_k^i \end{array}$$

Exploiting the conditions of the system it is possible to get that:

$$\begin{array}{rcl} u_j'(q_j) - u_i'(q_i) & = & u_k''(q_k)(q_k^j - q_k^i) + u_j''(q_j) \sum_{l \in S(j)} q_j^l - u_i''(q_i) \sum_{l \in S(i)} q_i^l \\ & = & u_k''(q_k)(q_k^j - q_k^i) + u_j''(q_j) \sum_{l \in S(j) \setminus S(i)} q_j^l - \sum_{l \in S(i)} (u_i''(q_i)q_i^l - u_j''(q_j)q_j^l) \\ & = & u_k''(q_k)(q_k^j - q_k^i) + u_j''(q_j) \sum_{l \in S(j) \setminus S(i)} q_j^l - |S(i)| \left(u_j'(q_j) - u_i'(q_i) \right) \end{array}$$

Which in turn implies the desired condition since:

$$(q_k^i - q_k^j) = \frac{u_j''(q_j)}{u_k''(q_k)} \sum_{l \in S(j) \setminus S(i)} q_j^l - (|S(i)| + 1) \frac{u_j'(q_j) - u_i'(q_i)}{u_k''(q_k)} > 0$$

It remains to be shown that $q_k^i = 0$ implies $q_k^j = 0$. If player i does not buy any consumption the claim is trivial since:

$$u_j'(q_j) - u_j''(q_j) \sum_{l \neq j} q_j^l > u_i'(q_i) \geq u_k'(q_k)$$

So suppose that player i buys the recall that $\emptyset \neq S(i) \subseteq S(j)$. To prove the claim it suffice to show that:

$$u_j'(q_j) - u_j''(q_j) \sum_{l \neq j} q_j^l > u_i'(q_i) - u_i''(q_i) \sum_{l \neq i} q_i^l$$

But as above notice that:

$$\begin{aligned} u_j'(q_j) - u_j''(q_j) \sum_{l \neq j} q_j^l - u_i'(q_i) + u_i''(q_i) \sum_{l \neq i} q_i^l &= \\ (u_j'(q_j) - u_i'(q_i))(|S(i)| + 1) - u_j''(q_j) \sum_{l \in S(j) \setminus S(i)} q_j^l &> 0 \end{aligned}$$

(2) By $u_i'(q_i) < u_j'(q_j)$ it follows directly that $q_i^j = 0$. ■

Assumption A2 *For any player $i \in V$ assume that $u_i = u$ and the coefficient of absolute risk aversion is weakly increasing:*

$$\frac{\partial \left(-u''(q)/u'(q)\right)}{\partial q} \geq 0$$

Proposition 13 *If A1 and A2 hold in any pure strategy equilibrium of the complete networked economy $q_i > q_j$ if and only if*

(1) $q_k^i \geq q_k^j$ and $q_k^i > 0$ implies $q_k^i > q_k^j$

(2) $q_i^k \leq q_j^k$ and $q_j^k > 0$ implies $q_i^k < q_j^k$

(3) $q_j^i \leq q_i^j$ and $q_j^i > 0$ implies $q_i^j = 0$

Proof. First it is shown that $q_i > q_j$ implies (1), (2) and (3). Conditions (1) and (3) follow directly from proposition 12 given that assumption A2 implies that $q_i > q_j$ if and only if $u'(q_i) < u'(q_j)$. The proof of condition (2) instead relies heavily on the symmetry implicit in A2. If $q_j^k > 0$ and $q_i^k = 0$ the claim is trivial. Also recall that by lemma 3 if $q_j^k = 0$ then $q_i^k = 0$. So suppose that both are positive $q_j^k > 0$ and $q_i^k > 0$. Then by optimality for the two trades it must be that:

$$u'(q_k) - u''(q_k) \sum_{l \in V(k)} q_k^l = u'(q_i) + q_i^k u''(q_i) = u'(q_j) + q_j^k u''(q_j) \geq 0$$

Thus rewriting and the two equalities one gets that:

$$q_i^k - q_j^k = \frac{(u_i'' u_j' - u_i' u_j'') + (u_k' - u_k'' \sum_{l \in V(k)} q_k^l)(u_j'' - u_i'')}{u_i'' u_j''} \tag{2}$$

Notice that the denominator is always positive. The second term in the numerator is negative since $q_j < q_i$ implies $u''(q_j) < u''(q_i)$ by $u''' > 0$. The first term is also negative since increasing absolute risk aversion and $q_j < q_i$ imply:

$$-\frac{u''(q_j)}{u'(q_j)} \leq -\frac{u''(q_i)}{u'(q_i)} \quad \Leftrightarrow \quad u''(q_i)u'(q_j) - u'(q_i)u''(q_j) \leq 0$$

Now the converse is proven. First suppose that $q_i, q_j < q_k = \max_l q_l$. Then since both i and j buy from k condition 2 needs to hold. Since assumption it must be that $q_i^k - q_j^k < 0$ it must be that:

$$\frac{(u_i''u_j' - u_i'u_j'') + (u_k' - u_k'' \sum_{l \in V(k)} q_k^l)(u_j'' - u_i'')}{u_i''u_j''} < 0 \tag{3}$$

But notice that assumptions A1 and A2 require that:

$$u''(q_j) > u''(q_i) \quad \Leftrightarrow \quad q_j < q_i \quad \Leftrightarrow \quad u'(q_i)u''(q_j) - u''(q_i)u'(q_j) < 0$$

Therefore since the denominator of 3 is positive and since both terms in the numerator have the same sign, they must both be negative which implies $q_j < q_i$.

Thus consider the case in which $\max\{q_i, q_j\} = \max_l q_l > \min\{q_i, q_j\}$. If $q_j > q_i$, then lemma 8 would require $q_i^j > 0$ which would contradict condition (3). Finally if $q_i = q_j = \max_l q_l$, then $q_k^i > 0$ implies $q_k^i = q_k^j$, since by lemma 2 both trade with k and:

$$\begin{aligned} u'(q_i) &= u'(q_k) + q_k^i u''(q_k) \\ u'(q_j) &= u'(q_k) + q_k^j u''(q_k) \end{aligned}$$

Which implies the desired condition since:

$$q_k^i - q_k^j = \frac{u'(q_i) - u'(q_j)}{u''(q_k)} = 0$$

Which contradicts condition (1). Thus A2 and conditions (1), (2), (3) imply $q_i > q_j$. ■

Proposition 14 If A1 and A2 hold in any outflow equilibrium of the complete networked economy $Q_i > Q_j$ if and only if $q_i > q_j$.

Proof. If $q_i > q_j$ in equilibrium then conditions (1)-(3) of the previous proposition holds. Condition (1) and the network being complete imply that $\sum_{k \in V(i)} q_k^i > \sum_{k \in V(j)} q_k^j$. While condition (2) requires that $\sum_{k \in V(j)} q_j^k > \sum_{k \in V(i)} q_i^k$. Thus one gets that:

$$Q_i - Q_j = \left(\sum_{k \in V(i)} q_k^i - \sum_{k \in V(j)} q_k^j \right) + \left(\sum_{k \in V(j)} q_j^k - \sum_{k \in V(i)} q_i^k \right) > 0$$

To prove the converse notice that if $Q_i > Q_j$ and $q_i < q_j$ then condition (1) implies $\sum_{k \in V(i)} q_k^i < \sum_{k \in V(j)} q_k^j$ and condition (2) implies $\sum_{k \in V(j)} q_j^k < \sum_{k \in V(i)} q_i^k$. Thus:

$$Q_i - Q_j = \left(\sum_{k \in V(i)} q_k^i - \sum_{k \in V(j)} q_k^j \right) + \left(\sum_{k \in V(j)} q_j^k - \sum_{k \in V(i)} q_i^k \right) < 0$$

A contradiction. Finally notice that if $Q_i > Q_j$ and $q_i = q_j$. Then since the problem of the two players is symmetric $q_i^k = q_j^k$ and $q_k^i = q_k^j$. Which implies that:

$$Q_i - Q_j = 0$$

Again a contradiction. ■

Large Economies and the Competitive Equilibrium

Proposition 15 *Consider a sequence of replica economies $\{G^r, Q^r, u^r\}_{r=1}^{\infty}$, if $V(i) \supseteq S$ for any $i \in B$ then any symmetric outflow equilibrium converges to the competitive equilibrium as the economy grows large.*

Proof. Since all copies of player $i \in V$ behave in the same way in the symmetric equilibrium of any replica economy, denote by $q_j^i(r)$ the amount of good traded by a copy of i to a copy of j in the r-replica. That is $q_j^i(r) = q_{j(t)}^{i(s)}(r)$ for any $i(s), j(t) \in V^r$.

Lemma 8 established that if $i \in \arg\min_{k \in V} u_k'(q_k(r))$ then any copy of player i never buys and sells to all players with strictly higher marginal utility. Thus $Q_i > q_i(r)$, unless the endowment is efficient. In which case $Q_i = q_i(r)$ because the endowment is competitive to begin with. Additionally since trade never equalizes marginal utility in the outflow model because of the price effects and since i has the lowest marginal utility, it must be that $u_i'(q_i(r)) < u_i'(q_i^*)$. Thus $Q_i > q_i(r) > q_i^*$. Which implies that $i \in S$. Then consider the player with $j \in \arg\max_{k \in V} u_k'(q_k(r))$. By lemma 2 player j only buys. Therefore unless the endowment is efficient $Q_j < q_j(r)$. Since j has the highest marginal utility for consumption in equilibrium it must also be that $u_j'(q_j(r)) > u_j'(q_j^*)$. Thus $Q_j < q_j(r) < q_j^*$. Which implies that $j \in B$.

By assumption it must be that $ij \in E$ since $i \in S$ and $j \in B$. Additionally by lemma 8 and since $u_i'(q_i(r)) < u_j'(q_j(r))$, it must be that $q_j^i(r) > 0$. Optimality for such trade requires that:

$$q_j^i(r) = \frac{u_i'(q_i(r)) - u_j'(q_j(r))}{u_j''(q_j(r))}$$

Notice that if $\lim_{r \to \infty} q_j^i(r) = 0$ were be satisfied, the claim would be proven, because the marginal utility of all players would be equalized in equilibrium.

By way of contradiction suppose that the contrary is true, $\lim_{r \to \infty} q_j^i(r) = \overline{q} > 0$. If so because j never sells:

$$\lim_{r \to \infty} q_j(r) = \lim_{r \to \infty} \left(Q_j + r \sum_{k \in V(j)} q_j^k(r) \right) = \infty$$

But leads to a contradiction since it implies that $\lim_{r \to \infty} u_j(q_j(r)) = 0$ and therefore:

$$\lim_{r \to \infty} q_j^i(r) = \lim_{r \to \infty} \frac{u_i'(q_i(r)) - u_j'(q_j(r))}{u_j''(q_j(r))} \leq \lim_{r \to \infty} \frac{u_i'(Q_i) - u_j'(q_j(r))}{u_j''(q_j(r))} = \frac{u_i'(Q_i)}{\lim_{r \to \infty} u_j''(q_j(r))} \leq 0$$

Where the first inequality holds since $u_i'(q_i(r)) \geq u_i'(Q_i)$ and the latter since u is increasing and concave.

Thus in the limit all marginal utilities are equalized and a unique price reigns in the economy, namely the marginal utility of consumption. Hence the equilibrium becomes competitive. ■

Proposition 16 *If the outflow equilibrium of a sequence of replica economies* $\{G^r, Q^r, u^r\}_{r=1}^{\infty}$ *converges to the competitive equilibrium, then outflow equilibrium resale vanishes as the economy grows large.*

Proof. If the equilibrium of the replica economy becomes competitive it must be that for any $i, j \in V$:

$$\lim_{r\to\infty} \left(u_i'(q_i(r)) - u_j'(q_j(r))\right) = 0$$

Then suppose that there is a player which in the limit has both a positive amount of inflows and a positive amount of outflows. That is $\exists i \in V$ such that:

$$\begin{cases} \lim\limits_{r\to\infty} \left(r \sum_{k\in V(i)} q_i^k(r)\right) > 0 \\ \lim\limits_{r\to\infty} \left(r \sum_{k\in V(i)} q_k^i(r)\right) > 0 \end{cases}$$

If so it must be that $q_j^i(r) > 0$ for some $j \in V$ if r is large enough. Optimality of such trade requires that:

$$q_j^i(r) = \frac{u_i'(q_i(r)) - u_j'(q_j(r)) - u_i''(q_i(r))\left(r \sum_{k\in V(i)} q_i^k(r)\right)}{u_j''(q_j(r))}$$

But if the equilibrium becomes competitive a contradiction arises:

$$\lim_{r\to\infty} q_j^i(r) = \lim_{r\to\infty} -\frac{u_i''(q_i(r))}{u_j''(q_j(r))}\left(r \sum_{k\in V(i)} q_i^k(r)\right) < 0$$

Hence as the economy grows large if i is a buyer in the limiting economy he cannot be a supplier. Thus resale vanishes. ■

Proposition 17 *If assumption A1 holds, if* $u_i''' > 0$ *and if all symmetric outflow equilibria of the limiting economy (if any) are non degenerate, then the limiting economy always possesses a symmetric outflow equilibrium.*

Proof. Since in the limiting economy the outflow markup vanishes. Revenues in each market are concave. Since the third derivative is positive costs of supplying units are convex. Therefore lemma 5 applies. ■

Proposition 18 *If sufficient conditions for existence are met and if for any replica* r *the economy possesses a unique symmetric outflow equilibrium, then in such equilibrium per-capita social welfare increases every time the economy is replicated.*

Proof. Define the total quantity sold from individuals of type i to individuals of type j in the symmetric equilibrium of an r-replica economy by $\overline{q}_j^i = rq_j^i$ The inequalities defining the complementarity problem at the symmetric equilibrium of an r-replica economy can be written in terms of such quantities by:

$$\begin{aligned} f_j^i(\mathbf{q}, \boldsymbol{\mu}|r) &= -u_j'(\overline{q}_j) - u_j''(\overline{q}_j)(\overline{q}_j^i/r) + u_i'(\overline{q}_i) - u_i''(\overline{q}_i) \textstyle\sum_{k\in V(i)}^{k} \overline{q}_i^k + \mu_i \geq 0 \\ f_i(\mathbf{q}, \boldsymbol{\mu}|r) &= \overline{q}_i \geq 0 \end{aligned}$$

Notice that in such system of equations the replica counter r appears only once. Moreover for $r = 1$ the conditions are those of the complementarity problem for the original economy. By assumption any replica economy possesses a unique equilibrium and conditions for existence are met. Therefore the Jacobian of the complementarity problem is positive definite at the unique symmetric equilibrium:

$$J_{T_r}(\overline{\mathbf{q}}, \boldsymbol{\mu}) = \nabla_{T_r} f(\overline{\mathbf{q}}, \boldsymbol{\mu}|r) > 0$$

where only the principal minor of Jacobian associated the active indices is considered. The indices active in the r-replica ate defined by:

$$T_r(\overline{\mathbf{q}}, \boldsymbol{\mu}) = \left\{ij{\in}E|\overline{q}^i_j > 0\right\} \cup \{i \in V|\mu_i > 0\}$$

Notice that by the implicit function theorem it must be that at the unique equilibrium of the r-replica:

$$\begin{aligned} \frac{\partial f}{\partial \overline{\mathbf{q}}}\frac{\partial \overline{\mathbf{q}}}{\partial r} + \frac{\partial f}{\partial \boldsymbol{\mu}}\frac{\partial \boldsymbol{\mu}}{\partial r} + \frac{\partial f}{\partial r} &= J_{T_r}(\overline{\mathbf{q}}, \boldsymbol{\mu})\frac{\partial(\overline{\mathbf{q}}, \boldsymbol{\mu})}{\partial r} + \frac{\partial f}{\partial r} = 0 \\ \frac{\partial(\overline{\mathbf{q}}, \boldsymbol{\mu})}{\partial r} &= -J_{T_r}(\overline{\mathbf{q}}, \boldsymbol{\mu})^{-1}\frac{\partial f}{\partial r} \end{aligned}$$

Moreover by the definition of the complementarity problem it must be that:

$$\frac{\partial f^i_j(\overline{\mathbf{q}}, \boldsymbol{\mu})}{\partial r} = \frac{u''_j(\overline{q}_j)\overline{q}^i_j}{r^2} \quad \text{and} \quad \frac{\partial f_i(\overline{\mathbf{q}}, \boldsymbol{\mu})}{\partial r} = 0$$

For notational convenience label players so that in the unique outflow equilibrium of the original economy $q_1 \geq q_2 \geq ... \geq q_V$. Also define:

$$\begin{aligned} z &= \left\{u''_j(\overline{q}_j)\overline{q}^i_j\right\}_{ij\in E} \quad \text{and} \quad R = \left\{r^{ij}_{kl}\right\}_{ij,kl\in E} \\ r^{ij}_{kl} &= \begin{cases} 1/r & \text{if} \quad ij = kl \\ 1 & \text{if} \quad j = k \, \cap \overline{q}^i_j > 0 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \end{aligned}$$

For such notation and letting $E_r(\overline{\mathbf{q}}) = \left\{ij{\in}E|\overline{q}^i_j > 0\right\}$, one gets that:

$$\begin{aligned} Rz &= \left\{u''_j(\overline{q}_j)(\overline{q}^i_j/r) + u''_i(\overline{q}_i)\textstyle\sum_{k\in V(i)} \overline{q}^k_i\right\}_{ij\in E} \\ \frac{\partial \overline{\mathbf{q}}}{\partial r} &= -\frac{1}{r^2}J_{E_r}(\overline{\mathbf{q}}, \boldsymbol{\mu})^{-1}z \end{aligned}$$

Where $J_{E_r}(\overline{\mathbf{q}}, \boldsymbol{\mu})^{-1}$ is the leading minor of $J_{T_r}(\overline{\mathbf{q}}, \boldsymbol{\mu})^{-1}$ associated with indexes in $E_r(\overline{\mathbf{q}})$. The matrix R is positive definite, since for an appropriate ordering of links it is lower triangular and because all elements on the main diagonal are positive. The matrix can be arranged in a triangular fashion for any profile of equilibrium flows, because goods do not cycle in the economy. Thus it is possible to order the trade flow matrix so that it is triangular. Now by differentiating per-capita social welfare with respect to r get that:

$$\begin{aligned}\frac{\partial S(\overline{\mathbf{q}})}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{V}\sum\nolimits_{i\in V} u_i(\overline{q}_i)\right) = \frac{1}{V}\sum\nolimits_{ij\in E}\frac{\partial \overline{q}^i_j}{\partial r}(u'_j(\overline{q}_j) - u'_i(\overline{q}_i)) = \\ &= -\frac{1}{V}\sum\nolimits_{ij\in E}\frac{\partial \overline{q}^i_j}{\partial r}\left(u''_j(\overline{q}_j)\overline{q}^i_j + u''_i(\overline{q}_i)\sum\nolimits_{k\in V(i)}\overline{q}^k_i\right) = \\ &= -\frac{1}{V}z'R'\frac{\partial \overline{\mathbf{q}}}{\partial r} = \frac{1}{Vr^2}z'R'J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})^{-1}z \geq 0\end{aligned}$$

Notice that the last expression is positive since it is a bilinear form and because both R' and $J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})$ are positive definite. In fact because are positive definite, consider the positive definite square root S of $J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})^{-1}$ (i.e. $J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})SS = I$) then $R'J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})^{-1} = S^{-1}(SR'S)S$. Therefore $R'J_{E_r}(\overline{\mathbf{q}},\boldsymbol{\mu})^{-1}$ and $SR'S$ have the same eigenvalues. Since $SR'S = S'R'S$, such matrix is positive definite and thus has only non-negative eigenvalues. The third equality uses the observation that $\partial\overline{q}^i_j/\partial r \neq 0$ implies that the first order condition holds with equality at the original allocation. If in fact $\partial\overline{q}^i_j/\partial r < 0$ then it must be that $\overline{q}_i > 0$, but $(r-1)q^i_j(r-1) > 0$ also implies $\overline{q}^i_j = rq^i_j(r) > 0$. Also if $\partial\overline{q}^i_j/\partial r > 0$ then $\overline{q}^i_j > 0$ and moreover since $(r-1)q_i(r-1) < 0$ first order conditions hold at r as well. ■

More on Existence

Lemma 20 *If assumption A1 and one of the following holds:*
(1) $u'''_i \geq 0$ *and* $2u''_i(q) + u'''_i(q)q' < 0$ *for any* $q \in [0,Q]$, any $q' \in [0, Q-q]$
(2) $u'''_i \leq 0$ *and* $u''_i(q) - u'''_i(q)q' < 0$ *for any* $q \in [0,Q]$, any $q' \in [0,q]$
Then player i's welfare *is concave.*

Proof. Let $M_l \subseteq V(i)$ denote the indices of the l^{th} leading minor of the Hessian matrix. Recall that $M_{l-1} \subseteq M_l$ for any $l \in \{2, ..., |V(i)|\}$. The determinant of any leading minor $M_l \subseteq V(i)$ of the Hessian matrix:

$$\det H_{M_l}(\mathbf{q}) = \left[1 + \sum_{j\in M_l}\frac{u''_i(q_i) - u'''_i(q_i)\sum_{k\in V(i)} q^k_i}{2u''_j(q_j) + u'''_j(q_j)q^i_j}\right]\prod_{j\in M_l} -\left[2u''_j(q_j) + u'''_j(q_j)q^i_j\right]$$

If all the terms in the sum are positive and those in the product are negative the determinants of all leading minors completely determined by the product and alternate in signs. This in turn implies that by Sylvester's criterion the Hessian matrix is negative definite. Assumption A1 plus either condition (1) or (2) guarantee that the terms in the sum be all positive and those in the product be negative. ■

Lemma 21 *Any outflow equilibrium solves* ***CP****. Further if individual welfare is pseudo-concave with respect to own outflows an allocation is an outflow equilibrium if and only if solves* ***CP****.*

Proof. The conditions of the **CP** problem are the Karush Kuhn Tucker first order necessary conditions for optimality of all players. Therefore they must satisfied in an outflow equilibrium. Otherwise a profitable deviation would exist for some player.

Since choice sets are non-empty and convex, if the individual welfare of all players is pseudo-concave in the outflows the first order conditions are necessary and sufficient for global optimality.

Thus in such scenario if an allocation satisfies the **CP** conditions then it must be an outflow equilibrium. ■

Theorem 22 *If utility functions are* C^3 *and if:*
(1) outflows are bounded and endowments are positive
(2) individual welfare is pseudo-concave with respect to own outflows
Then an outflow equilibrium exists. Moreover:
If all equilibria are non-degenerate, $\det J_T(q,\mu) > 0$ *at any equilibrium if and only if there is a unique equilibrium*

Proof. If outflows are bounded the boundary condition for the **CP** problem holds. In fact suppose that there exists a non-empty compact set $C \subset \mathbb{R}_+^E$ such that for any $\mathbf{q} \in \mathbb{R}_+^E \backslash C$:

$$-\frac{\partial w_i(\mathbf{q})}{\partial q_j^i} > 0 \text{ for any } j \in V(i) \text{ and any } i \in V$$

Suppose $Q_i > 0$ for each player for the moment and define for any individual $i \in V$:

$$\pi_i = \frac{|E|}{Q_i} \max_{\overline{\mathbf{q}} \in C} \max_{ij \in E} \frac{q_j^i \partial w_i(\overline{\mathbf{q}})}{\partial q_j^i}$$

Since C is compact and because individual welfare is pseudo-concave and two times continuously differentiable, π_i is finite. In fact $q_j^i(\partial w_i(\mathbf{q})/\partial q_j^i) = \infty$ implies that:

$$q_j^i u_j'(q_j) = \infty$$

Which cannot be if the utility is continuously differentiable on the real line. Given these definition and since $Q_i > 0$ the following set is compact:

$$K = \left\{(\mathbf{q}, \boldsymbol{\mu}) \in \mathbb{R}_+^{E+V} | \mathbf{q} \in C \text{ and } \boldsymbol{\mu} \leq \boldsymbol{\pi}\right\}$$

Therefore if $\mathbf{q} \in \mathbb{R}_+^E \backslash C$: the boundary condition is met since:

$$\sum_{ij \in E} q_j^i f_j^i(\mathbf{q}, \boldsymbol{\mu}) + \sum_{i \in V} \mu_i q_i = \sum_{ij \in E} -\frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} + \sum_{i \in V} \mu_i (Q_i + \textstyle\sum_{k \in V(i)} q_i^k) > 0$$

It remains to show that if $\mathbf{q} \in C$, but $\mu_i >$ then

$$\begin{aligned}
\sum_{ij \in E} q_j^i f_j^i(\mathbf{q}, \boldsymbol{\mu}) + \sum_{i \in V} \mu_i q_i \quad &\geq \quad \mu_i (Q_i + \textstyle\sum_{k \in V(i)} q_i^k) - \displaystyle\sum_{ij \in E} \frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} = \\
&> \quad \frac{|E|}{Q_i} \left(\max_{\mathbf{q} \in C} \max_{ij \in E} \frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} \right) (Q_i + \textstyle\sum_{k \in V(i)} q_i^k) - |E| \displaystyle\max_{ij \in E} \frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} \geq \\
&\geq \quad \left(\max_{\mathbf{q} \in C} \max_{ij \in E} \frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} \right) - \max_{ij \in E} \frac{q_j^i \partial w_i(\mathbf{q})}{\partial q_j^i} \geq 0
\end{aligned}$$

Additionally if outflow equilibria are non-degenerate all assumptions of the Kolstad and Mathiensen result hold and the proof is a direct application of that theorem.

Suppose instead that individual j's utility not defined whenever $q_j \leq c \leq 0$. If a tighter budget constraint must be adopted to make sure that:

$$q_j^i u_j'(q_j) < \infty$$

So that an equilibrium exists. Namely that $q_j - \max_{j \in V(i)} \left\{ q_j^i \right\} \geq c$. ■

Lemma 23 *If for any player $i \in V$ A1 holds, $w_i(q) = 0$ when $q_i < 0$, $u_i''' \geq 0$ and:*

$$\frac{\partial}{\partial q}\left(-\frac{u_i''(q)}{u_i'(q)}\right) > -\left(\frac{u_i''(q)}{u_i'(q)}\right)^2 \quad \text{for any } q \in [0, Q]$$

Then the costs are convex, the revenues are pseudo-concave and concave if strictly increasing.

Proof. Most of the proof of lemma 3 applies. The only part to be altered is the proof of pseudo-concavity of the revenues. Indeed notice that given the assumption revenues are not differentiable at $q_j = 0$. Indeed revenues form sales to j are flat at zero if $q_j < 0$, jump upward at $q_j = 0$ and remain positive thereafter. Thus since the revenues from a sale q_j^i are pseudo-concave if $q_j \geq 0$, they are globally pseudo-concave. Just as shown in 3. ■

Lemma 24 *If assumption A1 holds, $q_i - \max_{k \in V(i)} \{q_i^k\} \geq 0$, $u_i''' \geq 0$ and the coefficient of relative risk aversion of player $i \in V$ satisfies:*

$$\frac{\partial}{\partial q}\left(-\frac{q u_i''(q)}{u_i'(q)}\right) \geq 0 \textit{ for any } q \in [0, Q]$$

Then the costs are convex, the revenues are pseudo-concave and concave if increasing.

Proof. Because A1 holds and since $u_i''' \geq 0$ total costs are still convex. Now suppose that revenues from selling q_j^i are increasing at q_j. If so it must be that:

$$-\frac{u_j''(q_j)}{u_j'(q_j)} \leq \frac{1}{q_j^i}$$

But the assumption on the relative risk aversion coefficient and $q_j \geq q_j^i$ imply that:

$$-\frac{u_j'''(q_j)}{u_j''(q_i)} \leq -\frac{u_j''(q_j)}{u_j'(q_j)} + \frac{1}{q_j} \leq \frac{1}{q_j^i} + \frac{1}{q_j} \leq \frac{2}{q_j^i}$$

Revenues on the sale are concave if increasing since $2u_j''(q_j) + u_j'''(q_j) q_j^i \leq 0$.

It remains to be shown that revenues $\rho(q_j^i) = u_j'(q_j) q_j^i$ are pseudo-concave. Suppose that they are not. If so the exist $x, y \in \mathbb{R}_+$ such that $\rho'(x)(y - x) \leq 0$ and $\rho(y) > \rho(x)$. Also suppose that $\rho'(x) > 0$. But, because $y \leq x$ and $\rho \in C^3$, there exists $z \in [x, y]$ such that $\rho'(z) = 0$ and $\rho''(z) > 0$, which violates the condition that revenues be concave if increasing. Similarly if $\rho'(x) < 0$. ■

Large Markets Without Replica

Proposition 25 *Consider an increasing sequence $\{G^r, Q^r, u^r\}_{r \in \mathbb{N}}$, if the economy becomes com-*

petitive, then the amount of goods resold by any individual vanishes.

Proof. Consider an economy becomes competitive. If so $\lim_{r\to\infty}(p_j(r) - p_i(r)) = 0$ for any two players $i, j \in V^r$ for which $\lim_{r\to\infty} q_i(r) > 0$ and $\lim_{r\to\infty} q_j(r) > 0$. Suppose by contraddiction that $\lim_{r\to\infty} R_i(r) > 0$ for some player $i \in V$. Let $\mu_i(r) \geq 0$ denote the multiplier on the non-negativity constraint of player i. If so, first order optimality for a flow from i to $j \in V^r(i)$ requires:

$$\lim_{r\to\infty}(p_j(r) - p_i(r)) = \lim_{r\to\infty}\left(\mu_i(r) - u_i''(q_i(r))\sum\nolimits_{k\in V^r(i)} q_i^k(r) - u_j''(q_j(r))q_j^i\right) > 0$$

Which contraddicts the assumption that the economy becomes competitive. ■

Proposition 26 *Consider an increasing sequence $\{G^r, Q^r, u^r\}_{r\in\mathbb{N}}$, if $\exists \overline{r} \in N$ such that $V^r(i) \supseteq S^r$ for any $i \in B^r$ and $r > \overline{r}$ and if $\lim_{r\to\infty}|B^r| = \infty$, then any outflow equilibrium becomes competitive.*

Proof. Consider any increasing sequence that satisfies the stated assumption. First it shown that $\lim_{r\to\infty}|B^r| = \infty$ implies $\lim_{r\to\infty}|S^r| = \infty$. Indeed $\lim_{r\to\infty}|B^r| = \infty$ implies that the competitive equilibrium excess demand is unbounded:

$$\lim_{r\to\infty}\sum\nolimits_{i\in B^r}(q_i^*(r) - Q_i) = \lim_{r\to\infty}\sum\nolimits_{i\in S^r}(Q_i - q_i^*(r)) = \infty$$

Market clearing implies that the competitive equilibrium excess supply is umbounded as well. Therefore since consumption holdings are finite $\lim_{r\to\infty}|S^r| = \infty$.

Now it is shown that the economy becomes competitive. For the economy to become competitive a selection of the outflow equilibrium correspondence must coverge to the competitive equilibrium. If so, there is a sequence of outflow equilibria $\mathbf{q}(r) \in \mathbb{R}_+^{E^r}$ such that for any two players $i, j \in V^r$ for which $\lim_{r\to\infty} q_i(r) > 0$ and $\lim_{r\to\infty} q_j(r) > 0$:

$$\lim_{r\to\infty}(u_j'(q_j(r)) - u_i'(q_i(r))) = 0$$

Moreover since no resale can take place in a competitive economy and because all goods are traded at one price, flows satisfy:

$$\begin{aligned}
\lim_{r\to\infty}\left(Q_i - q_i^*(r) - \sum\nolimits_{k\in V^r(i)} q_k^i(r)\right) &= 0 \text{ for } \forall i \in \lim_{r\to\infty} S^r && \text{(a)}\\
\lim_{r\to\infty}\left(Q_i - q_i^*(r) + \sum\nolimits_{k\in V^r(i)} q_i^k(r)\right) &= 0 \text{ for } \forall i \in \lim_{r\to\infty} B^r && \text{(b)}\\
\lim_{r\to\infty}\left(q_j^i(r)\right) &= 0 \text{ for } \forall ij \in \lim_{r\to\infty} E^r && \text{(c)}
\end{aligned}$$

Consider any sequence of strategies $\mathbf{q}(r) \in \mathbb{R}_+^{E^r}$ that satisfies (a), (b) and (c). Such sequence of flows exists because $V^r(i) \supseteq S^r$ for any $i \in B^r$ and $V^r(i) \supseteq B^r$ for any $i \in S^r$ for $r > \overline{r}$ and because $\lim_{r\to\infty}|B^r| = \lim_{r\to\infty}|S^r| = \infty$.

No deviation from such strategies can lead to gains as the economy grows large. Indeed, no deviation $\overline{\mathbf{q}}^i(r) \in \mathbb{R}_+^{V^r(i)}$ by player i that satisfies (c), but not (a) and (b) is profitable in the limit. Since such deviation cannot affect prices in the limit economy, for player i it is still optimal to sell all units valued less than the competitive price. Thus only deviations that violate (c) remain to be checked. For any such deviation there exists r' and $j \in V^r(i)$ for $r \geq r'$ such that $\lim_{r\to\infty}\left(\overline{q}_j^i(r)\right) > 0$. For such deviation to be feasible in the limit it must be that player i consumes a positive amount of goods in the competitive economy $\lim_{r\to\infty} q_i^*(r) > 0$. The optimal deviation if of such type meets necessary optimality conditions of problem $\mathbf{CP}(i)$ all along the sequence. Thus if the optimal deviation violates (c) and for $\Delta_j^i(r) = \overline{q}_j^i(r) - q_j^i(r)$ it must be that:

$$\lim_{r\to\infty} u_i'(q_i(r) - \Delta_j^i(r)) = \lim_{r\to\infty}\left(u_j'(q_j(r) + \Delta_j^i(r)) + \overline{q}_j^i(r) u_j''(q_j(r) + \Delta_j^i(r))\right) < \lim_{r\to\infty} u_j'(q_j(r) + \Delta_j^i(r))$$

The equality holds because no resale takes place in the competitive limit economy. The inequality holds because (c) is violated. But, if $\lim_{r\to\infty} q_j(r) > 0$ the assumption on the strategies imply that $\lim_{r\to\infty}\left(u_i'(q_i(r)) - u_j'(q_j(r))\right) = 0$. Instead, if $\lim_{r\to\infty} q_j(r) = 0$ assumptions imply that $\lim_{r\to\infty}\left(u_i'(q_i(r)) - u_j'(q_j(r))\right) > 0$. Therefore it must be that:

$$\lim_{r\to\infty} u_i'(q_i(r) - \Delta_j^i(r)) > \lim_{r\to\infty} u_j'(q_j(r) + \Delta_j^i(r))$$

Which provides a contraddiction and implies that no deviation to $\lim_{r\to\infty}\left(\overline{q}_j^i(r)\right) > 0$ can ever be profitable. Thus the economy becomes competitive as it grows large.

It remains to be shown that not one, but all outflow equilibria converge to such outcome. By contraddiction suppose that there exists a sequence of outflow equilibria that does not converge to the competitive equilibrium. If so there exists a player whose marginal utility does not converge to the competitive value. Let $V_+^r = \{k \in V^r | q_k(r) > 0\}$ and let $i(r)$ be the player for which:

$$\left(u_i'(q_i(r)) - u_j'(q_j(r))\right) \leq 0 \text{ for } \forall j \in V_+^r$$

Such player belongs to S^r as r grows large, since no player in B^r can have a lower marginal utility than the individuals he buys from. Because $\lim_{r\to\infty} |B^r| = \infty$ and because $B^r \subseteq V_+^r \cap V^r(i(r))$ for r large, the number of players connected to $i(r)$ with equilibrium marginal utility strictly higher than $i(r)$ diverges to infinity, $\lim_{r\to\infty}\left|V_+^r \cap V^r(i(r))\right| = \infty$. Suppose first that player $i(r)$ is a source for r large enough. If so he must be selling a positive amount of consumption to all linked players with strictly higher marginal utility by lemma 9. Notice that the assumption on the non-competitiveness of limit outflow equiliubrium implies that for any $j \in \lim_{r\to\infty} B^r$:

$$\lim_{r\to\infty}\left(u_{i(r)}'(q_{i(r)}(r)) - u_j'(q_j(r))\right) < 0$$

Which implies that in that as r grows large $i(r)$ sells a positive amont of goods to all those players in equilibrium, $\lim_{r\to\infty}\left(q_j^i(r)\right) > 0$. But this is a contraddiction. Since $\lim_{r\to\infty}(q_i(r)) > 0$ and $Q_i < \infty$ by assumption, no player can sell a positive amount of goods to infinitely many players. Now it is shown that the amount of goods resold by $i(r)$ has to vanish so to be able to apply remark 9 in the limit, completing the proof. If for no player $\lim_{r\to\infty} q_j(r) = 0$ the result holds immediately, since only such players can sell to $i(r)$. Thus suppose that $V^r \backslash V_+^r$ is non-empty in the limit. Let $i^{\circ}(r)$ be the player for which:

$$\left(u'_{i^{\circ}}(q_{i^{\circ}}(r)) - u'_j(q_j(r))\right) \leq 0 \text{ for } \forall j \in V^r$$

Players in $V^r \backslash V^r_+$ sell to $i(r)$ only if $i^{\circ}(r)$ sells to $i(r)$, since their opportunity cost is higher. But if $i^{\circ}(r)$ sells to $i(r)$, then by remarks 10 $i^{\circ}(r)$ sells also to all players with lower marginal utility. Because $\lim_{r\to\infty} \left|V^r_+ \cap V^r(i^{\circ}(r))\right| = \infty$ and because $i^{\circ}(r)$ benefits from selling to all players in $V^r_+ \cap V^r(i^{\circ}(r))$, it must that the amount he sells to all, but finitely many players must vanish. In paritcular since infinitely may players have higher marginal utility than $i(r)$:

$$\lim_{r\to\infty} q^{i^{\circ}(r)}_{i(r)}(r) = 0^+$$

Necessary optimality conditions along the sequence would require that since for any $j \in V^r(i^{\circ}(r))$ with marginal utility bigger than $i(r)$:

$$u'_{i^{\circ}(r)}(q_{i^{\circ}(r)}(r)) + \mu_{i^{\circ}(r)} < u'_{i(r)}(q_{i(r)}(r)) < u'_j(q_j(r))$$

But this implies that the limit economy satisfies for all, but finitely many $j \in V^r(i^{\circ}(r))$ including $i(r)$:

$$\lim_{r\to\infty} u'_{i^{\circ}(r)}(q_{i^{\circ}(r)}(r)) + \mu_{i^{\circ}(r)} = \lim_{r\to\infty} u'_{i(r)}(q_{i(r)}(r)) = \lim_{r\to\infty} u'_j(q_j(r))$$

But since resale occurs only at positive markups this implies that in the limit $i(r)$ can sell only to finitely many players unless $\lim_{r\to\infty} R_{i(r)}(r) = 0$. Moreover because infinitely players have similar incentives to sell to the finitely many players buying goods at positive markups in the limit economy, suc players do not exist. Thus, since $\lim_{r\to\infty} R_{i(r)}(r) = 0$ no equilibrium that is not competitive can ever exist in the limit. $\blacksquare$