# Alternating-Offer Bargaining with Private Correlated Values

Anton Tsoy*

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**Abstract**

This paper studies an infinite-horizon bilateral bargaining model with alternating offers and private correlated values. The paper characterizes frequent-offer limits of common screening equilibria in which both parties make offers to screen the opponent's type, and all types of either party follow the same path of offers. Even in the limit when the correlation of values is nearly perfect, common screening equilibria exhibit two-sided screening dynamics and involve inefficient delay in contrast to the unique equilibrium outcome of the complete-information bargaining game. Segmentation equilibria, in which types partially separate themselves into segments by the initial offer, are also constructed. Most of the types in the segments trade in the first rounds, while types near the boundaries of the segments delay trade to convince the opponent that they belong to a segment with more favorable terms of trade. Segmentation equilibria are efficient in the limit as the correlation of values becomes nearly perfect, and establish the connection between the limit outcome of nearly perfect correlation and the complete information outcome. The model sheds light on the relative importance of various sources of inefficiency for different levels of correlation, the role of public and private information in bargaining, and the robustness of the complete information bargaining model to higher-order uncertainty about values.

**Keywords:** *bargaining, alternating offers, incomplete information, private correlated values, delay*

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# 1 Introduction

Bargaining is an important feature of many economic transactions, and differences in information about preferences are particularly important in determining bargaining efficiency and trade dynamics. Existing work on the theory of bargaining provides a framework for addressing these issues in the case of one-sided or independent private information, but lacks insights about the case of correlated private information. The latter case is relevant in many settings where values depend on both the unobserved "quality" of the good and on some idiosyncratic factors. For example, in over-the-counter markets for corporate bonds, the price a trader is willing to pay or accept for a particular bond depends on the default and liquidity risks associated with the bond as well as on the trader's portfolio strategy and hedging needs. Another example is the inter-dealer market for used cars.[1] The dealer's value of a particular car is determined by the condition of the car as well as the current state of the dealer's inventory and the preferences of the dealer's customer base. In these examples, the unobserved quality of the good is the common factor that drives the correlation of values, while the differences in values arise because of the dealers' business specifics. The differences in values can also arise because of discrepancies in the subjective evaluation of the good by parties. Evaluating risks associated with the bond or the condition of the used car is a complicated task, and the results of the expertise by the experienced dealers, even though close on average, can be quite different in each particular case.

This paper studies an infinite-horizon bargaining model with private correlated values and alternating offers. The correlation of values spans a variety of environments that are intermediate between perfectly correlated and independent values. I study two classes of equilibria: (1) common screening equilibria in which all types of either party follow the same path of offers, and (2) segmentation equlibria in which types partially separate by their initial offers. The analysis of these equilibrium classes allows one to answer several fundamental questions about efficiency and trade dynamics with correlated values. Most important of them are as follows.

- How efficiency and trade dynamics change as the correlation of values varies?
- Are equilibria efficient, and if not, what are the sources of inefficiency?
- Is the efficiency guaranteed at least as the correlation of values becomes nearly perfect and players are almost certain about each other's values? If not, are all equilibria inefficient in the limit of almost-perfect correlation?
- What is the role of public and private information in bargaining?

The important novel feature of the model is the correlation of values, and to stress the new aspects of the model, I will frequently refer to the limit case as correlation becomes almost

[1] In the wholesale used-car market, every year 15 million cars are sold in the United States, and in about 20% of the cases prices are determined by the over-the-phone alternating-offer bargaining (Larsen (2013)).

perfect. However, it should be emphasized that all results in the paper are obtained for a broad range of levels of correlation (including, for some specifications, independent values) which significantly broadens the applicability of the theory.

To address these questions, I consider the standard alternating-offer bargaining model with the following correlation of values. Players' types are uniformly distributed on a "diagonal stripe" inside the unit square, and players' values are strictly increasing functions of their own types (see Figure 1).[2] In this specification, values are positively correlated: a buyer (seller) type with a higher valuation (cost) assigns positive probability to an interval of seller (buyer) types with higher costs (valuations). In the OTC example, the buyer of the bond with a high value attributes part of it to low risks associated with the bond, and hence, knows that the seller's value of the bond should also be relatively high.[3] Similarly, the buyer of a used car who discovers during his or her inspection some engine defects, expects that the seller could also be aware of them, and hence, is more willing to sell the car.

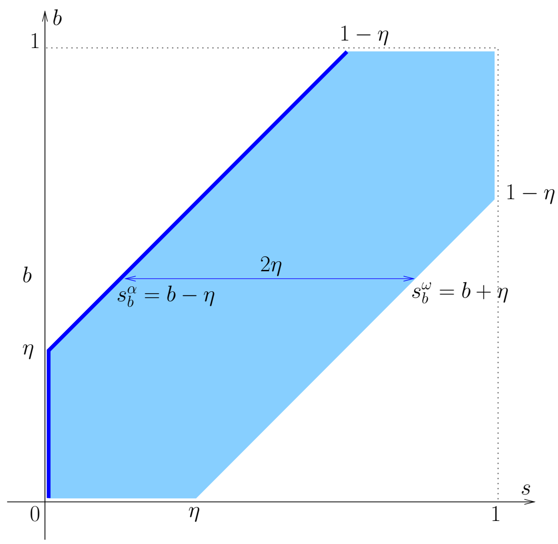



Figure 1: **Distribution of types.** Types $(s, b)$ are uniformly distributed on the diagonal stripe of width $2\eta$ inside the unit square. The bold line depicts the support of optimistic beliefs of the buyer.

The width of the diagonal stripe reflects the degree of correlation of types and the *individual*

[2]It is possible to match the empirical marginal distribution of values of both sides by varying the mappings from types into values. In this respect, the assumption of the uniform distribution on the diagonal stripe is not as restrictive as it might seem at first. However, this assumption does restrict the correlation of values. See Section 7 for further discussion of this assumption.

[3]To justify the assumption of private values in the OTC example, the model can be viewed as the model of bargaining between two brokers that trade on behalf of their clients. The clients value the asset directly which determines their willingness to pay or the lowest price they are willing to accept for the asset. The broker's payoff equals the difference between the value of his or her client and the actual price of trade. In this interpretation, the value of the asset of each broker is not derived directly from the asset, but given exogenously, justifying the private-values assumption. At the same time, brokers can still use the knowledge that the values of their clients are correlated during the bargaining process.

*uncertainty* of players about the type of the opponent. Players' individual uncertainty about the opponent's type is determined by the correlation of types, and in what follows, it is implied that the increase in the correlation is accompanied by the decrease in the individual uncertainty and vice versa. The more narrow the stripe, the higher the correlation of values, and the lower the individual uncertainty. In particular, in the limit when the stripe collapses into a diagonal line, types are almost perfectly correlated, the individual uncertainty is vanishingly small, and values are almost common knowledge. At the other extreme, values are independent when the stripe coincides with the unit square. I assume that for any realization of types, the gains from trade are positive. This assumption is realistic in secondary markets where participants trade to manage their liquidity. Bond sellers can be forced to liquidate their positions because of the urgent need to raise cash and so, other things equal, their value of the bond is lower than the value of buyers that are not hit by the liquidity shock.

The distribution of types in this paper is similar to the information structure commonly used in the global games literature (Morris and Shin (1998), Morris and Shin (2003)). To see the parallel, one can think of the types generated as follows. A *fundamental* $\omega$ is drawn from $[0,1]$ and players' types are determined by $b = \omega + \eta_B$ and $s = \omega + \eta_S$ where $\eta_b$ and $\eta_s$ are independent conditional on $\omega$ with bounded support $[-\frac{\eta}{2}, \frac{\eta}{2}] \cap [-\omega, 1-\omega]$. Types are mapped into values by strictly increasing functions which ensures the positive correlation of values.

The analysis of *common screening equilibria (CSEs)* reveals that for a wide range of correlation levels including almost-perfect correlation, equilibria can exhibit two-sided screening dynamics. Screening dynamics is common in bargaining models with incomplete information about values. In CSEs, all types on either side follow the *common screening policy.* The buyer (irrespective of type) follows an increasing sequence of price offers to screen seller types starting from the bottom of the type distribution, and the seller (irrespective of type) follows a decreasing sequence of counter-offers to screen buyer types starting from the top of the type distribution. In other words, types pool on price offers and separate by the time they accept the opponent's price offer. I characterize the dynamics of acceptance and price paths in CSEs for various levels of correlations of values in the limit as offers become frequent.[4] The characterization implies that such two-sided screening dynamics do not necessarily require players to be very uncertain about each other's values. In particular, the same paths of price offers of both sides can be a part of equilibrium for both high and low correlation of values, and the equilibria will differ only in the rate of acceptance.

Two-sided screening dynamics provide a realistic description of the bargaining process. In CSEs, both parties begin bargaining by making extreme offers, even though, when the individual uncertainty is small, both could know that such offers are not going to be accepted. As bargaining continues, parties make concessions and moderate their demands to more reasonable levels. Over

[4]Most of the results in the literature as well as in this paper are obtained in the limit of frequent offers. The qualifier "in the frequent-offer limit" is further omitted in the description of the results.

time the common screening policy narrows down the range of types possible in the game. In the OTC markets for corporate bonds or the market for used cars, it seems reasonable to assume that both sides have very precise information about each other's values. However, if one turns to the complete information game as in Rubinstein (1982) to describe the trade dynamics in this examples, then one comes to the unrealistic conclusion that the agreement should be immediate. In contrast, the model in this paper predicts that the two-sided screening dynamics can persist even when individual uncertainty about the value of the opponent is small.

Because of the two-sided screening dynamics, trade in CSEs is spread-out over time,[5] and inefficient delay arises for types in the middle of the support of the distribution. To understand the nature of this inefficiency and to see how the correlation of values affects the efficiency of CSEs, it is useful to distinguish three sources of inefficiency in CSEs. First, the surplus is dissipated through a channel analogous to the standard monopoly deadweight loss. In order to efficiently screen player types, each offer is targeted at a particular group of types, and the allocation is delayed for the rest of the types. The second source of inefficiency is signaling costs. Higher seller types and lower buyer types prefer to reject the offer of the opponent and continue screening to signal their value and convince the opponent to accept their screening offer. These inefficiencies were already present in the model with independent private values.[6]

The third source of inefficiency is novel and arises from the common screening. Since players use common screening policies, for buyer and seller types in the middle of the distribution support, a significant amount of time could pass before the common screening policy starts efficiently truncating the support of their beliefs about the opponent's types. Nevertheless, in CSEs, these types adhere to the offers in the common screening policy that are guaranteed to be rejected by their opponent. This could result in a significant CSE inefficiency, even when individual uncertainty is small.

The presence of the inefficiency due to the common screening brings me to a surprising discontinuity in equilibrium outcomes as the correlation becomes almost perfect. Both equilibrium behavior and efficiency are quite different for the case when the correlation is almost perfect analyzed in this paper and for the case of complete information studied in Rubinstein (1982). In the former model, a variety of screening patterns is possible and equilibria are not necessarily efficient due to the trade delay, while in the latter model, the unique equilibrium outcome is efficient and features an immediate equal split of the surplus. While the complete information bargaining model is known to be non-robust to the introduction of the incomplete information, the previous literature assumed big differences in the support of players' beliefs to obtain the discontinuity in the equilibrium outcomes.[7] In contrast, in this paper, as correlation becomes

[5] Unless the screening is trivial and price offers by both sides are equal from the onset.

[6] See Ausubel and Deneckere (1992b) for the analysis of these two sources of inefficiency in a model with independent values.

[7] The literature on the Coase Conjecture (Fudenberg, Levine, and Tirole (1985), and Gul, Sonnenschein, and Wilson (1986)) assumes a significant difference in the support of beliefs of the uninformed side, Abreu and Gul

almost perfect, the supports of players' beliefs become concentrated around the realized types.

It might be counter-intuitive at first that even in the case of vanishing individual uncertainty it is possible that types in the middle of the distribution follow the common screening policy. Indeed, with small individual uncertainty these types trade with a delay and can predict very accurately the time and price at which they will trade in the future. Moreover, they know that the opponent also is able to predict the terms of trade with great accuracy. The question is then why is it not possible for one of the sides to deviate and offer an immediate trade at the compromising price? The key to this is the *Contagious Coasian Property* of the *punishing equilibria* used to support the equilibrium path. In punishing equilibria, the punishing side holds the most optimistic beliefs consistent with the prior distribution over type of the opponent. For example, in the seller-punishing equilibrium, any buyer type puts probability one on the lowest seller type that she initially considered possible (in Figure 1 the support of optimistic beliefs of the buyer is depicted by a bold line). The Contagious Coasian Property states that in the frequent-offer limit, the utility of all types of the deviator in the punishing equilibrium is equal to the lowest utility achievable in any equilibrium. Intuitively, in the punishing equilibrium, even when individual uncertainty is very small and the switch to optimistic beliefs does not lead to a drastic updating of beliefs, the fact that all types simultaneously switch their beliefs allows the amplification of this effect and effectively punishes the deviator.

A natural next question is whether all equilibria of the model with correlated values are bound to be inefficient and if there is a way to avoid the inefficiency of common screening. To answer this question, I turn to the class of *segmentation equilibria* featuring drastically different trade dynamics and efficiency properties. In segmentation equilibria, types partially separate themselves into several segments by the initial price offer. After that, only types close to the boundaries of the segments remain in the game. Such types build a reputation for belonging to a segment with a more favorable price by delaying trade and insisting on that price. This process is similar to the war-of-attrition bargaining game in Abreu and Gul (2000) where rational types build reputation for being committed types.

As opposed to CSEs, in segmentation equilibria, there is no inefficiency of common screening, and the efficiency loss is only due to the standard deadweight loss and signaling costs. As a result, the efficiency properties of such equilibria are similar to that of the model with independent values: as individual uncertainty vanishes (correlation of values becomes perfect), the ex-ante probability of inefficient delay converges to zero. I use this result to construct a sequence of segmentation equilibria with vanishing individual uncertainty and time between bargaining rounds that approximates Rubinstein (1982)'s immediate equal split of the realized surplus. Along this sequence, the number of segments increases and the definition of segments becomes finer to guarantee the nearly equal split of the realized surplus.

(2000) assumes a significant difference in the behavior of committed and rational types, Feinberg and Skrzypacz (2005) assumes a significant difference in the support of the beliefs of the uninformed party.

The analysis of CSEs raises concerns about the applicability of the complete information bargaining game in settings with small individual uncertainty but large common uncertainty. This non-robustness is particularly relevant, as it is common in the economic literature to assume that once agents meet, trade is immediate at a price that splits the surplus proportionately.[8] The proportional split of the surplus is motivated by the *Nash (1950) bargaining solution.* Rubinstein (1982)'s complete-information game provides non-cooperative foundations for the Nash bargaining solution and shows the immediate trade result.

The segmentation equilibria partially address this robustness concern by providing a sequence of equilibria outcomes that converges to the complete information outcome as individual uncertainty vanishes. A more constructive approach is to study the implications of the two-sided screening dynamics. For this purpose, I provide a characterization of limits of CSEs as the individual uncertainty and time between offers vanish. The characterization is in terms of relatively tractable static incentive compatibility and individual rationality constraints. In applications, one can address robustness concerns by using the characterization to determine terms of trade instead of the Nash bargaining solution. In a companion paper, Tsoy (2014) applies the characterization of double limits to study the effect of strategic bargaining delay on asset prices and liquidity in OTC markets. The model with vanishing individual uncertainty also gives a testable empirical implication about the dependence of the trade delay on the quality of the good traded. Because of the two-sided screening dynamics, the delay is inverse U-shaped in quality, a prediction which differs from both the prediction of the complete information model and the bargaining models with one-sided private information.

The analysis of this paper stresses the role of public rather than private information in guaranteeing efficiency, which has not been emphasized in the previous literature. The private information of players is reflected in the individual uncertainty about the value of the opponent, while public information is reflected in the *common uncertainty*, i.e. how much information about values is common knowledge. The bargaining literature normally assumes that the information of agents is independent conditional on public information. Therefore, the reduction in individual uncertainty is equivalent to the reduction in public uncertainty about the values. This does not always accurately capture the reality. For example, OTC markets are known to be opaque, and only limited public information is available about assets. At the same time, professional traders rely both on publicly available information, like credit ratings and asset characteristics, and on their own information sources to evaluate the risks associated with the asset at hand. Therefore, the information of the traders is more refined when compared to public information. This aspect is captured in the present paper. There is a gap between the private and public information, as players know that the value of the opponent belongs to a subset of the values possible in the game.

[8]For example, the Nash bargaining solution was used to study the relationship between unemployment and search in the labor market (see Mortensen and Pissarides (1994)), liquidity in over-the-counter markets (see Duffie, Gârleanu, and Pedersen (2005)), renegotiation in contract theory (see Tirole (1999)), equilibrium selection in repeated games (see Miller and Watson (2013)).

Because of that, it is possible to disentangle the importance of private and public information to efficiency. As the individual uncertainty vanishes, the private information becomes infinitely precise, while the public information still remains relatively crude. The inefficiency of CSE limits demonstrates that even in the limit of vanishing individual uncertainty, the equilibria can have inefficient screening as long as the common uncertainty remains in the game. At the same time, I also show that letting common uncertainty vanish through more accurate public information leads to the efficiency of all equilibria.

The structure of the paper is as follows. Section 2 describes the game. Section 3 characterizes the limit equilibrium behavior in CSEs. Section 4 analyzes segmentation equilibria. Section 5 characterizes double limits of CSE outcomes as both the time between offers and individual uncertainty vanish and constructs a sequence of segmentation equilibria that converges to the complete-information outcome. Punishing equilibria that support equilibrium paths of CSEs and segmentation equilibria are studied in Section 6. Section 7 discusses the assumptions and gives directions for future research. To maintain continuity of the argument all proofs are relegated to the Appendix. Before proceeding further I describe the contribution of the paper to the earlier literature.

**Related Literature** The paper is most closely related to the literature on bargaining with asymmetric information about preferences. The bargaining game with infinite horizon and one-sided incomplete information has been extensively studied and by now is well understood. When the seller's cost is commonly known, the buyer's valuation is private information, and the seller cannot commit to a path of prices, the ability of the seller to earn profits beyond the competitive level depends crucially on the support of the distribution of the buyer's valuation. When there is a gap between the seller's cost and the lowest buyer's valuation, the inability of the seller to commit to future price offers drives seller offers down to the lowest buyer valuation. This result is known as the Coase conjecture (See Fudenberg, Levine, and Tirole (1985), Gul, Sonnenschein, and Wilson (1986), Grossman and Perry (1986), Gul and Sonnenschein (1988)). In the case of no gap, the Coasian path can be used as a punishment to prove a folk-theorem type of result. In this case, a variety of outcomes is sustainable in equilibrium including the outcome with the seller's profit close to the static monopoly profit (see Ausubel and Deneckere (1989a,b), Ausubel and Deneckere (1992a)). As with the Coase conjecture, the Contagious Coasian Property of the punishing equilibria proven in this paper shows that the punished side gets the lowest utility possible in any equilibrium in the frequent-offer limit. The important distinction is that in this paper, the assumption of bounded support restricts beliefs off the equilibrium path. In particular, for high correlation of types, only marginal updating is possible after deviations. This does not allow the application of earlier results and instead, I develop a contagion argument to prove the Contagious Coasian Property of the punishing equilibria.

In a recent work, Deneckere and Liang (2006) explored the case of interdependent values in a model with one-sided incomplete information. In their model, a fundamental determines values of both parties, but only one party is informed about the fundamental, while the other party holds prior beliefs about it that are commonly known. The equilibrium efficiency depends crucially on a static incentive constraint. When there is no efficient static mechanism, the trade happens in bursts between prolonged periods of almost no trade. This model was further studied in Gerardi, Hörner and Maestri (2013), and Fuchs and Skrzypacz (2013).[9] This paper differs from this literature in that both parties have private (correlated) information.

The literature on bargaining with two-sided incomplete information about values has thus far focused exclusively on the case of independent private values and one-sided offers.[10] Bargaining models with two-sided incomplete information are known to be prone to a multiplicity of equilibria, and the literature studied certain classes of equilibria in such a model. Cramton (1984, 1992), Cho (1990), and Ausubel and Deneckere (1992b) investigated the relationship between two-sided uncertainty and efficiency.[11] In the equilibrium analyzed in Cramton (1984), seller types initially pool on the same path of offers, but separate over time starting from the bottom of the type distribution. After the seller reveals her type, she quickly screens the buyer types as in the one-sided incomplete information game. Cho (1990) constructs a class of separating equilibria in which all seller types separate by price offers in the first round and continue to separate by offers in every round in the future. Ausubel and Deneckere (1992) shows that in the no-gap case, an outcome of almost no trade is possible along with relatively efficient monopoly equilibria. In the monopoly equilibria, all seller types, except a small subset at the bottom of the distribution, reveal themselves by offering a monopoly sales price. Such types trade in the first round and never trade after, since lowering the price would lead to the buyer switching to optimistic beliefs and would imply no trade for such seller types. It should be noted that many interesting equilibria in the model with one-sided offers are not guaranteed to have counterparts in the model with two-sided offers. Cramton (1992) allows for both two-sided offers and the strategic choice of delay by parties as in Admati and Perry (1987). Under intuitive-criterion-style refinement both sides use strategic delay to credibly signal their private information. The focus of this paper is on environments where players cannot commit to delay their offer.

The model in this paper is complementary to both the literature on bargaining with two-sided independent private values and on bargaining with one-sided incomplete information. It covers a wide range of environments in which values are correlated and private information exists, and both parties make offers.[12] The two equilibrium classes studied here offer drastically different

[9] An earlier analysis of this model is given in Vincent (1989).

[10] The exception is Ausubel and Deneckere (1993) which allowed offers by both sides and showed that the restriction to one-sided offers is ex-ante efficient for a variety of welfare weights.

[11] See also Fudenberg and Tirole (1983) for the analysis of the model with two bargaining rounds, and Chatterjee and Samuelson (1987) for a neat characterization of the bargaining dynamics under the additional restriction that the type and action space consist of only two types and two offers. Watson (1998) analyzes uncertainty about discount factors.

[12] In either market for corporate bonds of the inter-dealer car market, there is no a priori reason to assume

equilibrium dynamics and efficiency properties. In terms of trade dynamics and efficiency, CSEs are similar to equilibria constructed in Cramton (1984), where first the seller gradually reveals her type and then buyer types are screened. In terms of trade dynamics, segmentation equilibria are similar to the equilibria in Cho (1990) and Ausubel and Deneckere (1992), where the informed side reveals its private information early in the game by price offers.

The dynamics of segmentation equilibria is similar to the war-of-attrition dynamics in reputational bargaining of Abreu and Gul (2000) which is another two-sided offer and two-sided incomplete-information bargaining model. In their model, commitment types require a particular share of the surplus, and rational types mimic the behavior of the commitment types. This results in a war of attrition game in which the first side to reveal rationality accepts the terms of the opponent. Similarly, in segmentation equilibria, types near boundaries of the segments delay trade to convince their opponent that they belong to a segment with more favorable terms of trade. However, there are important differences in the dynamics as well. Abreu and Gul (2000) shows that in the unique frequent offer limit of equilibria in their model, rational players concede with probability one by some finite time. Unlike in their model, in this paper bargaining between rational types takes infinite time. The difference stems from the fact that without commitment types, it is not possible for bargaining to end in finite time since the utility of a rational player is discontinuous at this time, and a sufficiently patient player prefers to wait past this time.

The paper contributes to the literature exploring the effect of higher-order uncertainty on bargaining outcomes. Feinberg and Skrzypacz (2005) shows that the Coase conjecture is not robust to the introduction of second-order uncertainty. In particular, in the standard model with the buyer privately informed about his valuation, if the seller with some probability can know for sure that the valuation of the buyer is high, then under an "intuitive" refinement, the equilibrium outcome is necessarily inefficient. In this paper, I explore the robustness of the complete information game to the introduction of higher-order uncertainty. As the correlation of values becomes nearly perfect, the types of players converge (in the product topology) to the types in the complete-information game, however, the equilibrium behavior can be very different. Weinstein and Yildiz (2013) also shows that the complete-information game is not robust to the perturbations of higher-order beliefs. For this result, they construct artificial types, while in this paper, the type space of the model has a natural interpretation.

## 2 The Model

This section formally describes the model. A buyer and a seller meet to trade one unit of a good.[13] The seller's type $s$ and the buyer's type $b$ are jointly uniformly distributed on the diagonal stripe inside the unit square, $SB \equiv \{(s,b) \in [0,1]^2 : s - \eta \leq b \leq s + \eta\}$. The individual

that one side is better informed or has more commitment power in the negotiation, so symmetry is a natural assumption.

[13] Female pronouns are used to refer to the seller and male pronouns are used to refer to the buyer.

uncertainty parameter $\eta \in (0,1)$ controls the degree of correlation of types.[14] By varying $\eta$, the model spans a variety of environments. In applications where players have precise information about each other's values, $\eta$ could be thought of as small so types are highly correlated, while in applications where there is a great degree of heterogeneity in values due to idiosyncratic factors, $\eta$ could be close to one, so types are almost independent.

Given their types, players hold prior beliefs about their opponent's type. The prior beliefs of a seller of type $s$ are uniform on the interval $B_s \equiv [b_s^\alpha, b_s^\omega]$ where $b_s^\alpha \equiv \max\{0, s-\eta\}$ and $b_s^\omega \equiv \min\{1, s+\eta\}$. Analogously, the prior beliefs of a buyer of type $b$ are uniform on the interval $S_b \equiv [s_b^\alpha, s_b^\omega]$ where $s_b^\alpha \equiv \max\{0, b-\eta\}$ and $s_b^\omega \equiv \min\{1, b+\eta\}$. Players' types and their priors are illustrated in Figure 1.

The valuation of the good of a type $b$ buyer is $v(b)$, and the cost of selling the good for a type $s$ seller is $c(s)$, where $v : [0,1] \to \mathbb{R}$ and $c : [0,1] \to \mathbb{R}$ are strictly increasing, differentiable functions with derivatives bounded from below by some positive constant and bounded from above by $\ell > 0$.[15] Let $\xi \equiv \min\limits_{(s,b)\in SB} \{v(b) - c(s)\}$ be the minimal gains from trade possible in the game. I assume the gains from trade are positive for any buyer and seller type ($\xi > 0$), but note that this does not preclude the possibility that $c(1) > v(0)$, and hence, there does not in general exist a single price that gives non-negative utility to all types. Note also that gains from trade are not common knowledge due to the imperfect correlation of types. As a result, players have incentives to pretend that the gains from trade are small to get a better price. For $\eta < 1$, the opponent may detect that certain low gains are not possible.

Additionally, I impose the following mild technical condition on valuation and cost functions. A function $f$ on a compact set $X$ is *regular* if it is smooth and there exists $D > 0$ such that $\frac{1}{l!}\frac{d^l f(x)}{dx^l} < D$ for all $l \in \mathbb{N}$ . This condition is slightly stronger than analyticity, but many functions used in applications are regular (for example, all polynomial functions are regular).[16] I assume that $v$ and $c$ are regular.

Bargaining occurs in rounds $n \in \mathbb{N}$, and the length of the time interval between bargaining rounds is $\Delta > 0$. Players discount the future at the common discount rate $r > 0$. The seller is active in odd rounds, and the buyer is active in even rounds. An active player can either accept the last offer of the opponent or make a counter-offer. Once a price offer is accepted, the game ends and payoffs are determined. An *outcome* $(N\Delta, p)$ consists of the time of trade

[14] To focus on the novel features of the model, the extreme cases $\eta = 0$ and $\eta = 1$ are left out from the analysis. The case $\eta = 0$ has been studied in Rubinstein (1982). The analysis of the case $\eta = 1$ is simpler than the case $\eta \in (0,1)$, but requires a separate treatment in proofs. All results of the paper carry to this case.

[15] When types are independent, it is a standard result that types can be taken to be uniformly distributed on the unit interval without loss of generality. For any distribution of values, there is a transformation of the valuation and cost functions that preserves the distribution of values and changes the distribution of types into uniform on the unit interval. With correlated types this result is no longer true as no such transformation is guaranteed to preserves the correlation structure. In this paper, I consider a general class of valuation and cost functions, but restrict the distribution of types to be uniform. Relaxing this assumption is left for future research.

[16] For analytic function $f$ on a compact set $X$, there exists $D > 0$ such that $\frac{1}{l!}\frac{d^l f(x)}{dx^l} < D^l$ for all $l \in \mathbb{N}$ and all $x \in X$.

$N\Delta \leq \infty$ (where $N$ is the round of trade) and the price of trade $p$. The utility of type $b$ buyer is $e^{-r(N-1)\Delta}(v(b)-p)$ and the utility of type $s$ seller is $e^{-r(N-1)\Delta}(p-c(s))$.[17]

In any round $n$ by the beginning of which trade has not happened, a history $h^n$ is a sequence of rejected price offers up to round $n-1$. A (pure) strategy of the buyer $\sigma_b^n$ is a measurable function that maps any buyer type $b$ and history $h^n$ into the acceptance decision or a counter-offer. The posterior beliefs of the buyer $\mu_b^n$ is a measurable function that maps any buyer type $b$ and any history $h^n$ into a probability distribution over seller types. The strategy $\sigma_s^n$ and the posterior beliefs $\mu_s^n$ are defined analogously for the seller.[18]

A *Perfect Bayesian equilibrium,* which I further refer to simply as *equilibrium*, consists of a pair of strategy profiles $(\sigma_b^n, \sigma_s^n)$ and beliefs $(\mu_b^n, \mu_s^n)$ that satisfy sequential rationality and the following conditions on the updating of beliefs:

1. for any $(i,j) \in \{(b,s),(s,b)\}$, offer $p_n$, history $h^n$, if $\int_0^1 \mathbf{1}\{\sigma_j^n = p_n\} d\mu_i^n(j) > 0$ $\mu_i^{n+1}(\Theta) = \frac{\int_\Theta \mathbf{1}\{\sigma_j^n = p_n\} d\mu_i^n(j)}{\int_0^1 \mathbf{1}\{\sigma_j^n = p_n\} d\mu_i^n(j)}$ for measurable $\Theta$;
2. $\mu_b^n$ and $\mu_s^n$ do not change in even and odd rounds, respectively;
3. for any history $h^n$, $\mu_b^n \in \Delta(S_b)$ and $\mu_s^n \in \Delta(B_s)$.

This is a natural adaptation of the Perfect Bayesian equilibrium (Fudenberg and Tirole (1991)) to the environment with correlated values analyzed in this paper. Sequential rationality requires that after any history, players best respond to the strategy of the opponent given their posterior beliefs. The first condition on beliefs requires that beliefs be updated by Bayes' rule whenever possible, and the second is the standard "no signaling what you don't know" condition. The latter requirement implies that the correlation structure is common knowledge. Both on and off the equilibrium path, players put positive probability only on types of the opponent that lie in the support of their priors, players are certain that their opponent also puts positive probability only on a subset of the support of his/her prior beliefs, and the regress continues indefinitely.

# 3 Common Screening Equilibria

This section characterizes the dynamics of CSE frequent-offer limits and studies their efficiency. The approach is to temporarily turn to a related *concession game* analyzed in the next subsection. The concession game is a continuous-time counterpart of the described bargaining game except that players take price-offer paths as given and only choose the time at which they accept

[17]By convention, if trade does not occur in a finite number of rounds, $N=\infty$ and both players get a payoff of zero.

[18]It is standard in the bargaining literature to restrict attention to equilibria in pure strategies with the reservation that mixing is possible off the equilibrium path (see Gul, Sonnenschein and Wilson (1986), and Fudenberg, Levine, and Tirole (1985) for a discussion of mixing off the equilibrium path). In this paper mixing could be necessary only for seller type 0 and buyer type 1 off the equilibrium path of the punishing equilibrium analyzed in Section 6. With minor adjustments the results in this paper could be formulated to incorporate this possibility.

the opponent's offer. Equilibria of the concession game have a convenient analytic characterization presented in Theorem 1.

The main result of this section (Theorem 2) relates CSE frequent-offer limits to equilibria of the concession game. In general, the ability to choose price-offer paths puts additional restrictions on price paths and acceptance strategies in CSEs. However, as Theorem 2 shows, in the frequent-offer limit such restrictions on the CSE's equilibrium behavior are minimal and boil down to modified individual rationality constraints. Subsection 3.2 describes the new source of inefficiency associated with positive correlation of values, as well as the distinction between the public and private information that can be made in the present model. Subsection 3.3 highlights main steps of the proof of the Theorem 2. In particular, Lemma 2 is at the heart of all equilibria constructions in this paper.

### 3.1 Concession game

The concession game is defined as follows. Types of the buyer and the seller are drawn uniformly from $SB$ as in Section 2. There are continuously differentiable paths of buyer price offers $q_t^B : t \mapsto q_t^B$ and seller price offers $q_t^S : t \mapsto q_t^S$ such that $q_t^S \geq q_t^B$ for all $t \geq 0$. Players take as given paths of price offers and choose the time at which they accept their opponent's offer. Outcome $(T^c, q^c)$ consists of the time $T^c \in \bar{\mathbb{R}}_+$ and the price $q^c$ at which trade happens.[19] Given outcome $(T^c, q^c)$, the utility of buyer type $b$ is $e^{-rT^c}(v(b) - q^c)$, and the utility of seller type $s$ is $e^{-rT^c}(q^c - c(s))$. Strategies are acceptance times $t_B^*(b)$ and $t_S^*(s)$ for each type $b$ buyer and type $s$ seller, respectively. For any types $b, s$ and strategies $t_B^*(b), t_S^*(s)$, the outcome is determined by $T^c = \min\{t_B^*(b), t_S^*(s)\}$, and $q^c = q^B_{t_B^*(b)}$ if $t_B^*(b) \leq t_S^*(s)$ and $q^c = q^S_{t_S^*(s)}$ if $t_S^*(s) < t_B^*(b)$.[20]

I make three assumptions about the price paths. First, no player gets negative utility from his/her offer being accepted, i.e. $q_\infty^S \geq c(1)$ and $q_\infty^B \leq v(0)$. Second,

$$c^{-1}(q_\infty^B) - v^{-1}(q_\infty^S) \geq \eta, \tag{1}$$

where $q_\infty^B \equiv \lim_{t\to\infty} q_t^B$ and $q_\infty^S \equiv \lim_{t\to\infty} q_t^S$ are the limits of price paths as $t \to \infty$.[21] Condition (1) guarantees that all gains from trade can eventually be realized through one of the players accepting the opponent's offer.[22] Third, there exists $\hat{T} \in \bar{\mathbb{R}}_+$ such that seller price path $q_t^S$ is strictly decreasing on $[0, \hat{T}]$, buyer price path $q_t^B$ is strictly increasing on $[0, \hat{T}]$, and price paths are constant after $\hat{T}$. The monotonicity of offers is fairly natural, and it reflects the fact that over time parties converge in their demands. Observe that the concession game is static, even

[19] I use notation $\mathbb{R}_+ \equiv [0, \infty)$ for a set of positive reals, and $\bar{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$.

[20] In equilibria that I analyze, players assign probability zero to ties, and the tie-breaking rule can be specified arbitrarily.

[21] In what follows, I define $x_\infty \equiv \lim_{t\to\infty} x_t$ whenever the limit exists.

[22] To see this, notice that the set of types that get negative payoffs from accepting any opponent's offer is a subset of $[0, v^{-1}(q_\infty^S)]$ for the buyer and a subset of $[c^{-1}(q_\infty^B), 1]$ for the seller. By $c^{-1}(q_\infty^B) - v^{-1}(q_\infty^S) \geq \eta$, $\big([c^{-1}(q_\infty^B), 1] \times [0, v^{-1}(q_\infty^S)]\big) \cap SB = \emptyset$ giving the desired conclusion.

though payoffs are determined by a dynamic procedure. I define an equilibrium of the game as follows.

**Definition 1.** *An* equilibrium *of the concession game is a tuple* $(t_B^*(b), t_S^*(s), q_t^B, q_t^S)$ *such that given price paths* $q_t^S$ *and* $q_t^B$ *and the strategy of the opponent (given by* $t_S^*(s)$ *or* $t_B^*(b)$*), and acceptance times* $t_B^*(b)$ *and* $t_S^*(s)$ *are optimal.*

I restrict the analysis of equilibria to monotone strategies as defined next.[23]

**Definition 2.** *Acceptance strategies* $t_B^*(b)$ *and* $t_S^*(s)$ *are* monotone *if there exist processes* $b_t^* : t \mapsto b_t^*$ *and* $s_t^* : t \mapsto s_t^*$ *such that*

1. $t_B^*(b) \equiv \inf\{t : b_t^* = b\}$ *and* $t_S^*(s) \equiv \inf\{t : s_t^* = s\}$,[24]
2. *for some* $T_B, T_S \in \bar{\mathbb{R}}_+$, $b_t^*$ *is strictly decreasing for* $0 \leq t \leq T_B$ *and constant for* $t \geq T_B$, *and* $s_t^*$ *is strictly increasing for* $0 \leq t \leq T_S$ *and constant for* $t \geq T_S$.

*Say that* $b_t^*$ *and* $s_t^*$ *are* smooth monotone strategies *if, additionally,* $b_t^*$ *and* $s_t^*$ *are continuous and a.e.-continuously differentiable on* $[0, T)$.

Monotone strategies specify the highest type $b_t^*$ of the buyer and the lowest type $s_t^*$ of the seller remaining in the game at time $t$. I use $t_B^*(b)$ and $b_t^*$ interchangeably to refer to the monotone strategy of the buyer, and analogously, I use both $t_S^*(s)$ and $s_t^*$ for the monotone strategy of the seller. The strict monotonicity of $b_t^*$ and $s_t^*$ implies that there are no periods with no acceptance until times $T_B$ and $T_S$, respectively, when players stop accepting their opponent's offers. During "quiet" periods, price offers that are not accepted can be specified arbitrarily, as long as no types choose to accept them. The focus of this section is on the relationship between the dynamics of price paths and acceptance strategies and so the strict monotonicity is necessary to pin down such a relationship. The next theorem characterizes equilibria in smooth monotone strategies.

**Theorem 1.** *Suppose* $(b_t^*, s_t^*, q_t^B, q_t^S)$ *is an equilibrium of the concession game in smooth monotone strategies. Then the following conditions hold*

1. *There exists a time* $T \in \bar{\mathbb{R}}_+$ *such that*

$$b_T^* = b_{s_T^*}^{\alpha} \text{ and } q_T^B \leq q_T^S \text{ with equality if } T < \infty. \tag{2}$$

2. *For all* $t \in [0, T)$,

$$\begin{aligned} r\left(v(b_t^*) - q_t^S\right) &= \lambda_t^S\left(q_t^S - q_t^B\right) - \dot{q}_t^S, & (3)\\ r\left(q_t^B - c(s_t^*)\right) &= \lambda_t^B\left(q_t^S - q_t^B\right) + \dot{q}_t^B; & (4)\end{aligned}$$

[23]The restriction to monotone strategies is common in Bayesian games with a continuum of types. Pure-strategy equilibria in monotone strategies has been studied by Athey (2001), McAdams (2003), Reny (2011) in the context of auctions and by Van Zandt and Vives (2007) in the context of games with strategic complementarities.

[24]By convention, $\inf \emptyset = \infty$.

*where* $\lambda_t^B \equiv -\frac{\dot b_t^*}{b_t^* - b^\alpha_{s_t^*}} \mathbf{1}\left\{ b^\omega_{s_t^*} \geq b_t^* \right\}$ *and* $\lambda_t^S \equiv \frac{\dot s_t^*}{s^\omega_{b_t^*} - s_t^*} \mathbf{1}\left\{ s^\alpha_{b_t^*} \leq s_t^* \right\}$.

*Conversely, if a tuple* $(b_t^*, s_t^*, q_t^B, q_t^S)$ *of smooth monotone strategies and price paths satisfies conditions* (2)*,* (3)*,* (4)*, then* $(b_t^*, s_t^*, q_t^B, q_t^S)$ *is an equilibrium of the concession game.*

Theorem 1 justifies the validity of the first-order approach for the analysis of equilibria in smooth monotone strategies. To see this, consider the problem of a type $b$ buyer. Suppose that the seller uses smooth monotone strategy $s_t^*$ and price paths are $q_t^B$ and $q_t^S$. Let $F_t^S(b) \equiv \frac{\max\{\min\{s_t^*, s_b^\omega\} - s_b^\alpha, 0\}}{s_b^\omega - s_b^\alpha}$ be the CDF of the seller's acceptance time evaluated by type $b$ buyer and $f_t^S(b)$ be the corresponding density function. Type $b$ buyer chooses $t \in \bar{\mathbb{R}}_+$ to maximize his expected utility,

$$u^B(t,b) = \int_0^t e^{-ru}\left(v(b) - q_u^B\right) f_u^S(b) du + (1 - F_t^S(b)) e^{-rt}\left(v(b) - q_t^S\right),$$

and the first-order condition for his problem is

$$r(v(b) - q_t^S) = \frac{f_t^S(b)}{1 - F_t^S(b)}\left(q_t^S - q_t^B\right) - \dot q_t^S. \tag{5}$$

Function $u^S(t,s)$ and the problem of the type $s$ seller are defined and analyzed analogously. Condition (3) is the first-order condition (5) evaluated at $b = b_t^*$. It describes the incentives of the threshold type of the buyer. The buyer balances the cost due to discounting (left-hand side), and the benefit from the possible concession of the seller (the first term on the right-hand side) and from the change in the seller price offer (the second term on the right-hand side). The first-order condition in (5) is only a necessary condition for optimality of the monotone strategy $b_t^*$, and one is still left to prove sufficiency.

Notice that $\eta$ enters into conditions (3) – (4) through terms $\lambda_t^S$ and $\lambda_t^B$, and there are two opposite effects of $\eta$ on the acceptance strategies. First, $\lambda_t^S$ and $\lambda_t^B$ are positive only when the threshold types are closer than distance $\eta$ to each other. For fixed paths of price offers, a higher $\eta$ implies that the intensity of the acceptance becomes positive earlier for both players. This gives players additional incentives to delay trade and slows down the acceptance. Second, denominators in $\lambda_t^S$ and $\lambda_t^B$ are increasing in $\eta$, and so, for higher $\eta$, the acceptance by a certain mass of opponent types has a smaller effect on the expected payoff of the player. Hence, to make threshold types indifferent between immediate acceptance and marginal delay, the acceptance strategy of the opponent should be more rapid for higher $\eta$.

Conditions similar to (3) arise in the dynamic screening models. When intensities of acceptance are equal to zero, the single crossing property of payoffs immediately implies the global optimality of the threshold acceptance strategy. This is the case for example in the Coasian literature. In this paper, the global optimality of the buyer's threshold acceptance strategy is more intricate and is guaranteed by the monotonicity of the seller's strategy and the structure of the correlation of types. In particular, they imply that $u^B(t,b)$ satisfies the smooth strict single-crossing difference property in $(-t,b)$ on the relevant set of types and times (Lemma 6 in the Appendix).[25] By Theorem 4.2 in Milgrom (2004), this together with the monotonicity of $b_t^*$ and the fact that it satisfies (3) implies that global optimality is guaranteed (Lemma 7 in Appendix). Intuitively, higher types of the buyer are more impatient both because of the higher valuation and the fact that they assign lower probability to their offer being accepted.[26]

Theorem 1 reduces the analysis of equilibria of the concession game to the problem of finding a solution to a system of differential equations. In general, one should expect that there are many equilibria in the concession game. By Picard-Lindelöf theorem, if price paths $q_t^S$ and $q_t^B$ have uniformly (over $t$) bounded derivatives and $q_\infty^S > q_\infty^B$ , then there exist $b_t^*$ and $s_t^*$ that solve the system (3)-(4). To guarantee that they constitute an equilibrium, we need to check that processes $b_t^*$ and $s_t^*$ are monotone.[27] For this, it is sufficient that prices do not change too fast: $\ddot{q}_t^S \geq 0$ and $\ddot{q}_t^B \leq 0$ before the time when intensity of acceptance for threshold types becomes positive, and after this time $r(v(b_t^*) - q_t^S) + \dot{q}_t^S \geq 0$ and $r(q_t^B - c(s_t^*)) - \dot{q}_t^B \geq 0$.[28]

Before moving on to the analysis of the bargaining game, I illustrate via example the equilibrium behavior in the concession game.

**Example** To illustrate the effect of individual uncertainty and price paths on the trade dynamics and efficiency, consider the following model. Utilities are linear in types given by $v(b) = b+\xi$, $c(s) = s$. Let $r = 10\%$ and $\xi = \frac{2}{3}$. Consider price paths

$$q_t^S = 1 + \frac{1}{3}e^{-rt} \text{ and } q_t^B = \frac{2}{3} - \frac{1}{3}e^{-rt}. \tag{6}$$

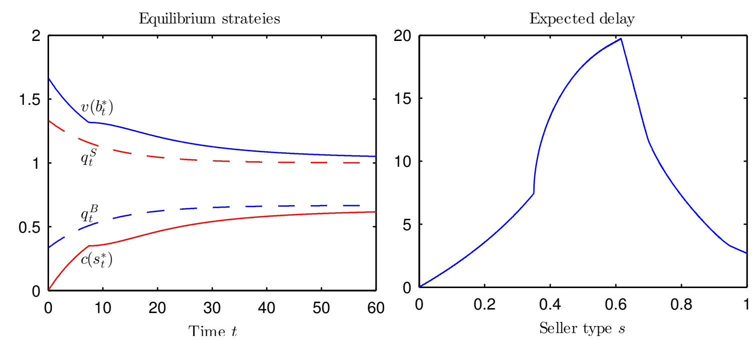
By the requirement that $q_t^S > c(1)$ and $q_t^B < v(0)$, $\eta \in (0,1)$. In Figure 2, I depict equilibrium strategies and expected trade delay for the seller for $\eta = .3$ and $\eta = .01$. Notice that there is a kink in the acceptance strategy around time 7.4 for $\eta = .3$ and 13.6 for $\eta = .01$. At this point, the intensities of $\lambda_t^S$ and $\lambda_t^B$ become positive for threshold types and give additional benefits for delaying the acceptance. As a result, both sides slow down the concession after this time.

[25] Function $u^B(t,b)$ satisfies the *smooth strict single-crossing difference property in* $(-t,b)$ if for $t < t'$ and $b < b'$, $u^B(t,b) > u^B(t',b)$ implies that $u^B(t,b') > u^B(t',b')$, and $u^B(t,b) \geq u^B(t',b)$ implies that $u^B(t,b') > u^B(t',b')$, and in addition, whenever $\frac{d}{dt}u^B(t,b) = 0$ for any $d > 0$, $\frac{d}{dt}u^B(t,b+d) \leq 0$ and $\frac{d}{dt}u^B(t,b-d) \geq 0$ (see Milgrom and Shannon (2004)).

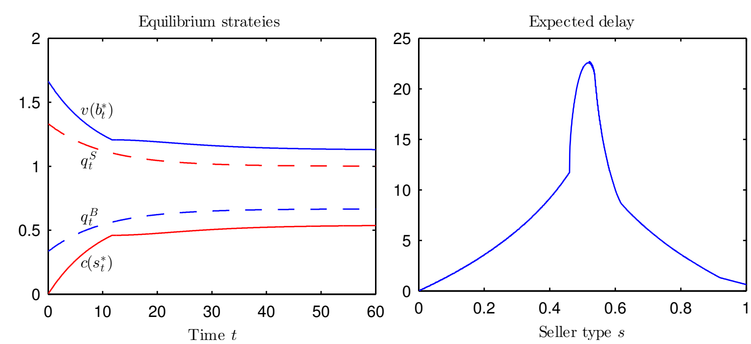
[26] A condition similar to (3) also arises in Fuchs and Skrzypacz (2010) that studies the screening model with the possibility of arrival of another buyer. The uninformed seller makes screening offers to a privately informed buyer, and after the arrival of the second buyer, the seller runs an English auction. In Fuchs and Skrzypacz's model, the condition reflecting the incentives of the buyer is given by the condition (3) with the term $\lambda_t^S(q_t^S - q_t^B)$ replaced by the constant arrival rate multiplied by the type-dependent drop in the utility of the buyer from the arrival. In contrast, in this paper the likelihood of the acceptance is type-dependent. Fuchs and Skrzypacz (2010) makes assumptions about the exogenously-defined buyer's utility upon arrival to guarantee that the first-order conditions are sufficient.

[27] The transversality condition $b_T^* = b_{s_T^*}^\alpha$ follows again from the Picard-Lindelöf theorem.

[28] A simple condition on price paths that ensures it is that price paths stop changing after the time when intensities of acceptance becomes positive.




(a) $\eta = .3$




(b) $\eta = .01$

Figure 2: Equilibrium strategies for $\eta = .3$ and $\eta = .01$. Left panels depict price paths and values of threshold types in the acceptance strategy. Right panels depict expected trade delay for the seller as a function of seller type.

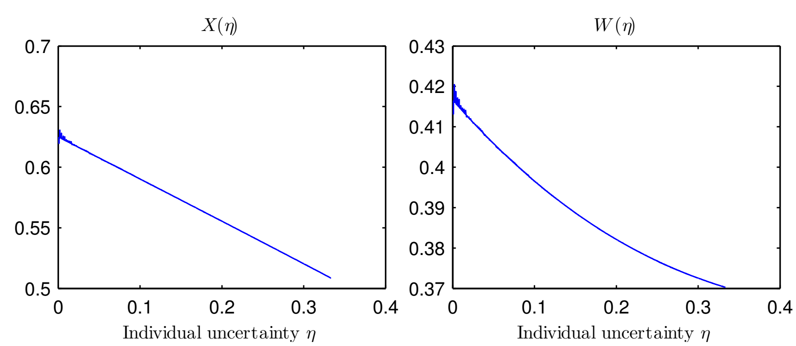



Figure 3: Expected discount on the surplus ($X$) and expected discounted surplus from trade ($W$) as a function of $\eta$ in the equilibrium of the concession game.

Since for lower $\eta$, the bandwidth of opponent types whose acceptance affects the incentives of the player is smaller, the time when $\lambda_t^S$ and $\lambda_t^B$ become positive is larger for smaller $\eta$. The expected delay is inverse U-shaped. For low seller types, the expected delay is low because they accept the buyer offer early in the game. For the high seller types, the expected delay is low because they put probability one on high buyer types that accept early in the game. The expected delay is highest in the middle of the type distribution, as for these types the opponent can accept the their price offer with positive probability and this gives additional incentives to them to delay the acceptance. In Figure 3, I present two efficiency measures: the expected discount on the surplus $X = \mathbb{E}\left[e^{-rT^c}\right]$ and the expected discounted surplus from trade $W = \mathbb{E}\left[e^{-rT^c}(v(b) - c(s))\right]$. Clearly, $X \leq 1$ and $W \leq \mathbb{E}\left[v(b) - c(s)\right] = \frac{1}{2}$. We can see that the efficiency is decreasing in $\eta$, however, equilibria are not fully efficient as $\eta$ vanishes. In particular, even as $\eta \to 0$, around 37% of the surplus is dissipated due to inefficient trade delay.

### 3.2 Characterization of CSEs

Equilibria in the concession game are appealing because of their analytic tractability. However, the assumption that price paths are fixed seems far from innocuous at first sight. Next, I present the central result of this section justifying this assumption. Even if different types of the same player are allowed to offer different price offers, there are equilibria in the bargaining game, in which they choose not to do so, and all types follow a given path of offers. I first define the class of CSEs.

**Definition 3.** Common screening equilibria (CSEs) *are equilibria of the bargaining game in which on-path equilibrium strategies are described by the tuple* $(b_n, s_n, p_n^B, p_n^S)$ *which satisfies the following properties.*

1. *A path of seller offers* $p_n^S$ *changes only in odd rounds, and in any odd round* $n$*, all seller types that do not accept the buyer's offer make counter-offer* $p_n^S$*.*[29] *All buyer types follow a sequence of offers* $p_n^B$*, which changes only in even rounds.*

2. *Sequence* $p_n^S$ *is (weakly) decreasing, sequence* $p_n^B$ *is (weakly) increasing, and* $p_n^B < v(0)$*,* $p_n^S > c(1)$ *for all* $n$*.*

3. *There is a non-increasing sequence of threshold buyer types* $b_n$ *and a non-decreasing sequence of threshold seller types* $s_n$*. In even rounds, all remaining buyer types above* $b_n$ *accept the seller's offer* $p_{n-1}^S$*, and in odd rounds all remaining seller types below* $s_n$ *accept buyer's offer* $p_{n-1}^B$*, so long as there have been no deviations from price paths* $p_n^S$ *and* $p_n^B$ *in the past.*

4. $c^{-1}(p_\infty^B) - v^{-1}(p_\infty^S) \geq \eta$.

[29] In the text, I refer to a sequence $\{x_n\}_{n=1}^{\infty}$ by its member $x_n$ and to a continuous time process $\{x_t\}_{t\geq 0}$ by its member $x_t$.

*A CSE is* active *if on the equilibrium path a positive mass of remaining buyer or seller types accepts the opponent's offer in every round up to some $\bar{N} \leq \infty$ and no types remain after $\bar{N}$.*

On-path equilibrium strategies in CSEs are the discrete-time analogues of the strategies and price paths in concession game. In a CSE, both sides screen the opponent's type and all types on either side use a common screening policy, i.e. they follow the same sequence of offers. The restriction that all price offers of the seller are above the highest costs of the seller (and a symmetric restriction on buyer price offers) guarantees that types never get negative utility from pooling on offers. On the equilibrium path, the seller makes decreasing price offers and screens buyer types starting from the top of the distribution, and the buyer screens the seller types via an increasing sequence of price offers starting from the bottom of the distribution. The property that higher buyer types accept the seller's offer earlier than lower types (and the reverse for the seller) is referred to in the bargaining literature as a *skimming property.* The skimming property greatly simplifies the Bayesian updating of beliefs. In any round $n$, the posterior beliefs of any remaining type $b$ buyer is a truncation of the uniform distribution on $S_b$ at the bottom at $s_n$, and symmetrically, the beliefs of any remaining type $s$ seller is a truncation of the uniform distribution on $B_s$ at the top at $b_n$.[30]

Subsequently, I define the limit of CSEs as the round length $\Delta$ converges to zero. First, I extend strategies in the discrete-time game to continuous time. For any sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$, say that a function $f_t$ is an *extension of $\{f_n\}_{n\in\mathbb{N}}$ to a continuous domain* if $f_t|_{t=n\Delta} = f_n$ for all $n \in \mathbb{N}$, and $f_t$ is linear on each interval $[(n-1)\Delta, n\Delta]$.[31] The following definition formalizes the notion of convergence.

**Definition 4.** *A sequence $(b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta})$ of CSEs indexed by $\Delta \to 0$ has a* smooth limit *if*

1. *processes $b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta}$ converge pointwise to continuous, a.e.-continuously differentiable limit processes $b_t^*, s_t^*, q_t^B, q_t^S$, respectively;*

2. *$T = \limsup_{\Delta\to 0} T_\Delta$, where $T \equiv \inf\{t \geq 0 : b_{t'}^* = b_t^*$ and $s_{t'}^* = s_t^*$ for all $t' \geq t\}$ and $T_\Delta \equiv \inf\{t \geq 0 : b_{t'}^\Delta = b_t^\Delta$ and $s_{t'}^\Delta = s_t^\Delta$ for all $t' \geq t\}$;*

3. *$b_T^* = \lim_{\Delta\to 0} b_{T_\Delta}^\Delta$ and $s_T^* = \lim_{\Delta\to 0} s_{T_\Delta}^\Delta$.*

*The tuple $(b_t^*, s_t^*, q_t^B, q_t^S)$ is called the* smooth limit *of the sequence.*

[30] As in the analysis of the concession game, the monotonicity restrictions in CSEs guarantee global optimality of the on-path strategies. I additionally require that in CSEs an analogue of condition (1) holds. This way I focus on the inefficiencies that arise due to the timing of the acceptance, but not due to unrealized gains from trade.

[31] To distinguish CSE on-path strategies $b_n, s_n, p_n^B, p_n^S$ from their extensions $b_t, s_t, p_t^B, p_t^S$, respectively, I use time index $t$ instead of round index $n$, whenever I refer to the extensions. Additionally, since the characterization of CSE limits is in terms of equilibria of the concession game, with a slight abuse of notation, I use the same notation for the CSE limit as for strategies in the concession game.

Condition 1 in Definition 4 implies that in the limit, no positive mass of types accepts the opponent's price offer in any arbitrarily short interval of time, and moreover prices do not change drastically. Condition 2 guarantees that the limit preserves information about when the trade ends with certainty. Condition 3 ensures that the sets of accepting types $(b^{\Delta}_{T_\Delta}, 1]$ and $[0, s^{\Delta}_{T_\Delta})$ do not collapse in the limit.[32]

In contrast to the concession game, in the bargaining game players choose price offers that they make instead of following a given price path. This places additional restrictions on price paths and acceptance strategies. These restrictions are formulated in terms of continuation utilities of players in the concession game, and it is useful to denote by $\mathcal{U}^B_t(b)$ and $\mathcal{U}^S_t(s)$ the continuation utilities at time $t$ of buyer type $b$ and seller type $s$, respectively, in an equilibrium of the concession game. More precisely, for $t \le t^*_B(b)$, let $\mathcal{U}^B_t(b) \equiv \frac{e^{rt}}{1-F^S_t(b)}\left(u^B(t^*_B(b), b) - u^B(t, b)\right)$, and for $t \le t^*_S(s)$, let $\mathcal{U}^S_t(s) \equiv \frac{e^{rt}}{1-F^B_t(s)}\left(u^S(t^*_S(s), s) - u^S(t, s)\right)$.

The next lemma gives weak restrictions on equilibrium price offers in the bargaining game that arise from the fact that it is common knowledge among players that valuations belong to the interval $[v(0), v(1)]$ and costs belong to the interval $[c(0), c(1)]$.

**Lemma 1.** *In any equilibrium and after any history,*

1. *any buyer's offer above* $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$ *is accepted by the seller, and the buyer never accepts any offer higher than* $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$*;*

2. *any seller's offer below* $\frac{v(0)+e^{-r\Delta}c(0)}{1+e^{-r\Delta}}$ *is accepted by the buyer, and the seller never accepts any offer lower than* $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$*.*

The interpretation of Lemma 1 is as follows. Suppose that the seller manages to convince the buyer that she has the highest possible costs, $c(1)$, and the buyer's valuation turns out to be $v(1)$, thus maximizing the size of the surplus. Then the outcome would be as in the unique subgame perfect equilibrium of the complete information game with valuation $v(1)$ and cost $c(1)$ analyzed by Rubinstein (1982). In such an equilibrium, the seller makes offer $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$ and rejects any offer below $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$, and the buyer makes offer $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$ and rejects any offer above $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$. By Lemma 1, the seller cannot get a higher payoff than in the scenario described. Moreover, the buyer always has the option to trade immediately at price $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$ by admitting that he has the highest valuation $v(1)$ and by recognizing that the seller has the highest costs $c(1)$.

Observe that Lemma 1 implies that when the range of $v$ and $c$ is getting smaller, the expected delay in any equilibrium decreases.[33] Hence, reducing the common uncertainty increases

[32] The Appendix provides examples that show that conditions 2 and 3 differ.

[33] Indeed, for any $b$ it holds that

$$v(0) - \frac{c(1) + e^{-r\Delta}v(1)}{1 + e^{-r\Delta}} \le v(b) - \frac{c(1) + e^{-r\Delta}v(1)}{1 + e^{-r\Delta}} \le \mathbb{E}_b\left[e^{-rT}\left(v(b) - p\right)\right] \le \left(v(1) - \frac{c(0) + e^{-r\Delta}v(0)}{1 + e^{-r\Delta}}\right)\mathbb{E}_b\left[e^{-rT}\right],$$

the efficiency of any equilibrium. However, as will be shown in this section, the reduction in individual uncertainty does not necessarily have the same effect.

Lemma 1 together with the fact that players can always reject any offer implies that in the frequent-offer limit, seller type $s$ gets at least her *reservation utility* $\max\left\{\frac{v(0)+c(0)}{2}-c(s),0\right\}$, and *reservation utility* of buyer type $b$ is $\max\left\{v(b)-\frac{v(1)+c(1)}{2},0\right\}$. This translates into the following restriction on the utilities that players get in equilibrium in the concession game. For all $t\in[0,T)$ and all $b$ and $s$,

$$\mathcal{U}_t^B(b)\geq\max\left\{v(b)-\frac{v(1)+c(1)}{2},0\right\},\tag{7}$$

$$\mathcal{U}_t^S(s)\geq\max\left\{\frac{v(0)+c(0)}{2}-c(s),0\right\}.\tag{8}$$

The next theorem shows that in the limit of frequent offers, conditions (7) and (8) are the only restrictions that the ability to choose price offers puts on the equilibrium price paths and acceptance strategies. In particular, it establishes that under additional generic conditions on equilibrium strategies, the sets of active CSE smooth limits and equilibria in smooth monotone strategies of the concession game coincide.

**Theorem 2** (Characterization of CSEs)**.** *Suppose a sequence of active CSEs indexed by $\Delta\to 0$ has a smooth limit. Then the smooth limit of the sequence constitutes an equilibrium in the concession game, and in addition, satisfies conditions (7) and (8).*

*Conversely, suppose an equilibrium of the concession game in smooth monotone strategies $(b_t^*,s_t^*,q_t^B,q_t^S)$ and date $T$ specified in condition (2) are such that $b_\infty^*\in(0,1)$, $s_\infty^*\in(0,1)$, $c(s_T^*)<q_T^B\leq q_T^S<v(b_T^*)$, and strict versions of inequalities (7) and (8) hold for all $t\in[0,T)$,$b$,$s$. Then there exists a sequence of active CSEs indexed by $\Delta\to 0$ with a smooth limit $(b_t^*,s_t^*,q_t^B,q_t^S)$.*

Theorem 2 sheds light on the limit dynamics of trade and sources of inefficiency in CSEs. In CSEs, players simultaneously screen each other's types and this two-sided screening dynamics is possible for a variety of $\eta$. Importantly, for a given price path, there can be a variety of individual uncertainty levels $\eta$ that are consistent with the given price path, and one can determine $\eta$ only by observing the frequency of acceptance at each price.

Since all types on each side make use of a common screening policy, there is a tension between interests of different types. In particular, higher types of the seller would like the screening policy to go more thoroughly through the higher buyer types, while lower types of the seller would prefer that the higher types be skipped altogether (as they assign zero probability to such types) and that the screening focus on lower buyer types. One would expect that the common screening policy reaches a compromise between interests of different types. As will be shown later, very harsh punishment is available even in the limit as $\eta$ vanishes and so there is no need for such

and so, $\mathbb{E}_b\left[e^{-rT}\right]$ converges to one as the range of values decreases.

compromise. In fact, there is a "pecking order" of types with higher types of the seller and lower types of the buyer receiving priority. That is, at the beginning of bargaining, high types of the buyer are screened, which is in the interest of the high seller types. Over time, the screening policy reaches buyer types in the middle of the type distribution, which gives positive profit to the seller types in the middle of the distribution. However, because they spend a certain amount of time making screening offers to high types that are rejected with certainty, the profit from the screening includes the discount due to a delayed start of the effective screening. Since it may take a significant amount of time until the common screening policy becomes efficient for the lower seller types, they might prefer to accept the less favorable buyer offer instead of waiting for their "turn to screen". In fact, the common screening policy may never reach the seller types at the bottom, as they all have accepted earlier some offer of the buyer. The ordering of buyer types is reversed with lower buyer types screening first, and higher buyer types accepting some seller offer.

The equilibrium conditions help us understand the sources of inefficiency in the model. Two standard sources of inefficiency are reflected in conditions (3) and (4). For example, consider equation (3), which describes the evolution of threshold buyer types. A more rapid decrease in seller price offers $q_t^S$ leads to higher $b_t^*$ and creates an inefficient delay. This is the standard deadweight loss from screening. If the seller were not discriminating, then $q_t^S$ would not change and this would lead to a lower $b_t^*$, and hence, more rapid trade.

To see the second inefficiency due to signaling, consider the likelihood $\lambda_t^S$ that the buyer's offer is accepted. In equation (3), an increase in $\lambda_t^S$ results in higher threshold buyer type $b_t^*$. By delaying trade, the buyer signals the seller that his valuation is low and further delay could be costly to the seller. The stronger the impact of such a signal on the seller's behavior (higher $\lambda_t^S$), the greater the incentives of the buyer to signal by inefficiently delaying trade.

The model also has a third new source of bargaining inefficiency created by the fact that there is a pecking order of types and it could take a long time until the common screening policy becomes efficient for the types in the middle of the type range. To see this effect, observe that a seller of type $s$ expects positive profit from her screening offers only after time $t$ when $b_t^* \leq b_s^\omega$, and buyer types in the support of her beliefs start accepting the seller's screening offers. Suppose type $s$ is such that the first time $t$ when $b_t^* \leq b_s^\omega$ is finite. Until this time, seller type $s$ follows the common screening path $q_t^S$, even though she knows that such offers are rejected with certainty. As a result, the delay for seller type $s$ is increased by the amount of time it takes to screen buyer types above $b_s^\omega$.

Theorem 2 has an important empirical implication. It is possible that for a wide range of $\eta$, bargaining may start from offers that are far from the equal division of the realized surplus and that feature the two-sided screening dynamics. In this case, trade can be significantly delayed. As one decreases individual uncertainty (for fixed price paths), the sources of inefficiency shift from the standard deadweight loss and signaling costs to the inefficiency of common screening.

As a result, even in the limit as $\eta$ is vanishingly small, the equilibria are far from efficient.

The distinction between private and public information allows the interpretation of the model in this paper to be the description of the trade within segments determined by the public information. In the OTC example, credit ratings divide the bond market into several segments: prime, investment grade, and non-investment grade bonds. Traders can evaluate risks more accurately associated with a particular bond and use credit ratings as a starting point to trade a finer distinctions of risk within each bond segment. The model implies that the transparency, as opposed to the sophistication of traders, is crucial for the efficient functioning of the market.

The analysis of CSEs reveals the non-robustness of the model with independent private information to higher-order uncertainty. Suppose that one tries to predict the outcome of a particular trade and only observes the scope of private information of players (beliefs of players). Then the application of the model with independent private information can be misleading as it ignores the inefficiency of common screening. In the context of the model, observing only $b$ and $s$ as well as beliefs $\mu_b^0 \in \Delta(S_b)$ and $\mu_s^0 \in \Delta(B_s)$ may not be sufficient to make the prediction about the equilibrium outcome. A correct approach would be to start with all pairings of the buyers and sellers in the market. Then one finds the smallest sets $\hat{B}$ and $\hat{S}$ of buyer and seller types such that for any type $b' \in \hat{B}$, $S_{b'}$ intersects with $B_{s'}$ for some seller types $s' \in \hat{S}$, and the analogous condition for the seller. This will give the proxy for the common uncertainty in the market, and one can proceed from that to study possible CSEs. By similar logic, the model with perfect information is not robust to higher-order uncertainty if one admits that both sides can have some small amount of private information.

### 3.3 Proof Sketch of Theorem 2

I next describe the main methodological contribution of this paper. To show that an equilibrium $(b_t^*, s_t^*, q_t^B, q_t^S)$ satisfying the conditions of Theorem 2 can be obtained as a smooth limit of the sequence of CSEs, I construct a sequence of CSEs in grim trigger strategies. Equilibria in grim trigger strategies consist of two ingredients: the *main path* and the *punishment path*. Players start the game by following the main path and continue to follow it unless a detectable deviation occurs. Detectable deviations from the CSE equilibrium path trigger the punishment, and players switch to the punishing path for the deviating side given by punishing equilibria analyzed in detail in Section 6. By the Contagious Coasian Property of punishing equilibria (Theorem 6), as $\Delta \to 0$, the utility of any type of the deviator in the punishing equilibrium converges uniformly to the lowest utility possible in the equilibrium which, in conjunction with the strict versions of inequalities (7) and (8), allows us to support the main path. In this subsection, I focus on the steps in the construction of the main path.

The construction of the main path is based on the approximation of differential equations (3) and (4) by difference equations. For $T < \infty$, there is an approximating sequence of strategies $(b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta})$ that converges uniformly to $(b_t^*, s_t^*, q_t^B, q_t^S)$ as $\Delta \to 0$. By uniform convergence, deviations from the main path can be deterred by the threat of switching to the punishing path. Extending the construction to the case $T = \infty$ is important for two reasons. First, it allows one to cover the screening dynamics in which price offers of the buyer and the seller never converge ($q_\infty^S > q_\infty^B$). More importantly, it is key in the construction of a different class of segmentation equilibria analyzed in Section 4. The difficulty in the case $T = \infty$ is that it is no longer possible to construct a uniform approximation of equilibrium strategies as was the case for $T < \infty$. I circumvent this difficulty as follows. Consider an on-equilibrium-path history of the CSE in which it is revealed that buyer types are below some $b_0$ and seller types are above $s_0$. Define the *continuation CSE* as the continuation equilibrium after such a history in which the strategies in the continuation equilibrium are described as in the definition of CSE. Lemma 2 constructs particular continuation CSEs with $T = \infty$ in which, on the equilibrium path, price offers are constant over time, and the mass of the remaining types can be arbitrarily small. Given this result, an equilibrium in which negotiation continues indefinitely is approximated with an equilibrium in which after a certain time $T'$ price offers become constant. For times before $T'$, a uniform approximation of $(b_t^*, s_t^*, q_t^B, q_t^S)$ is available, and I can proceed as in the case $T < \infty$.

**Lemma 2.** *Suppose $b_0 \in (0, 1-\eta]$, $s_0 \in [b_0 - \eta, b_0 + \eta) \cap [\eta, 1)$, $P^B$, $P^S$ satisfy*

$$\max\left\{c\left(s_{b_0}^\omega\right), \frac{v(0)+c(0)}{2}\right\} < P^B < P^S < \min\left\{v\left(b_{s_0}^\alpha\right), \frac{v(1)+c(1)}{2}\right\}, . \tag{9}$$

*Then for all $\Delta$ sufficiently small, there exists an active continuation CSE such that*

1. *$b_0$ and $s_0$ are the highest buyer type and the lowest seller type, respectively, remaining in the game,*
2. *$p_n^B = P^B$ and $p_n^S = P^S$ for all $n \in \mathbb{N}$,*
3. *$\max\{b_{n-1} - b_n, s_n - s_{n-1}\} < \Delta C$ for all $n \in \mathbb{N}$, where $C$ is a constant independent of $\Delta$.*

Lemma 2 constructs a continuation CSE that starts from the moment that only buyer types below $b_0$ and seller types above $s_0$ remain in the game. There are two price offers $P^B$ and $P^S$ that are made on the equilibrium path, and each player decides whether to accept the less favorable offer of the opponent, or to delay acceptance in the hope that the opponent will accept earlier.[34] In every round, a positive mass of types of the active player accepts. Condition (9) ensures that for all types remaining in the game the utility from accepting the opponent's offer

[34] The equilibrium constructed in Lemma 2 is similar to equilibria in war of attrition game. See Fudenberg and Tirole (1991) for a survey of the literature on the war of attrition. Abreu and Gul (2000) establishes a connection between reputational bargaining and the war of attrition. Krishna and Morgan (1997) analyzes the war of attrition with affiliated values as an auction form in which the winning bidder pays the highest losing bid and losing bidders pay their bids. The literature on the war of attrition has a different payoff structure and is mostly formulated in continuous time, so I was not able to build on the techniques used in this literature.

exceeds their reservation utility.[35] Given that there is a positive difference in players' payoffs from trading at $P^B$ or $P^S$, bargaining necessarily continues indefinitely.[36] The last property of the continuation equilibrium constructed in Lemma 2 guarantees that in the limit concession happens continuously.[37]

It should be mentioned that the equilibrium construction in Lemma 2 is significantly harder for the case of correlated values when compared to the case of independent values ($\eta = 1$). For the intuition of differences in the analysis, compare the incentives of the threshold types in the case $\eta = 1$ and $\eta < 1$. To keep the threshold buyer type indifferent between accepting and rejecting the current offer, the probability of seller acceptance in the next round should be sufficiently high. When types are independent ($\eta = 1$), it is possible to vary this probability from 0 to 1 by varying the threshold seller type in the next round. In this case, for any initial choice of $b_2$, it is possible to recursively construct the subsequent thresholds. However, when types are correlated ($\eta < 1$), there is an upper bound on the probability of seller acceptance evaluated by buyer type $b$. This comes from the fact that for the buyer type $b$, all seller types in the interval $[s_\infty, s_b^\omega]$ never accept the buyer's price offer. In this case, if the construction begins from an arbitrary choice of the first threshold types $b_2$ and proceeds recursively, it can happen that in some round $n$ there is no threshold type of the active player in round $n+1$ that makes the threshold type of the active player in round $n$ indifferent between acceptance and delay. Nevertheless, Lemma 2 establishes that it is possible to find an initial threshold type $b_2$ so that the recursive construction of thresholds is possible.

## 4 Segmentation Equilibria

In CSEs, as individual uncertainty vanishes, common-screening inefficiency becomes a dominant source of the surplus dissipation. In this section, I analyze a very different class of equilibria which I call segmentation equilibria. In this class of equilibria, common-screening inefficiency is eliminated shortly after bargaining begins and a positive mass of types trades in the first instance, and after that, trade is gradual for the remaining types. This contrasts with the trade dynamics and efficiency properties of CSEs.

[35] To understand the requirement on $b_0$ and $s_0$ in Lemma 2, observe that in the sufficiency part of Theorem 2, it holds $b^*_\infty \in (0,1)$ and $s^*_\infty \in (0,1)$, and together with condition (2), this implies $b^*_\infty \in (0, 1-\eta)$ and $s^*_\infty \in (\eta, 1)$. I use Lemma 2 to construct a continuation CSE in which the remaining buyer types are below $b_0$ and the remaining seller types are above $s_0$, and $b_0$ and $s_0$ are close to $b^*_\infty$ and $s^*_\infty$, respectively. Therefore, I place the restriction $b_0 \in (0, 1-\eta]$ and $s_0 \in [\eta, 1)$. The requirement $s_0 \in [b_0 - \eta, b_0 + \eta)$ guarantees that starting from the first round in the continuation equilibrium both sides assign positive probability to the acceptance of their offer in the next round. This makes the concession continuous in the limit $\Delta \to 0$ with no positive mass of types accepting in any instant of time, and in particular, implies that the bound on $\max\{b_{n-1} - b_n, s_n - s_{n-1}\}$ in Lemma 2 holds.

[36] Otherwise, for sufficiently small $\Delta$, players would prefer to marginally delay acceptance before the final date. This would give a discontinuous gain in the payoff, making the acceptance at times close to the final date suboptimal.

[37] Indeed, in an interval of length $\Delta$ at most mass $\Delta C$ of types concedes and so the speed of acceptance is bounded above by $C$.

The idea behind segmentation equilibria is to construct equilibria such that with their initial offers, players establish common knowledge of a relatively narrow range of players' types which eliminates the inefficiency of common screening. As a result, the efficiency of segmentation equilibria resembles the efficiency of the model with independent values: as individual uncertainty decreases, efficiency increases. I also require that offers on the equilibrium path can come only from a particular set. This enables the interpretation that types endogenously self-select into segments associated with a particular price. This motivates the following definition.

**Definition 5.** Segmentation equilibria *satisfy the following two conditions on the equilibrium path.*

- *After the first two rounds, every remaining type assigns positive probability to his/her offers being accepted.*
- *Only offers from finite sets* $Q_Z^S = \{q_1^S, \dots, q_Z^S\}$ *and* $Q_Z^B = \{q_1^B, \dots, q_Z^B\}$ *are made, where* $q_z^S \in \left(\frac{v(0)+c(0)}{2}, \frac{v(1)+c(1)}{2}\right)$ *and* $q_z^B \in \left(\frac{v(0)+c(0)}{2}, \frac{v(1)+c(1)}{2}\right)$ *for all* $z$*.*

The next theorem constructs segmentation equilibria in which each segment is associated with a particular price offer of the seller $q_z^S$ and the buyer $q_z^B$, and segments are ordered by the level of price offers.

**Theorem 3.** *Fix an integer* $Z$*, an increasing sequence of offers* $\{q_z^B\}_{z=1}^Z$*, and increasing sequences* $\{b^z\}_{z=1}^Z$ *and* $\{s^z\}_{z=1}^Z$ *of buyer and seller types such that* $b^0 = s^0 = 0$*,* $b^Z = s^Z = 1$ *and*

1. $s^z = b^z + \eta$ *and* $c(s^z) < q_z^B < v(b^{z-1})$ *for* $z = \overline{1, Z-1}$*,*
2. $b^{z+1} - b^z > 4\eta$ *for* $z = \overline{1, Z-2}$*.*

*Then for sufficiently small* $\Delta$*, there exists a segmentation equilibrium with* $Z$ *segments and buyer price offers* $\{q_1^B, \dots, q_Z^B\}$ *such that*

- *there is no almost sure upper bound on the delay on the equilibrium path, i.e. for any positive* $t$*,* $\mathbb{P}(N\Delta > t) > 0$*;*[38]
- *ex-ante probability of delay longer than two rounds is bounded above by* $\frac{4\eta(Z-1)}{2-\eta}$*.*

In segmentation equilibria, a positive mass of types trades in the first instant followed by the gradual acceptance of the remaining types, and the mass of the remaining types becomes small as $\eta$ decreases. Therefore, unlike CSEs, segmentation equilibria are relatively efficient and have a burst of trade in the beginning of bargaining.

I first describe informally how trade happens in the limiting case of low individual uncertainty and then give the outline of the construction of segmentation equilibria. When $\eta \approx 0$ and

[38] Recall that $N$ is the round when trade happens in equilibrium.

$\Delta \approx 0$, then $b^z \approx s^z$ and $q_z^S \approx q_z^B$. Types $b^z$ are the boundaries of segments, and buyer/seller types in $(b^{z-1}, b^z)$ belong to segment $z$. All of those types, except for an $\eta-$neighborhood around boundaries, trade almost immediately at the price $q_z^B$ corresponding to the segment. The inefficiency of the segmentation equilibria stems from the types in the $\eta$-neighborhood of the segment boundaries. Such types have incentives to delay trade and form reputations for belonging to a segment with more favorable terms of trade. Hence, these types could continue bargaining for an arbitrarily long time. Because, the inefficient delay happens only for types in the $\eta$-neighborhood of segment boundaries, inefficiency decreases with $\eta$ and increases with the number of segments $Z$, which is reflected in the upper bound $\frac{4\eta(Z-1)}{2-\eta}$ on the probability that the delay is longer than two rounds.

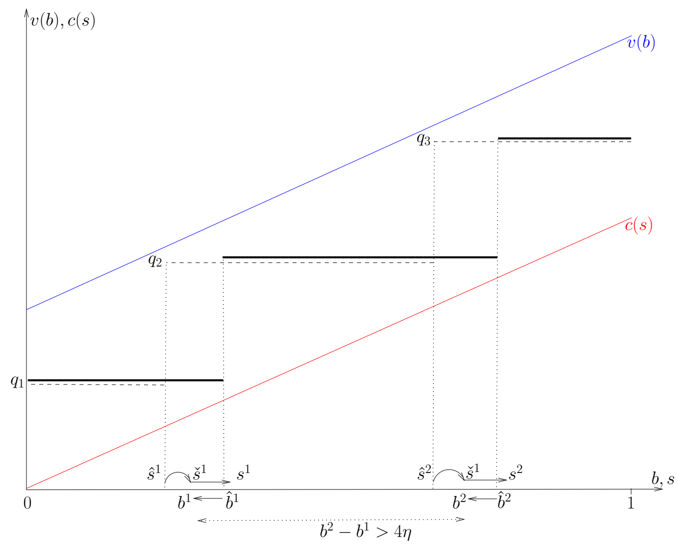



Figure 4: **Segmentation equilibrium.** The dashed line depicts offers of each seller type in the first round, the solid line depicts counter-offers that each buyer type makes in the second round if he rejects the seller price offer. Straight arrows depict gradual acceptance and arched arrows depict a positive mass of seller types accepting in the third round.

More formally, I construct the following segmentation equilibria illustrated in Figure 4. The dashed line depicts the first offer of the seller, and the bold line depicts the counter-offer of the buyer if he does not accept the offer of the seller. While price offers $q_z^S$ and $q_z^B$ are different for $\Delta > 0$, when $\Delta \to 0$, the difference vanishes, and one can interpret that each segment is associated with a particular price offer. The price offers are such that if either of the first two offers is accepted, both sides get positive utility (conditions 1 in Theorem 3). The intervals of types that make the same offer are sufficiently big (condition 2 in Theorem 3) so that on the equilibrium path, any type either knows what offer the opponent will make or expects that the opponent will make one of two offers. Notice that the two conditions place restrictions on the number of segments for a given $\eta$. On the one hand, there should be enough segments to ensure that trade is individually rational for every realization of types. On the other hand, the segments should not be too close to each other. Given the assumption of strict gains from trade (recall $\xi > 0$), for $\eta$ sufficiently small, it is possible to construct the segments that satisfy conditions 1 and 2 of Theorem 3.

The acceptance strategies are described as follows. In the second round, buyer types above $\hat{b}^z$ accept any equilibrium offer of the seller, and in the third round seller types below $\check{s}^z$ accept any equilibrium offer of the buyer. When offers are frequent, this initial trade happens almost immediately. If the game has not ended during this initial period, and the first two offers were $q_z^S$ and $q_{z-1}^B$, then remaining types are playing the war-of-attrition game described in Lemma 2 with $b_0 = \hat{b}^z$ and $s_0 = \check{s}^z$ and price offers $P^S = q_z^S$ and $P^B = q_{z-1}^B$. In the limit of frequent offers, the gradual acceptance is described by a decreasing $b_t^z \in [b^z, \hat{b}^z, ]$ and an increasing $s_t^z \in [\check{s}^z, s^z]$. Acceptance strategies $b_t^z$ and $s_t^z$ are described by the following pair of differential equations:

$$r\left(v(b_t^z) - q_z^S\right) = \frac{\dot{s}_t^z}{b_t^z - s_t^z + \eta}\left(q_z^S - q_{z-1}^B\right), \tag{10}$$

$$r\left(q_{z-1}^B - c(s_t^z)\right) = -\frac{\dot{b}_t^z}{b_t^z - s_t^z + \eta}\left(q_z^S - q_{z-1}^B\right); \tag{11}$$

with transversality conditions $b^z = \lim_{t\to\infty} b_t^z$ and $s^z = \lim_{t\to\infty} s_t^z$. Equations (10) and (11) are analogous to those in the characterization of the equilibria for the concession game under constant price paths and when both players assign positive probability to their offer being accepted in the next round.

Notice that a segmentation equilibrium with just two segments is the common screening equilibrium with constant price paths. Hence, Theorem 3 gives both conditions for existence of common screening and segmentation equilibria. For the common screening equilibria, they boil down to $c^{-1}(v(0)) - v^{-1}(c(1)) > \eta$.[39] However, segmentation equilibria with more than two segments exist for a wider range of parameters. In particular, for any specification of $v$ and $c$, if $\eta$ is sufficiently small, one can construct corresponding prices and segment boundaries that satisfy conditions Theorem 3.

Theorem 3 could be reformulated to allow for segmentation to happen over time rather than all at once in the first two rounds. Together with Theorem 2, this suggests a rich description of possible equilibrium behavior. Intervals of gradual trade and (common) screening, as in CSEs, are interrupted by rounds in which remaining types split into endogenous segments, trade bursts, and common uncertainty is drastically reduced.

**Example (continued)** Finally, continuing the example in Section 3 with $\eta = .08$, I illustrate numerically the efficiency and trade dynamics of segmentation equilibria. I first construct a

[39]To see this, for $\varepsilon > 0$ sufficiently small, let $q_1^B = v(0) - \varepsilon$ and $q_2^B = c(1) + \varepsilon$. By the condition, for sufficiently small $\varepsilon$, it is possible to choose some $b^1 \in (v^{-1}(c(1)), c^{-1}(v(0)) - \eta)$ and $s^1 = b^1 + \eta$ such that $c(s^1) < v(0) - 2\varepsilon < q_1^B < v(0)$ and $c(1) < q_2^B < c(1) + 2\varepsilon < v(b^1)$, and so, by Theorem 3, there exists a segmentation equilibrium with two segments.

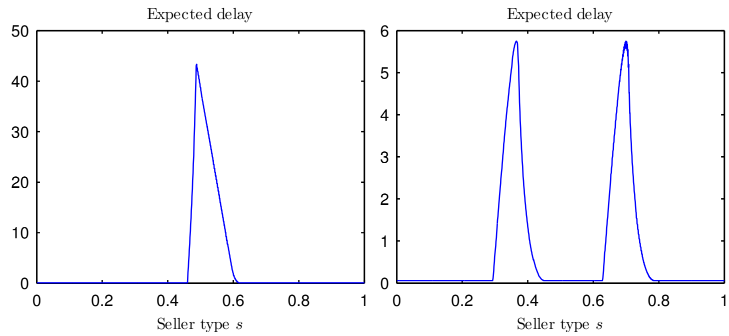



Figure 5: Expected delay for the seller in segmentation equilibria with two (left panel) and three segments (right panel).

segmentation equilibrium with just two segments and price offers $q_1^B = \frac{1}{2}$ and $q_2^B = \frac{7}{6}$. Price offers $q_1^B$ and $q_2^B$ are the mid-points between the first and the last price offer of the buyer and the seller, respectively. The first panel in Figure 5 depicts the expected delay as a function of the seller type. Compared to the converging price paths (panel b in Figure 2), there is an atom of trade in the first instance of time which decreases to zero the expected delay for high and low types of the seller. For types in the middle, however, the expected delay may increase, because of the higher difference in price offers between segments compared to the difference in price offers $q_t^S$ and $q_t^B$ in the common screening equilibria.

Next, consider the segmentation equilibrium with three segments. Let $b^1 = \frac{1}{3} - \frac{\eta}{2}$ and $b^2 = \frac{2}{3} - \frac{\eta}{2}$ (correspondingly, $s^1 = \frac{1}{3} + \frac{\eta}{2}$ and $s^2 = \frac{2}{3} + \frac{\eta}{2}$), and $q_1^B = \frac{1}{2}$, $q_2^B = \frac{5}{6}$, $q_3^B = \frac{7}{6}$ (see Figure 4 for an illustration of strategies in such an equilibrium). The expected delay in this equilibrium is depicted on the second panel in Figure 5. The figure illustrates that most of types trade in the first instance of time, while the types near boundaries of segments delay trade. Notice that while the increase in the number of segments decreases the efficiency, as there are more types that are involved in the inefficient war-of-attrition type of game near segment boundaries, the decrease in the difference in price offers between segments reduces the inefficiency. As a result, in the example, there is only a small difference in the efficiency between the segmentation equilibrium with two segments ($W = .61$) and three segments ($W = .63$).

Figure 6 presents the expected discount on the surplus and the expected discounted surplus as a function of $\eta$ for the segmentation equilibrium with three segments. Given the restrictions of Theorem 3, the range of $\eta$ for which this segmentation equilibrium exists is $(0, \frac{1}{12})$. Figure 6 shows that the equilibria become more efficient with the decrease in individual uncertainty.

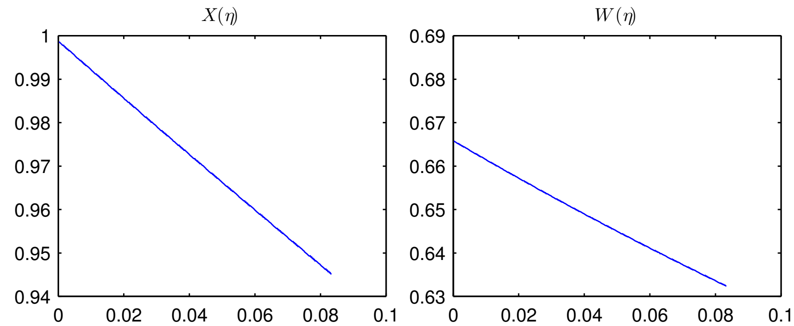



Figure 6: Efficiency of segmentation equilibria with three segments.

# 5 Vanishing Individual Uncertainty

Section 3 shows the remarkable discontinuity of the complete-information model which raises a concern about its applicability in environments with a big difference between private and public information. Because of the common screening inefficiency, a variety of equilibria with two-sided screening dynamics is possible even in the limit of vanishing individual uncertainty. In this section, I offer two ways to address this concern. First, in a different class of segmentation equilibria the gap between private and public information is eliminated endogenously. This property of segmentation equilibria allows me to restore the convergence to the complete-information outcome in the next subsection. Second, in Subsection 5.2, limits of CSEs as both $\Delta$ and $\eta$ converge to zero are characterized in terms of familiar incentive compatibility constraints, that are imposed on high buyer types and low seller types, and individual rationality constraints. This way, one can address the non-robustness concern directly by studying implications of two-sided screening dynamics. In a related paper (Tsoy (2014)), I take this route and study the effect of trade delay on the liquidity of OTC markets.

## 5.1 Approximation of the Complete-Information Outcome

Segmentation equilibria allow me to restore the connection between the complete-information bargaining game (Rubinstein (1982)) and the limit of the model in this paper as the correlation of values becomes perfect. Define the *complete-information outcome* to be $\left(0, \frac{v(b)+c(s)}{2}\right)$. The next theorem constructs a sequence of segmentation equilibria with increasingly fine definition of segments such that outcomes of these equilibria approximate the complete-information outcome.

**Theorem 4.** *There exists a sequence of segmentation equilibria indexed by $(\Delta, \eta) \to (0, 0)$ such that outcomes $(N\Delta, p)$ of segmentation equilibria converge in probability to the complete-information outcome $\left(0, \frac{v(b)+c(s)}{2}\right)$, i.e. for any $\varepsilon > 0$ there exists a segmentation equilibrium in* *the sequence such that*

$$\mathbb{P}\left(N\Delta > \varepsilon \text{ and } \left|p - \frac{v(b)+c(s)}{2}\right| > \varepsilon\right) < \varepsilon.$$

To prove Theorem 4, I apply Theorem 3 to construct segmentation equilibria with $Z \sim \frac{1}{\sqrt{\eta}}$ segments and prices $q_z^B = \frac{v(b^{z-1})+c(s^z)}{2}$. As $\eta \to 0$, the probability of any given delay is bounded from above by $\frac{4\eta(Z-1)}{2-\eta} \sim \sqrt{\eta}$ and converges to zero. The length of each segment $\sqrt{\eta}$ also converges to zero and so $q_z^B$ is close to the equal division for types in each segment $z$.

## 5.2 Double Limits of Common Screening Equilibria

I start by defining interim CSE outcomes. Recall that the outcome consists of the time $N\Delta$ and price $p$ at which trade occurs. For any CSE and buyer type $b$, define the discounted probability of allocation by $P^B(b) \equiv \mathbb{E}\left[e^{-r\Delta N}|S_b, \sigma_b\right]$ and the discounted transfer by $X^B(b) \equiv \mathbb{E}\left[e^{-r\Delta N}p|S_b, \sigma_b\right]$.[40] Define functions $P^S$ and $X^S$ for the seller analogously. The *interim outcome* $(P^B, X^B, P^S, X^S)$ is the expected outcome of each player after the type of the player is realized but before the type of the opponent is known.

**Theorem 5.** *Suppose a sequence of CSEs is indexed by $(\Delta, \eta) \to (0,0)$. Then CSE interim outcomes $(P^B, X^B, P^S, X^S)$ converge over subsequence for almost all types to $(\bar{P}^B, \bar{X}^B, \bar{P}^S, \bar{X}^S)$ that satisfies the following conditions:*

1. *$\bar{P}^B(\omega) = \bar{P}^S(\omega) = \bar{P}(\omega)$ and $\bar{X}^B(\omega) = \bar{X}^S(\omega) = \bar{X}(\omega)$ for $\omega \in [0,1]$.*
2. *For all $\omega, \omega' > \omega^*$,*

$$\begin{aligned} \bar{P}(\omega)v(\omega) - \bar{X}(\omega) &\geq \bar{P}(\omega')v(\omega) - \bar{X}(\omega') \geq 0, & (12)\\ \bar{P}(\omega)v(\omega) - \bar{X}(\omega) &\geq \max\left\{0, v(\omega) - \frac{v(1)+c(1)}{2}\right\}. & (13) \end{aligned}$$

3. *For all $\omega, \omega' < \omega^*$,*

$$\begin{aligned} \bar{X}(\omega) - \bar{P}(\omega)c(\omega) &\geq \bar{X}(\omega') - \bar{P}(\omega')c(\omega) \geq 0, & (14)\\ \bar{X}(\omega) - \bar{P}(\omega)c(\omega) &\geq \max\left\{0, \frac{v(0)+c(0)}{2} - c(\omega)\right\}. & (15) \end{aligned}$$

4. *Left and right limits of $\bar{X}(\omega)$ exist at $\omega^*$ and $\bar{X}(\omega^*+) \geq \bar{X}(\omega^*-)$.*[41]

[40] The expectations are taken conditional on the event that buyer type $b$ follows the equilibrium strategy $\sigma_b$, a seller type is drawn from a uniform distribution on $S_b$ and the seller follows the equilibrium strategy $\sigma_s$.

[41] For a monotone function $f$, I use notation$f(x+) \equiv \lim_{x' \to x+} f(x')$ for the right limit of $f$ at point $x$ and $f(x-) \equiv \lim_{x' \to x-} f(x')$ for the left limit of $f$ at point $x$ (which exists by monotonicity of $f$).

*Conversely, for any* $(\bar{P}^B, \bar{X}^B, \bar{P}^S, \bar{X}^S)$ *that satisfies conditions 1-4 and, in addition, satisfies*

$$\bar{P}(\omega) > 0 \text{ for all } \omega \neq \omega^* \text{ and } \bar{P}(\omega^*+) = \bar{P}(\omega^*-), \tag{16}$$

*there is a sequence of* $(\Delta, \eta) \to (0, 0)$ *and a corresponding sequence of CSEs such that CSE interim outcomes converge to* $(\bar{P}^B, \bar{X}^B, \bar{P}^S, \bar{X}^S)$ *for almost all types.*

The characterization in Theorem 5 describes limit interim outcomes of CSEs in terms of incentive and individual rationality constraints used in the mechanism design literature.[42] The first condition gives the following interpretation. There is a *quality* of the good $\omega$ which completely determines the time and price at which trade happens. Conditions (12) and (14) are standard incentive compatibility constraints required to hold only for the high buyer types (above $\omega^*$) and low seller types (below $\omega^*$). These conditions reflect the fact that in CSEs for small $\eta$ the time of trade is determined by the buyer's acceptance if realized types are sufficiently high, and by the seller's acceptance if realized types are sufficiently low. Conditions (13) and (15) are individual rationality constraints, adjusted for the fact that by Lemma 1, in the bargaining game the equilibrium price of trade should lie within the interval $\left[\frac{v(0)+c(0)}{2}, \frac{v(1)+c(1)}{2}\right]$ in the limit of frequent offers. Condition 3 reflects the fact that in CSEs seller price offers are always greater than buyer price offers.

Theorem 5 has two important implications. First, it gives the relationship between the trade delay and the quality of the good. By the standard argument from mechanism design literature (see Myerson (1981)), the incentive compatibility constraints (12) and (14) imply that $\bar{P}(\omega)$ is decreasing for $\omega > \omega^*$ and increasing for $\omega < \omega^*$. Therefore, by Theorem 5, trade delay is inverse U-shaped in quality. Goods with quality closer to the extremes of the quality range are traded faster, as their quality is revealed quickly in the two-sided screening process. On the contrary, for goods with quality in the middle of the quality range, trade can be significantly delayed because of the inefficiency of common screening. This provides a testable empirical implication of the model with vanishing individual uncertainty. It differs from the implications of the standard models of one-sided screening where the relationship between delay and quality is monotone, and the complete information model which has immediate agreement.

Second, the analysis of the double limits of CSEs provides a modeling tool for studying the implications of trade delay. In applied work, a non-trivial trade delay can be a realistic and desirable feature of the model. A natural way to incorporate trade delay into the model is by introducing incomplete information. However, in many cases, this considerably complicates the analysis and could turn out to be intractable in the context of a more general framework. Theorem 5 suggests an approach that combines the tractability of the complete-information model

[42] This approach of describing limit bargaining outcomes in terms of mechanism-design constraints was previously used in Ausubel and Deneckere (1989a) and Ausubel, Cramton and Deneckere (2001) to study the model with one-sided incomplete information, and in Gerardi, Hörner, and Maestri (2013) to study the model with interdependent values.

and the realistic trade dynamics of incomplete-information models. Specifically, if one assumes that the individual uncertainty about values is negligible but there is large common uncertainty, then one can use the characterization in Theorem 5 to describe the trade dynamics in such an environment. The benefit of this approach is that it combines realistic two-sided screening dynamics and at the same time allows for a relatively simple expression for terms of trade through the quality of good. Tsoy (2014) takes this approach to incorporate bargaining delay into a standard search model. While introducing two-sided private information in this setting would lead to a complicated functional fixed-point problem, the characterization in Theorem 5 allows me to avoid this complication and leads to a tractable model which I use to study the difference in the effect of bargaining delay and search delay on liquidity and asset prices.

# 6 Punishing Equilibria

This section introduces and analyzes *punishing equilibria.* Punishing equilibria successfully deter deviations from equilibrium paths of CSEs and segmentation equilibria because of the *Contagious Coasian Property*: in the limit of frequent offers, in the punishing equilibrium the utility of the deviator converges uniformly (over all types of the deviator) to the lowest utility possible in any equilibrium. In this section, I focus on the seller-punishing equilibrium[43], and I further refer to it as simply the punishing equilibrium. The description and construction of the punishing equilibria is a bit cumbersome, and in Subsection 6.1 I state the main result of the section, describe informally the equilibrium behavior in the punishing equilibria and give the intuition for why the Contagious Coasian Property obtains. I then proceed in Subsections 6.2 and 6.3 to carefully describe strategies in the punishing equilibrium and to show the existence in Theorem 7. Finally, in Subsection 6.4, I sketch the proof of the result that punishing equilibria possess Contagious Coasian Property.

## 6.1 Contagious Coasian Property

The seller punishing equilibrium is an equilibrium of the game in which seller types hold their original beliefs, while buyer types hold *optimistic beliefs* and put probability one on the lowest seller type in the support of his beliefs $S_b$. More precisely, buyer types put probability one on the lowest seller type in the support of their prior beliefs, i.e.

$$\mu_b^n(s_b^\alpha) = 1 \tag{17}$$

[43]The buyer-punishing equilibrium is analyzed analogously.

for all histories $h^n$ with some seller-detectable deviation.[44] Beliefs described in (17) are a natural counterpart of optimistic beliefs commonly used in the bargaining literature (see Rubinstein (1985)). In the punishing equilibrium, all buyer types pool on the lowest price offer possible in the game and accept seller offers according to endogenously-determined *willingness to pay*. Seller types make price offers to screen buyer types. Since offers do not affect the beliefs of the buyer, each seller type chooses her offers optimally given the buyer's willingness to pay. However, a buyer's willingness to pay depends on the buyer's expectations about the seller's future offers and so, both willingness to pay and the seller screening policy are determined in equilibrium simultaneously. The main result is the following property of the seller punishing equilibrium.

**Theorem 6** (Contagious Coasian Property). *For any $\varepsilon > 0$, there exists $\bar{\Delta}$ such that for all $\Delta \leq \bar{\Delta}$, the continuation utility of any seller type $s$ in the seller punishing equilibrium is at most $\max\left\{\frac{v(0)+c(0)}{2} - c(s), 0\right\} + \varepsilon$ for all $s$.*

For $\eta = 1$, all types of the buyer put probability one on the seller type 0 and Theorem 6 states the Coasian property of the punishing equilibrium. As $\Delta \to 0$, seller type 0 loses all monopoly power and allocates to all buyer types at the lowest price. Surprisingly, even for small $\eta$, in the punishing equilibrium the seller gets her reservation utility in the frequent-offer limit.[45] Even though the buyer types become only marginally optimistic, the coordination of all buyer types on the optimistic beliefs creates the connection between the screening policies of different seller types. Low screening offers of seller type 0 force seller types slightly above 0 to make low price offers, as the big fraction of the buyer types that they face belongs to $[0, \eta]$ and expects almost immediate allocation at the price close to $\frac{v(0)+c(0)}{2}$ from seller type 0. This leads buyer types slightly above $\eta$ to expect a low price offer and, in turn, forces a larger set of seller types to make price offers close to $\frac{v(0)+c(0)}{2}$. This way even the seller types that are significantly far from seller type 0 are forced to make low price offers.

The intuitive contagion mechanism described above is more delicate than it might seem at first glance. As offers become more frequent, the seller screens more thoroughly in the sense that cut-offs of the seller screening strategy become closer together. Hence, seller types slightly above seller type 0 spend an increasing number of rounds selling to buyer types above $\eta$. If such time is positive in the limit, then it is possible that the limit of willingness to pay of the

[44] Such beliefs could be justified by the following trembles in the model with a finite number of types and finite grid of price offers. Seller's and buyer's types come from $\{k/K\}_{k=1}^{K}$ for some integer $K$. Suppose price offers come from a discrete set $\mathtt{P}$. Seller type $s$ trembles with probability $(1-s)^m/2$ for some integer $m$ and conditional on trembling chooses a price offer uniformly from $\mathtt{P}$. As $m \to \infty$, the probability of tremble converges to zero. Yet, conditional on the buyer type $b$, the probability that the tremble comes from seller type $s_b^\alpha$ is $\frac{(1-s_b^\alpha)^n}{(1-s_b^\alpha)^m + \sum_{s \in S_b \setminus s_b^\alpha}(1-s)^m} \to 1$ as $m \to \infty$, since $\frac{1-s}{1-s_b^\alpha} < 1$.

[45] Observe that unlike on the CSE equilibrium path, in the punishing equilibrium all types on the punishing side benefit from coordinating on optimistic beliefs. Every type of punishing player (subjectively) gets the highest possible utility in the frequent-offer limit.

buyer types would be higher than $\frac{v(0)+c(0)}{2}$. In fact, this happens for the buyer types that put probability one on the seller types with costs above $\frac{v(0)+c(0)}{2}$. However, as Theorem 6 shows that even though the limit of willingness to pay increases for such buyer types, it does not go above $c(s_b^\alpha)$. Given that the buyer's willingness to pay is at its lowest possible level in the limit of frequent offers, the seller's utility approaches the reservation utility as $\Delta \to 0$.

## 6.2 Description of strategies

Since the optimistic beliefs of the buyer might exclude the realized seller's type, the buyer and the seller may have different expectations regarding the path of play.[46] I refer to the path of play expected by the seller in the punishing equilibrium as the *equilibrium path of the punishing equilibrium.*

**Buyer on-path strategy.** All buyer types pool on the lowest acceptable price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ (cf. Lemma 1). Buyer type $b$ accepts any price offer less than or equal to his *willingness to pay* $P(b)$ which is left-continuous and strictly increasing in $b$. Since $P(b)$ is strictly increasing, for any history $h^n$ without buyer deviations, there exists a buyer type $\beta \in [0,1]$ such that only buyer types in the interval $[0,\beta]$ remain in the game. Whenever $\beta \geq b_s^\alpha$, posterior beliefs of seller type $s$ are uniform on $B_s \cap [0,\beta]$.

**Seller on-path strategy.** The seller faces the static demand function given by $P(b)$ and makes price offers to screen buyer types based on their willingness to pay. Notice that it is never optimal for the seller to offer a price in $[P(b), P(b+))$, if $b$ is point of discontinuity of $P(b)$.[47] Let $\hat{P}(b)$ be a right-continuous function that is equal to $P(b)$ in all continuity points of $P(b)$. Then the strategy of the seller could be equivalently represented as follows. Given the highest remaining buyer type $\beta$, seller type $s > 0$ chooses a cut-off buyer type $t_\beta(s)$ and allocates to all remaining buyer types above $t_\beta(s)$. To reach this goal, the seller should make offer $\hat{P}(t_\beta(s))$.[48]

The strategy of seller type 0 differs from the rest of the seller types, due to the fact that a positive mass of buyer types in $[0,\eta]$ puts probability one on seller type 0. Seller type 0 (and only this seller type) accepts buyer price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$, whenever the highest buyer

[46]For example, suppose that buyer type $b$ and seller type $s \in S_b \backslash s_b^\alpha$ are realized. In the punishing equilibrium, beliefs of buyer type $b$ assign probability one to type $s_b^\alpha$. If the punishing equilibrium strategies prescribe different actions for seller types $s$ and $s_b^\alpha$, then buyer type $b$ will observe seller's deviations from the expected path of play. In turn, the seller takes into account the fact that the buyer could perceive her action as a deviation from the equilibrium strategy.

[47]Indeed, alternatively the seller could offer price $P(b+)$ and still sell the good to all buyer types above $b$, but at a higher price.

[48]It might be tempting to define $P(b)$ as a right-continuous function and this way avoid the necessity to introduce auxiliary function $\hat{P}(b)$. This, however, is not possible. To see this, suppose that every seller type does not screen and allocates to buyer type $b_s^\alpha$ in the first round. Then

$$P(b) = \begin{cases} (1-e^{-r\Delta})v(b) + e^{-r\Delta}\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}, & \text{for } b \in [0,\eta], \\ (1-e^{-2r\Delta})v(b) + e^{-2r\Delta}P(\max\{b-2\eta,0\}), & \text{for } b \in (\eta,1]. \end{cases}$$

It is easy to see that such a function is not right-continuous (e.g. at point $b = \eta$).

type remaining in the game is below some $\bar{\beta} \in (0, \eta]$. Given the highest remaining buyer type $\beta \in (\bar{\beta}, \eta]$, seller type 0 allocates to buyer types above $t_\beta(0)$.

Before moving on to the description of strategies off-path, I state the optimality conditions that on-path strategies of the punishing equilibrium should satisfy. The problem of seller type $s$ could be formulated recursively. Let bounded function $R_\beta(s)$ for $\beta \in [b_s^\alpha, 1]$ be the profit of seller type $s > 1$ from selling to buyer types in $[b_s^\alpha, \beta]$. Then $R_\beta(s)$ satisfies Bellman equation[49]

$$R_\beta(s) = \sup_{b \in B_s \cap [0,\beta]} \left\{ (\beta - b)(\hat{P}(b) - c(s)) + e^{-2r\Delta} R_b(s) \right\}. \tag{18}$$

Denote by $T_\beta(s)$ the set of maximizers of the right-hand side of (18). A seller strategy $t_\beta(s)$ is a *best-reply* to buyer strategy $P(b)$, if $t_\beta(s) = \inf T_\beta(s)$ for all $s$ and $\beta \geq b_s^\alpha$. A special role in the analysis is played by the first cut-off buyer type chosen by seller type $s$, which I denote by $t(s) \equiv t_{b_s^\omega}(s)$.

For screening strategy $t_\beta(s)$ of the seller, the willingness to pay $P(b)$ for $b \in (\eta, 1]$ is given by

$$P(b) = (1 - e^{-2r\Delta})v(b) + e^{-2r\Delta} \hat{P}(t(s_b^\alpha)) \tag{19}$$

The interpretation of (19) is as follows. The expectation of buyer type $b$ about future screening offers of the seller is determined by the screening policy of seller type $s_b^\alpha$. Buyer type $b$ in the interval $(\eta, 1]$ believes that he is the highest buyer type in the support of beliefs of seller type $s_b^\alpha$. If the seller makes price offer $P(b)$, then in the next screening round, buyer type $b$ will be the highest buyer type remaining in the game. Hence, buyer type $b$ will expect to buy the good in the next round at price $\hat{P}(t(s_b^\alpha))$. Equation (19) states that buyer type $b$ is simply indifferent between accepting price offer $P(b)$ and getting utility $b - P(b)$, and rejecting $P(b)$ and accepting price offer $P(t(s_b^\alpha))$ in the following round of screening.

As with seller type 0, the willingness to pay of buyer types in the interval $[0, \eta]$ differs from that of the rest of the buyer types. Both on and off the equilibrium path of the punishing equilibrium, it is determined by some strictly increasing and left-continuous function $P^0(b)$.

**Strategies off-path.** If the buyer makes a price offer different from $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ or $\beta < b_s^\alpha$, seller type $s$ switches to optimistic beliefs and assigns probability one to the highest buyer type in the support of her prior belief, i.e.

$$\mu_s^n(b_s^\omega) = 1 \tag{20}$$

for all histories $h^n$ with both seller- and buyer-detectable deviations. Lemma 13 in the Appendix describes equilibrium strategies when both players have optimistic beliefs. This result is based on the analysis of a bargaining game with complete information (Rubinstein (1982)).

Seller deviations from the equilibrium strategies in the punishing equilibrium are ignored. If

[49] The value function is defined only on the set $\{(\beta, s) \in BS : b_s^\alpha \leq \beta\}$. Outside of this set, seller $s$ detects that state $\beta$ is achieved as a result of buyer deviation and switches to the optimistic belief (20) as specified below.

buyer type $b$ rejects a seller price offer lower than $P(b)$, then the seller detects such deviation only if $b > \beta + 2\eta$. In this case, the continuation play is as in Lemma 13. If $\beta < b \leq \beta + 2\eta$, then such deviation is not detected and buyer type $b$ makes price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$, and accepts any price offer less than $P_\beta(b) \equiv (1 - e^{-2r\Delta})v(b) + e^{-2r\Delta}P(t_\beta(s_b^\alpha))$, which now depends also on the highest remaining buyer type $\beta$. This completes the description of the strategies in the punishing equilibrium.

## 6.3 Existence

In this subsection, I show the existence of the punishing equilibrium. The proof of the existence is constructive, and the key to its construction is to show that willingness to pay $P(b)$ and screening policy $t_\beta(s)$ satisfying (18) and (19) exist. The next theorem presents the result.

**Theorem 7.** *For all sufficiently small $\Delta$, there exist $t_\beta(s)$ and $P(b)$ such that they satisfy* (19) *and $t_\beta(s)$ is the solution of the optimization problem in equation* (18)*.*

I now sketch the main steps for the construction of the punishing equilibrium. The construction is carried out starting from the bottom of the type distribution. I first analyze strategies of seller type 0 and buyer types in $[0, \eta]$ that put probability one on this seller type. This is the model with one-sided incomplete information and alternating offers, and the following result is standard in the literature.[50]

**Lemma 3.** *For all sufficiently small $\Delta$, there exists a sequential equilibrium in a game between seller type 0 and buyer types in $[0, \eta]$, in which on the equilibrium path*

1. *the buyer makes price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ and accepts seller price offers according to left-continuous and strictly increasing willingness to pay function $P^0(b)$ ;*

2. *there exists $\bar{\beta} \in [0, \eta]$ such that if the highest remaining buyer type is below $\bar{\beta}$, then seller type 0 accepts the buyer price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$;*

3. *given the highest remaining buyer types $\beta \in (\bar{\beta}, \eta]$, seller type 0 allocates to buyer types above $t_\beta(0)$ in the current round.*

*Moreover, for any $\varepsilon > 0$ the first price offer of seller type 0 does not exceed $\frac{v(0)+c(0)}{2} + \varepsilon$ for $\Delta$ sufficiently small.*

One detail worth mentioning is that despite the fact that in the punishing equilibrium, seller type 0 follows a pure strategy on the equilibrium path, off-the-equilibrium path mixing might be necessary (see footnote 18). This possibility can easily be incorporated into the analysis. For notation simplicity, I will assume that the seller screening strategy in Lemma 3 is pure.

[50] See Grossman and Perry (1986), Gul and Sonnenschein (1988).

Strategies for the rest of the types are constructed via the iterative algorithm that runs as follows. Buyer types in $[0, \eta]$ put probability one on seller type 0 and have willingness to pay $P^0(b)$. By Lemma 18 in the Appendix, all seller types allocate to at least a mass $c(\eta, \Delta)$ of buyer types in the first round of screening where $c(\eta, \Delta)$ is a constant as specified in Lemma 18. Hence, it is sufficient to know the willingness to pay of buyer types in $[0, \eta]$ to construct the screening policy of seller types in $[0, c(\eta, \Delta)]$. Moreover, buyer types in $[\eta, \eta + c(\eta, \Delta)]$ put probability one on sellers in the interval $[0, c(\eta, \Delta)]$. In Step 1 of the algorithm, screening policy $t_\beta(s)$ for seller types in $[0, c(\eta, \Delta)]$ and willingness to pay $P(b)$ for buyer types $[\eta, \eta + c(\eta, \Delta)]$ is constructed. The algorithm continues "climbing up" the types with an increment $c(\eta, \Delta)$. Choose $I$ as the smallest integer such that $Ic(\eta, \Delta) \geq 1 - \eta$.

**Iterative Algorithm**

*Input:* Define $\pi^0(b) = \begin{cases} P^0(b), \text{ for } b \in [0, \eta], \\ v(b), \text{ for } b \in (\eta, 1]. \end{cases}$

*Execute Step $i, i = 1, \ldots, I+1$.*

*Step $i$.* Construct a best-reply $\tau^i_\beta(s)$ to $\pi^{i-1}(b)$. Construct $\pi^i(b)$ by

$$\pi^i(b) = \begin{cases} \pi^{i-1}(b), & \text{for } b \in [0, \eta + (i-1)c(\eta, \Delta)], \\ (1 - e^{-2r\Delta})v(b) + e^{-2r\Delta}\hat{\pi}^{i-1}(\tau^i(s^\alpha_b)), & \text{for } b \in (\eta + (i-1)c(\eta, \Delta), \eta + ic(\eta, \Delta)], \\ v(b), & \text{for } b \in (\eta + ic(\eta, \Delta), 1]; \end{cases}$$

where $\hat{\pi}^{i-1}(b)$ denotes the right-continuous function that coincides with $\pi^{i-1}(b)$ at all continuity points of $\pi^{i-1}(b)$.

*Output:* $P(b) = \pi^{I+1}(b)$, $t_\beta(s) = \tau^{I+1}_\beta(s)$.

By construction, $t_\beta(s)$ is a best-reply to $P(b)$, and one is left to verify that $P(b)$ is the optimal acceptance strategy for the buyer and that it is optimal for buyer types to pool on $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$. The former is proven in Lemma 20 in the Appendix, and the argument uses the monotonicity in $s$ of seller screening strategy $t_\beta(s)$ and the monotonicity of $P(b)$. The proof of the latter is based on the Contagious Coasian Property proven in the next subsection.

## 6.4 Proof Sketch of Theorem 6

This subsection proves the Contagious Coasian Property. The frequent-offer limit of the punishing equilibria gives all seller types their reservation utility level independent of $\eta$. At the same time, all optimistic buyer types expect to acquire the good in the first round of the seller's screening at the price that converges to the lowest (type specific) price. The former property allows us to support a wide range of equilibrium behavior in CSEs and segmentation equilibria.

The latter property gives the final step in the proof of the existence of punishing equilibria, as it deters deviations of the buyer from pooling on the price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ in the punishing equilibrium. Therefore, the seller-punishing equilibrium is a natural candidate for deterring deviations from the equilibrium path: it simultaneously punishes all types of seller as harshly as possible, and rewards all types of buyer by the greatest amount possible.

Theorem 6 is a corollary of the following result describing the limit of willingness to pay functions as $\Delta \to 0$. Consider a sequence of punishing equilibria indexed by $\Delta \to 0$. For each $\Delta$, let $(P^{\Delta}(b), t^{\Delta}_{\beta}(s))$ be equilibrium path strategies of the punishing equilibrium for round length $\Delta$. Then the limit of $P^{\Delta}(b)$ is given by the following theorem.

**Theorem 8.** *There exists a subsequence of the sequence $P^{\Delta}(b)$ that converges uniformly to $P^*(b) = \max\left\{\frac{v(0)+c(0)}{2}, c(s^{\alpha}_{b})\right\}$ as $\Delta \to 0$.*

The proof of Theorem 8 is broken down into three steps. In each step the limit of the willingness to pay function $P^*(b)$ for a separate category of buyer types is analyzed. Let $s^+$ be the seller type for whom $c(s^+) = \frac{v(0)+c(0)}{2}$ holds. In the first step it is shown that for buyer types in $[0, \eta]$ and seller type 0 equilibrium behavior exhibits Coasian dynamics. Namely, as $\Delta \to 0$, the first offer of seller type 0 is close to the buyer's demand $\frac{c(0)+v(0)}{2}$. In the second step buyer types in $(\eta, b^{\omega}_{s^+}]$ and seller types in $(0, s^+]$ are analyzed, and the last step covers the remaining types. The difference between these two cases is that seller types below $s^+$ have positive expected profit from the lowest buyer type in the support of their beliefs when they face limit willingness to pay function $P^*(b)$, while for seller types above $s^+$, such profit is zero.

**Step 1.** The final statement in Lemma 3 implies that $P^*(b) = \frac{v(0)+c(0)}{2}$ for $b \in [0, \eta]$.

**Step 2.** The next lemma shows that if the limit function $P^*(b)$ is increasing at $\hat{b}$, then it is equal to the reservation price of the seller type $s^{\alpha}_{\hat{b}}$.[51]

**Lemma 4.** *Suppose that for some $\hat{b} \in (0, 1)$, $P^*(\hat{b}) > c(s^{\alpha}_{\hat{b}})$. Then there exists $\phi > 0$ such that $P^*(b)$ is constant for all $b \in (\hat{b} - \phi, \hat{b} + \phi)$.*

Lemma 4 means that function $P^*(b)$ could be increasing at $b$ only if buyer type $b$ expects the seller to make the first offer close to $c(s^{\alpha}_{b})$. In other words, function $P^*(b)$ could have jumps only at points where $P^*(b) = c(s^{\alpha}_{b})$. This implies that $P^*(b) = \frac{v(0)+c(0)}{2}$ for $b < b^{\omega}_{s^+}$.

**Step 3.** It is more intricate to find the limit of the screening policy for seller types above $s^+$ for the following reason. For seller type $s < s^+$, it can be shown that a positive mass of buyer types in $B_s$ has willingness to pay close to $\frac{v(0)+c(0)}{2}$ and so the profit from allocating to all remaining buyer types at a price close to $\frac{v(0)+c(0)}{2}$ is positive. Suppose seller type $s$ delays trade at price $\frac{v(0)+c(0)}{2}$ to sell at a price exceeding $\frac{v(0)+c(0)}{2}$. Such a delay should be sufficiently large to guarantee that buyer type $b^{\omega}_{s}$ buys in the first round of screening. Then for any $\varepsilon > 0$, it is possible to construct an alternative screening policy that accelerates the trade at prices above

[51] An increasing function $f(x)$ is *increasing at point $x$* if for all $\phi > 0$, $f(x + \phi) > f(x - \phi)$.

$\frac{v(0)+c(0)}{2}$ and loses at most $\varepsilon$ on such trades. The advantage of such a policy is that it allows the seller to allocate to all buyer types with the willingness to pay $\frac{v(0)+c(0)}{2}$ sooner. Since the profit from such buyer types is strictly positive, for sufficiently small $\varepsilon$ such an alternative screening policy is preferred by the seller, giving a contradiction.

The reasoning above is not valid for seller types above $s^+$. These seller types eventually decrease their screening offers to a level close to their costs. Hence, they could have incentives to spend a significant amount of time screening buyer types that will bring them positive profit. The next lemma is key for establishing that the time seller types above $s^+$ screen buyer types is enough to keep $P^*(b)$ just above $c(s_b^\alpha)$.

**Lemma 5.** *The function $P^*(b)$ is continuous.*

Together with Lemma 4, Lemma 5 implies that $P^*(b) = c(s_b^\alpha)$ for $b \geq b_{s^+}^\omega$ and completes the proof of Theorem 8.

# 7 Conclusion

This paper studies implications of the correlation of values for trade dynamics and efficiency in a standard bargaining model with alternating offers and an infinite horizon. The analysis of the CSEs shows that models with independent values overlook an important source of inefficiency due to common screening. This new source of inefficiency becomes predominant in environments where there is a big gap between private and public information. Many secondary markets are known for their opaqueness and, at the same time, sophistication of their participants. This paper shows that for a wide range of individual uncertainty levels, bargaining in such markets may feature two-sided screening dynamics which is a realistic description of the actual negotiation process. I also show that there are equilibria in which common screening inefficiency is eliminated shortly after bargaining begins. In segmentation equilibria, types self-select into endogenous segments by their initial price offers. For small individual uncertainty, most of the types in such equilibria trade immediately after the first offers, and only a small mass of types located at the boundaries of segments continues bargaining. I characterize CSE limits in terms of equilibria in smooth, monotone strategies of the concession game (for smooth limits), and incentive compatibility and individual rationality constraints (double limits). Both of them are intuitive and analytically tractable.

There are two assumptions that are important to the analysis in this paper. First is the distribution of types and, specifically, the bounded support of beliefs. One might wonder in particular what would happen if the buyer's and seller's types are distributed with a positive, continuous density on the unit square with most of the density concentrated on the diagonal stripe. The crucial difference lies in the conjectures supporting the equilibrium path. When beliefs of players have full support, the optimistic beliefs of the buyer put probability one on the lowest seller type ($s = 0$) after the seller's deviation. Then the initial correlation of values is virtually erased off-the-equilibrium path. Even if beliefs of each buyer type are initially concentrated on a very narrow range of seller types, after the deviation, because of the full-support assumption, each buyer type puts probability one on the seller type that he initially considered very unlikely. This is an undesirable feature of the model that studies the implications of the correlation of values, and one might try to refine off-path beliefs by requiring that they preserve the information about the correlation of types. The model in this paper is essentially doing this by restricting the support of beliefs, and hence, preserving the correlation of types even off-path.

In terms of the analysis, the construction of the punishing path is greatly simplified by the full-support assumption, and it boils down to the Coase conjecture. Therefore, the CSE equilibria can be constructed even without the bounded-support assumption. However, the construction of the segmentation equilibria does not go through in the case of full support of beliefs. An interesting topic for future research is whether under the full-support assumption it is possible to construct equilibria that feature the separation by price offers and are efficient in the limit of vanishing individual uncertainty.

The second important assumption of the model is that I allow for optimistic beliefs after deviations. To motivate the restriction to particular equilibrium classes, Cramton (1984) and Cho (1990) use certain monotonicity conditions to restrict the out-of-the-equilibrium-path beliefs and in particular, to rule out optimistic beliefs. Their conditions seem less compelling in the environment with heterogeneous beliefs. With heterogeneous beliefs, describing conjectures that satisfy the monotonicity condition is a daunting task, as one must carefully specify beliefs for every type of the punishing player. The task is much simpler with independent values, as all types of player have the same beliefs about the opponent's type. From this point, optimistic conjectures are more intuitive than the conjectures satisfying certain monotonicity constraints.

Apart from the simplicity of optimistic beliefs, it is a priori unclear whether they will have the same bite when the support of beliefs is restricted. In particular, simulations in the Online Appendix reveal that for a fixed length of the bargaining rounds $\Delta$, decreasing $\eta$ leads to a higher utility of seller types in the punishing equilibria exceeding the reservation utility. I show an interesting robustness property of the optimistic beliefs. Optimistic beliefs are known to be capable of supporting a great variety of outcomes when the support of beliefs is big (Rubinstein (1985)). In this paper, even when optimism is restricted in such a way that players update their beliefs only marginally, optimistic beliefs are still efficient in supporting equilibrium paths due to the Contagious Coasian Property of the punishing equilibria. The techniques developed in this paper may be useful in the analysis of other dynamic models with correlated types. In particular, the analysis of punishing equilibria could be extended to a model with interdependent values.[52]

[52]By an interdependent-values environment I mean that the values of players are determined by an unobserved fundamental, and players receive signals about the fundamental.

Extending the equilibrium analysis in this paper to an interdependent-values environment is an exciting topic for future research.

This paper provides a useful benchmark for future research by suggesting that to get sharper predictions additional restrictions are required. A natural development of the model is to explore the predictions of the model in the presence of outside options, as in Fuchs and Skrzypacz (2007), or to endogenize the length of bargaining rounds and use an intuitive criterion style refinement, as in Admati and Perry (1987) and Cramton (1992). Finally, similarly to Tsoy (2014) one can apply the characterization of double limits of CSEs to study implications of bargaining delay for efficiency and dynamics of labor markets and macroeconomic models.

# 8 Appendix

In the Appendix, I present the proofs in the order of their appearance in the paper. I use the following additional notation throughout the Appendix. Let $\Sigma \equiv \max_{(s,b)\in SB}\{v(b) - c(s)\}$ be maximal gains from trade possible in the game. I denote $\delta \equiv e^{-r\Delta}$ to be the players' discount factor and the frequent-offer limit ($\Delta \to 0$) corresponds to the limit of patient players $\delta \to 1$. For any real $a$ and $b$, denote $a \vee b \equiv \max\{a, b\}$ and $a \wedge b \equiv \min\{a, b\}$. To unify the notation, whenever I refer to a sequence of equilibria, I reserve index $j$ to indicate magnitudes arising in the $j$'s equilibrium in the sequence. In particular, I use superscript $j$ to denote functions in the $j$'s equilibrium, and subscript $j$ to denote variables that I introduce in the analysis of the $j$'s equilibrium.[53]

## 8.1 Proofs for Section 3

### 8.1.1 Concession Game

**Lemma 6.** *If $s^*_t$ is a monotone seller strategy, then $u^B(t, b)$ on $TB \equiv \{(t, b) : b \in [0, 1], t \in [0, t^*_S(s^\omega_b)]\}$ satisfies the smooth strict single-crossing difference property in $(-t, b)$.*[54,55]

*Proof of Lemma 6.* Suppose that the acceptance strategy of the seller $t^*_S(s)$ (or alternatively $s^*_t$) is monotone. Consider buyer types $b < b'$, times $t < t' \le t^*_S(s^\omega_b) \le t^*_S(s^\omega_{b'})$, and suppose a type $b$ buyer prefers to accept at time $t$ rather than time $t'$. If $s^*_{t'} < s^\alpha_b$, then the probability that the buyer's offer is accepted before time $t'$ is zero for both $b$ and $b'$ and so buyer type $b'$ strictly prefers to accept at time $t$ by the single-crossing property of $e^{-rt}(v(b) - q^S_t)$. Suppose that $s^*_{t'} \ge s^\alpha_b$. Let $\varphi(b) = \frac{s^\omega_b - s^*_{t'} \wedge s^\omega_b}{s^\omega_b - s^*_t \vee s^\alpha_b}$ be the probability that a type $b$ buyer assigns to the event that the seller does not accept the buyer's offer before time $t'$ conditional on the fact that she has not accepted by time $t$. Notice that $\varphi(b) < \varphi(b')$. The following two claims prove the strict single-crossing property of $u^B(t, b)$.

*Claim* 1. *Suppose*

$$v(b) - q^S_t \ge (1 - \varphi(b)) \int_{s^*_t \vee s^\alpha_b}^{s^*_{t'} \wedge s^\omega_b} e^{-r\left(t^*_S(s) - t\right)} \left(v(b) - q^B_{t^*_S(s)}\right) \frac{ds}{s^*_{t'} \wedge s^\omega_b - s^*_t \vee s^\alpha_b} + \varphi(b) e^{-r(t'-t)}(v(b) - q^S_{t'}). \tag{21}$$

[53] For example, in the analysis of punishing equilibria, I consider a sequence of seller punishing equilibira as $\delta_j \to 1$. In such a sequence, $(P^j(b), t^j_\beta(s))$ denote on-path equilibrium strategies in $j$'s punishing equilibrium, and in the proof of the Contagious Coasian Property, I introduce types $b_j$,$\beta_j$,$s_j$,$\sigma_j$ and quantities $K_j, L_j, x_{Kj}, x_{Lj}$.

[54] I only provide statements for the buyer, and symmetric statements are true for the seller.

[55] Observe that for any $b$, $u^B(t, b)$ is constant for $t > t^*_S(s^\omega_b)$, as buyer type $b$ expects that the seller accepts by time $t^*_S(s^\omega_b)$ with probability one. Hence, the restriction to set $TB$ is necessary to guarantee strict inequalities in the definition of the strict single crossing property.

*Then*

$$v(b)-q_t^S > (1-\varphi(b'))\int_{s_t^*\vee s_{b'}^\alpha}^{s_{t'}^*\wedge s_{b'}^\omega} e^{-r\left(t_S^*(s)-t\right)}\left(v(b)-q^B_{t_S^*(s)}\right)\frac{ds}{s_{t'}^*\wedge s_{b'}^\omega - s_t^*\vee s_{b'}^\alpha}+\varphi(b')e^{-r(t'-t)}(v(b)-q_{t'}^S). \tag{22}$$

*Proof.* Choose $\tilde{s}$ so that $\frac{\tilde{s}-s_t^*\vee s_b^\alpha}{s_b^\omega - s_t^*\vee s_b^\alpha} = \frac{s_{t'}^*\vee s_{b'}^\alpha - s_t^*\vee s_{b'}^\alpha}{s_{b'}^\omega - s_t^*\vee s_{b'}^\alpha}$. Then I have the following sequence of inequalities,

$$(1-\varphi(b))\int_{s_t^*\vee s_b^\alpha}^{s_{t'}^*\wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)}\left(v(b)-q^B_{t_S^*(s)}\right)\frac{ds}{s_{t'}^*\wedge s_b^\omega - s_t^*\vee s_b^\alpha}+\varphi(b)e^{-r(t'-t)}\left(v(b)-q_{t'}^S\right) \geq$$

$$(1-\varphi(b'))\int_{s_t^*\vee s_b^\alpha}^{\tilde{s}} e^{-r\left(t_S^*(s)-t\right)}\left(v(b)-q^B_{t_S^*(s)}\right)\frac{ds}{s_{t'}^*\wedge s_{b'}^\omega - s_t^*\vee s_{b'}^\alpha}+\varphi(b')e^{-r(t'-t)}\left(v(b)-q_{t'}^S\right) \geq$$

$$(1-\varphi(b'))\int_{s_t^*\vee s_{b'}^\alpha}^{s_{t'}^*\wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)}\left(v(b)-q^B_{t_S^*(s)}\right)\frac{ds}{s_{t'}^*\wedge s_{b'}^\omega - s_t^*\vee s_{b'}^\alpha}+\varphi(b')e^{-r(t'-t)}\left(v(b)-q_{t'}^S\right).$$

The first inequality follows from the $\varphi(b') > \varphi(b)$ and $q^B_{t_S^*(s)} \leq q_{t'}^B \leq q_{t'}^S$ for all $t_S^*(s) \leq t'$. To get the second inequality, observe that by monotonicity of $q_t^B$ and $t_S^*(s)$, for all $s \leq s'$, $t_S^*(s) \leq t_S^*(s')$ and $q^B_{t_S^*(s)} \leq q^B_{t_S^*(s')}$ and so, function $e^{-rt_S^*(s)}\left(v(b)-q^B_{t_S^*(s)}\right)$ is decreasing in $s$. Moreover, since $s_t^*\vee s_b^\alpha < s_t^*\vee s_{b'}^\alpha$ and $\tilde{s} < s_{t'}^*\vee s_{b'}^\alpha$, the uniform distribution on $[s_t^*\vee s_{b'}^\alpha, s_{t'}^*\vee s_{b'}^\alpha]$ first-order stochastically dominates the uniform distribution on $[s_t^*\vee s_b^\alpha, \tilde{s}]$, and the inequality follows from the definition of the first-order stochastic dominance. *Q.E.D.*

Since $\varphi(b') > 0$ and $v(b)$ is strictly increasing, by substituting $b'$ instead of $b$ in (22), strict inequality results and so type $b'$ strictly prefers to accept at time $t$. By an analogous argument, I can make the following claim, which completes the proof of Lemma 6.

*Claim* 2. *Suppose*

$$v(b')-q_t^S \leq (1-\varphi(b'))\int_{s_t^*\vee s_{b'}^\alpha}^{s_{t'}^*\wedge s_{b'}^\omega} e^{-r\left(t_S^*(s)-t\right)}\left(v(b')-q^B_{t_S^*(s)}\right)\frac{ds}{s_{t'}^*\wedge s_{b'}^\omega - s_t^*\vee s_{b'}^\alpha}+\varphi(b')e^{-r(t'-t)}(v(b')-q_{t'}^S). \tag{23}$$

*Then*

$$v(b)-q_t^S < (1-\varphi(b)) \int_{s_t^* \vee s_b^\alpha}^{s_{t'}^* \wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q^B_{t_S^*(s)}\right) \frac{ds}{s_{t'}^* \wedge s_b^\omega - s_t^* \vee s_b^\alpha} + \varphi(b) e^{-r(t'-t)}(v(b)-q_{t'}^S). \tag{24}$$

*Claim* 3. $\frac{d}{dt}u^B(t,b)$ is decreasing in $b$ on $TB$.

*Proof.* Differentiating $u^B(t,b)$ in $t$, results in the sign of the derivative being the same as the sign of

$$(q_t^S - q_t^B)\frac{\dot{s}_t^*}{s_b^\omega - s_t^*}\mathbf{1}\left\{s_b^\alpha \leq s_t^*\right\} - r\left(v(b) - q_t^S\right) + \dot{q}_t^S.$$

The expression is decreasing in $b$ on $TB$, which verifies the condition. □

**Lemma 7.** *If $b_t^*$ is a smooth monotone strategy in the concession game that satisfies* (3) *for some smooth monotone strategy of the seller $s_t^*$, then $b_t^*$ is a best-reply to $s_t^*$.*

*Proof of Lemma 7.* In the proof I verify conditions of Theorem 4.2 in Milgrom (2004). Lemma 6 verifies that $u^B(t,b)$ satisfies the smooth strict single-crossing difference condition. Function $u^B(t,b)$ is continuously differentiable in $t$ for fixed $b$ by the definition of $u^B(t,b)$ and the fact that $s_t^*$ is a smooth monotone strategy, and $q_t^B$ and $q_t^S$ are continuously differentiable. The envelope condition $u^B(t_1^*,1) - u^B(t_b^*,b) = \int_b^1 \frac{d}{db}u^B(t_b^*,b)db$ follows from condition (3). □

*Proof of Theorem 1.* The discussion following Theorem 1 shows that conditions (3) and (4) are necessary conditions of equilibria in smooth monotone strategies. Observe that by (1), eventually all gains from trade are realized and so, $b_T^* = b^\alpha_{s_t^*}$. Moreover, if $T < \infty$ and $q_T^B < q_T^S$, then the type of player that accepts at time $T$ can delay acceptance by a short period of time and still have the opponent accept his/her offer. This will give him/her strict gain in the utility due to $q_T^S > q_T^B$ which is a contradiction. Therefore, condition (2) is necessary.

Conversely, suppose $b_t^*$ and $s_t^*$ are given by equations (3) and (4), and the boundary condition (2). Then $b_t^*$ and $s_t^*$ specify acceptance strategies for all types and, by Lemma 7, they are mutual best replies and so constitute the equilibrium of the concession game.

□

### 8.1.2 Preliminary Results about CSEs

The following two examples clarify the difference between conditions 2 and 3 in Definition 4. In both examples suppose that $s_n^\Delta = 0$ for all $n \in \mathbb{N}$ and $\Delta > 0$ and so, $s_t^* = 0$ for all $t \geq 0$. In the first example, suppose that for some $T' > 0$, $b_n^\Delta = \frac{T'-n\Delta}{T'} + \frac{n\Delta}{T'}\Delta$, for $n \in \mathbb{N} \cap \left[0, \frac{T'}{\Delta}\right)$, and $b_n^\Delta = \Delta e^{-(n\Delta - T')}$, for $n \in \mathbb{N} \cap \left[\frac{T'}{\Delta}, \infty\right)$. Then $T_\Delta = \infty$, but for all $t \geq T'$, $b_t^\Delta \to 0$ as $\Delta \to 0$ and so, $T = T'$. Hence, condition 2 is not satisfied, however, $b^\Delta_{T_\Delta} = b_T^* = 0$ and condition 3 holds. In the second example, suppose that $b_n^\Delta = \frac{1}{2}\left(1 + e^{-n\Delta}\right)$, for $n \in \mathbb{N} \cap [0, \Delta^{-2})$, and

$b_n^\Delta = \frac{1}{2}\left(1+e^{-\Delta^{-1}}\right)e^{-n\Delta+\Delta^{-2}}$, for $n \in \mathbb{N} \cap [\Delta^{-2}, \infty)$. Then $b_{T_\Delta}^\Delta = 0$, but for all $t \geq 0$, $b_t^\Delta \to \frac{1}{2}\left(1+e^{-t}\right)$ as $\Delta \to 0$ and so $b_T^* = \frac{1}{2}$. Hence, condition 3 is not satisfied, but $T_\Delta = T = \infty$ and condition 2 holds.

In a CSE, denote by $U_n^B(b)$ and $U_n^S(s)$ expected continuation utilities in round $n$ of a type $b$ buyer and a type $s$ seller, respectively, and by $U_t^B(b)$ and $U_t^S(s)$ their extensions to a continuous domain. For CSE strategies $b_n$ and $s_n$, denote by $n_b \equiv \inf\{n : b_n \leq b\}$ and $n_s \equiv \inf\{n : s_n \geq s\}$ rounds of acceptance of a type $b$ buyer and a type $s$ seller, respectively. The following lemma is the counterpart of Lemma 6 for the bargaining game, and its proof replicates the proof of Lemma 6.

**Lemma 8.** *Suppose $p_n^B$ and $p_n^S$ are price paths as in the definition of the CSE. If $s_n$ satisfies the skimming property, then $U_n^B(b)$ on $NB = \left\{(n,b) : b \in [0,1], n = \overline{1, n_{s_b^\omega}}\right\}$ satisfies the strict single crossing property in $(n,b)$. Analogously, if $b_n$ is a monotone buyer strategy, then $U_n^S(s)$ on $NS = \{(n,s) : s \in [0,1], n = \overline{1, n_{b_s^\alpha}}\}$ satisfies the strict single crossing property in $(n,-s)$.*

I next state the necessary conditions for the optimality of strategies $b_n$ and $s_n$ in the active CSE that reflects the indifference of threshold types between accepting in the current round and delaying acceptance until the next active round.

**Lemma 9.** *Suppose $(b_n, s_n, p_n^B, p_n^S)$ describe an active CSE. Then for all even $n \leq \bar{N}$,*

$$v(b_n) - p_n^S = \delta\alpha_n^S\left(v(b_n) - p_{n+1}^B\right) + \delta^2\left(1-\alpha_n^S\right)\left(v(b_n) - p_{n+2}^S\right) \tag{25}$$

*where*

$$\alpha_n^S = \begin{cases} \frac{s_{n+1} - \max\{s_{n-1}, s_{b_n}^\alpha\}}{s_{b_n}^\omega - \max\{s_{n-1}, s_{b_n}^\alpha\}}, & \textit{if } s_{b_n}^\alpha \leq s_{n+1}, \\ 0, & \textit{otherwise}, \end{cases} \tag{26}$$

*and for all odd $n \leq \bar{N}$,*

$$p_n^B - c(s_n) = \delta\alpha_n^B\left(p_{n+1}^S - c(s_n)\right) + \delta^2\left(1-\alpha_{n+2}^S\right)\left(p_{n+2}^B - c(s_n)\right) \tag{27}$$

*where*

$$\alpha_n^B = \begin{cases} \frac{\min\{b_{n-1}, b_{s_n}^\omega\} - b_{n+1}}{\min\{b_{n-1}, b_{s_n}^\omega\} - b_{s_n}^\alpha}, & \textit{if } b_{s_n}^\omega \geq b_{n+1}, \\ 0, & \textit{otherwise}. \end{cases} \tag{28}$$

*Proof.* The left-hand side of equation (25) gives the utility of buyer type $b_n$ from accepting the seller's offer $p_n^S$. The right-hand side of equation (25) gives the utility of buyer type $b_n$ from delaying acceptance until the next active round. Then in round $n+1$, the seller accepts the buyer price offer $p_{n+1}^B$ with probability $\alpha_n^S$ (according to the beliefs of buyer type $b_n$) and with complementary probability in round $n+2$ buyer type $b_n$ accepts offer $p_{n+2}^S$. Notice that the probability $\alpha_n^S$ is the probability of acceptance of the offer in the next round for the threshold type of the buyer. Condition (27) is derived by analogous argument. $\square$

The following result shows that necessary conditions (25) and (27) are also sufficient. Say that a tuple $(b_n, s_n, p_n^B, p_n^S)$ is a *common screening strategy profile* if it satisfies conditions of CSE on-path strategies.

**Lemma 10.** *Suppose a tuple $(b_n, s_n, p_n^B, p_n^S)$ is a common screening strategy profile such that after some $\bar{N} \leq \infty$ no types remain in the game. If condition (25) holds for all rounds $n \leq \bar{N}$, then $b_n$ is a best reply to $s_n$. Symmetrically, if condition (27) holds for all rounds $n \leq \bar{N}$, then $s_n$ is a best-reply to $b_n$.*

*Proof.* Consider a buyer of type $b$ who accepts in round $n_b \leq \bar{N}$ and $n \leq \bar{\mathrm{N}}$. By condition (25), a buyer type $b_n$ is indifferent between accepting $p_n^S$ and delaying the acceptance until $n+2$. By Lemma 8, all buyer types above $b_n$ strictly prefer to accept in round $n$ rather than delay acceptance until round $n+2$, and all buyer types below $b_n$ strictly prefer to delay acceptance until round $n+2$ to accepting in round $n$. When $n < n_b$, then $b_n > b$ and so a buyer of type $b$ prefers to accept in round $n+2$ rather than in round $n$. When $n > n_b$, then $b_n < b$ and so a buyer of type $b$ prefers to accept in round $n$ rather than in round $n+2$. Therefore, $n_b$ is the optimal acceptance time for a buyer of type $b$. □

*Proof of Lemma 1.* Let $\bar{p}$ be the supremum of equilibrium price offers accepted by the buyer and $\bar{p}^B$ be the supremum of equilibrium price offers made by the buyer. I show that $\bar{p} \leq \frac{v(1)+\delta c(1)}{1+\delta}$ and $\bar{p}^B \leq \delta\bar{p} + (1-\delta)c(1) \leq \frac{\delta v(1)+c(1)}{1+\delta}$, which proves the first statement of the lemma. The second statement of the lemma is a symmetric statement for the seller and is proven analogously.

*Claim* 4. $\bar{p}^B \leq \delta\bar{p} + (1-\delta)c(1)$.

*Proof.* Suppose, by contradiction, that this is not the case. Then for any $\gamma > 0$ there is a history such that some buyer type makes an offer higher than $\bar{p}^B - \gamma/2$. Consider a deviation of this buyer type to $\bar{p}^B - \gamma$. Such a price offer is accepted by the seller with probability one only if $\bar{p}^B - \gamma - c(s) > \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ for all $s \in [0,1]$. This is indeed the case whenever $\gamma < \min\{1-\delta^2, \bar{p}^B - \delta\bar{p} - (1-\delta)c(1)\}$ which is possible for small $\gamma$, as the right-hand side of the inequality is positive. Given that price offer $\bar{p}^B - \gamma$ is accepted, the buyer prefers to deviate to price offer $\bar{p}^B - \gamma$ rather than make price offer $\bar{p}^B - \gamma/2$, which is a contradiction. Therefore, in equilibrium, the buyer never makes any offer higher than $\delta\bar{p} + (1-\delta)c(1)$. *Q.E.D.*

*Claim* 5. $\bar{p} \leq \frac{v(1)+\delta c(1)}{1+\delta}$

*Proof.* Suppose, by contradiction, that $\bar{p} > \frac{v(1)+\delta c(1)}{1+\delta}$. Then for any $\gamma > 0$ there is a history such that some seller type $s$ makes a price offer $\tilde{p} \in (\bar{p} - \gamma, \bar{p}]$ that is accepted by some buyer of type $b$. Consider a deviation by the buyer to a counter-offer $p_d$. For such a deviation not to be profitable it is necessary that $p_d - c(s) \leq \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ and $\delta(v(b) - p_d) \leq v(b) - \tilde{p}$ for some $s$ and $b$. If this were not the case, then all seller types would prefer to accept price offer $p_d$ (by the first inequality), and all buyer types would prefer such a counter-offer to accepting $\tilde{p}$ (by the second inequality). Then $\frac{1}{\delta}(\tilde{p} - (1-\delta)v(b))) \leq c(s) + \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ for some

$s$ and $b$, from which it follows that $\frac{1}{\delta}(\tilde{p}-(1-\delta)v(1))) \leq \max\{\delta\bar{p}+(1-\delta)c(1), \delta^2\bar{p}^B+(1-\delta^2)c(1))\}$. The maximum in the right-hand side is equal to $\delta\bar{p}+(1-\delta)c(1)$. Indeed, if it were not the case, then $\bar{p} < \delta\bar{p}^B + (1-\delta)c(1) \leq \delta(\delta\bar{p} + (1-\delta)c(1)) + (1-\delta)c(1)$ or $\bar{p} < c(1)$, which contradicts $\bar{p} > \frac{v(1)+\delta c(1)}{1+\delta}$. Hence, $\tilde{p} - (1-\delta)v(1)) \leq \delta^2\bar{p} + \delta(1-\delta)c(1)$ or $\frac{\tilde{p}-\delta^2\bar{p}}{1-\delta^2} \leq \frac{v(1)+\delta c(1)}{1+\delta}$. The left-hand side is greater than $\bar{p} - \frac{\gamma}{1-\delta^2} > \frac{v(1)+\delta c(1)}{1+\delta} - \frac{\gamma}{1-\delta^2}$. Since $\gamma$ was chosen arbitrarily, this gives a contradiction. *Q.E.D.* $\square$

### 8.1.3 Proof of Theorem 2. Necessity

Consider a sequence of active CSEs indexed by $j \to \infty$ with a smooth limit and such that $\Delta_j \to 0$ as $j \to \infty$. I show that the smooth limit satisfies conditions (2), (3) and (4) (and hence, constitutes an equilibrium in the concession game by Theorem 1) and conditions (7) and (8). The latter follow immediately from Lemma 1. Notice also that since $p_n^B$ and $p_n^S$ are monotone by Definition 3, so are their limits $q_t^B$ and $q_t^S$.

*Claim* 6. *Condition (2) holds.*

*Proof.* Suppose, by contradiction, that $b_T^* > b^\alpha_{s_T^*}$. By condition 3 in Definition 4, for any $\varepsilon \in \left(0, \frac{b_T^* - b^\alpha_{s_T^*}}{3}\right)$ and $\Delta_j > 0$ sufficiently small, $b_T^j > b_T^* - \varepsilon$ and $b^\alpha_{s_T^j} < b^\alpha_{s_T^*} + \varepsilon$. Consider seller type $s_T^j$ and any time $t$. The continuation utility of seller $s_T^j$ at time $t$ from following equilibrium strategy is bounded above by

$$\frac{\min\left\{b_t^j, b^\omega_{s_T^j}\right\} - b_T^j}{\min\left\{b_t^j, b^\omega_{s_T^j}\right\} - b^\alpha_{s_T^j}} \left(v(b_t^j) - c(s_T^j)\right).$$

Since $\min\{b_t^j, b^\omega_{s_T^j}\} - b^\alpha_{s_T^j} \geq \min\{b_T^j - b^\alpha_{s_T^j}, \eta\} > \min\{b_T^* - b^\alpha_{s_T^*} - 2\varepsilon, \eta\} > \min\{\varepsilon, \eta\} > 0$, the upper bound converges to zero as $t \to \infty$. An analogous upper bound (converging to zero as $t \to \infty$) can be derived for buyer type $b_T^j$. This is in contradiction with condition 4 of Definition 3, which requires that over time price offers converge enough so that gains from trade can be realized through the acceptance of one of the parties.

Now suppose $T < \infty$, but $q_T^S > q_T^B$. By condition 2 of the Definition 4, for any $\varepsilon > 0$, $T_j < T + \varepsilon$. By the continuity of $q_t^B$ and $q_t^S$, for $\varepsilon$ small enough, $q_t^S - q_t^B > \frac{q_T^S - q_T^B}{2}$ for all $t \in [T-\varepsilon, T+\varepsilon]$ and so, for $\Delta_j$ sufficiently small, $p_t^{Sj} > p_t^{Bj} + \frac{q_T^S - q_T^B}{4}$ for all $t \in [T-\varepsilon, T+\varepsilon]$. Suppose buyer type $b_{T-\varepsilon}^j$ deviates by rejecting $p_{T-\varepsilon}^{Sj}$ and waiting for $2\varepsilon$ until the seller accepts some price offer of the buyer. Type $b_{T-\varepsilon}^j$ gets utility of at least $\min_{t \in [T-\varepsilon, T+\varepsilon]} e^{-2r\varepsilon}\left(v(b_{T-\varepsilon}^j) - p_t^{Bj}\right)$. On the other hand, from following the equilibrium strategy, type $b_{T-\varepsilon}^j$ gets $v(b_{T-\varepsilon}^j) - p_{T-\varepsilon}^{Sj}$. For $\varepsilon$ small enough, such deviation is profitable, which is a contradiction. This proves condition (2). *Q.E.D.*

*Claim 7. Conditions (3) and (8) hold.*

*Proof.* For any $t < T$, let $\tau_t \equiv 2\Delta_j \left\lfloor \frac{t}{2\Delta_j} \right\rfloor$. By Lemma 9, condition (25) holds for all even $n \leq \bar{N}$, which can be rewritten for any $\tau_t$ as follows:

$$v\left(b^j_{\tau_t}\right) - p^{Sj}_{\tau_t} = e^{-r\Delta_j}\alpha^{Sj}_{\tau_t}\left(v\left(b^j_{\tau_t}\right) - p^{Bj}_{\tau_t+\Delta_j}\right) + e^{-2r\Delta_j}\left(1-\alpha^{Sj}_{\tau_t}\right)\left(v\left(b^j_{\tau_t}\right) - p^{Sj}_{\tau_t+2\Delta_j}\right).$$

Subtracting $e^{-2r\Delta_j}\left(v\left(b^j_{\tau_t}\right) - p^{Sj}_{\tau_t}\right)$ from both sides and dividing by $2\Delta_j$ results in

$$\frac{1-e^{-2r\Delta_j}}{2\Delta_j}\left(v\left(b^j_{\tau_t}\right) - p^{Sj}_{\tau_t}\right) = e^{-r\Delta_j}\frac{\alpha^{Sj}_{\tau_t}}{2\Delta_j}\left(v\left(b^j_{\tau_t}\right) - p^{Bj}_{\tau_t+\Delta_j}\right) - e^{-r\Delta_j}\left(v\left(b^j_{\tau_t}\right) - p^{Sj}_{\tau_t}\right) + e^{-2r\Delta_j}\frac{p^{Sj}_{\tau_t} - p^{Sj}_{\tau_t+2\Delta_j}}{2\Delta_j}.$$

Taking $\Delta_j \to 0$ results is condition (3), where convergence is guaranteed by the definition of the smooth limit and by the continuity of function $v$. The derivation of equation (4) for buyer price offers is symmetric. *Q.E.D.*

□

### 8.1.4 Proof of Lemma 2

I reduce the problem of finding a CSE with price offers that are constant over time to a mathematical problem of finding a positive trajectory which satisfies a particular recursive system. The following lemma is a key mathematical fact in the proof of Lemma 2.

**Lemma 11.** *Consider $b_\infty \in (0, 1-\eta)$, $s_\infty = b_\infty + \eta$, $P^B$, $P^S$ that satisfy*

$$\max\left\{c\left(s_\infty\right), \frac{v(0)+c(0)}{2}\right\} \quad < \quad P^B < P^S < \min\left\{v\left(b_\infty\right), \frac{v(1)+c(1)}{2}\right\}. \tag{29}$$

*There exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$ there are positive trajectories $x_k$ and $y_k$ that satisfy recursive system*

$$\begin{cases} x_{k+1} = (1-\alpha^B(y_{k+1}))x_k - \alpha^B(y_{k+1})y_{k+1}, \\ y_{k+1} = (1-\alpha^S(x_k))y_k - \alpha^S(x_k)x_k, \\ b_\infty + x_k \leq s_\infty - y_k + \eta; \end{cases} \tag{30}$$

*where $\alpha^B(y) \equiv \frac{(1-\delta^2)(P^B - c(s_\infty - y))}{\delta(P^S - c(s_\infty - y)) - \delta^2(P^B - c(s_\infty - y))}$ and $\alpha^S(x) \equiv \frac{(1-\delta^2)(v(b_\infty + x) - P^S)}{\delta(v(b_\infty + x) - P^B) - \delta^2(v(b_\infty + x) - P^S)}$. Moreover, for all $k \in \mathbb{N}$,*

$$\max\{x_{k-1} - x_k, y_{k-1} - y_k\} \quad < \quad (1-\delta)C, \tag{31}$$

*where $C$ is a constant that does not depend on $\delta$.*

*Proof of Lemma 11.* Observe that if $x_k$ and $y_k$ are given for $k \geq k_0$, then by (30), I can construct $x_k$ and $y_k$ for $k < k_0$. The following claim shows that it is sufficient to construct $x_k$ and $y_k$ that are positive starting from some $k_0$.

*Claim* 8. *If trajectories $x_k$ and $y_k$ satisfying* (30) *are positive starting from some $k_0$, then $x_k$ and $y_k$ are positive for all $k \in \mathbb{N}$.*

*Proof.* By rearranging terms in the first equation of (30), $x_k = \frac{x_{k+1}+\alpha^B(y_{k+1})y_{k+1}}{1-\alpha^B(y_{k+1})}$. Observe that $\alpha^B(y) \in (0,1)$ for $y > 0$ and so $x_k$ is positive whenever $x_{k+1}$ and $y_{k+1}$ are positive. Analogously, it can be shown from the second equation of (30) that $y_k$ is positive whenever $x_{k+1}$ and $y_{k+1}$ are positive. *Q.E.D.*

*Claim* 9. *For given $x_{k_0}$ and $y_{k_0}$, there is $K(x_{k_0}, y_{k_0})$ such that $k_0 \leq K(x_{k_0}, y_{k_0})$.*

*Proof.* First, observe that $x_k$ and $y_k$ are decreasing whenever they are positive. Indeed, for all $k \in \mathbb{N}$, I have $x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) > 0$ and similarly, $y_{k-1} - y_k > 0$. Next, from (30), for all $k \leq k_0$,

$$x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) \geq \alpha^B(y_{k_0})(x_{k_0} + y_{k_0}) > c_1 \tag{32}$$

for some $c_1 > 0$ where I made use of the fact that $\alpha^B(y)$ is increasing and $x_k$ and $y_k$ are decreasing sequences. Suppose for any $K \in \mathbb{N}$, I could construct sequences $x_k(K)$ and $y_k(K)$ such that $x_K(K) = x_{k_0}$ and $y_K(K) = y_{k_0}$. From (32), for $K$ sufficiently large, $b_\infty + x_0(K) > s_\infty - y_0(K) + \eta$, which contradicts (30). *Q.E.D.*

Let $V^B \equiv v(b_\infty) - P^S$, $V^S \equiv P^B - c(s_\infty)$ and $\Delta P \equiv P^S - P^B$. The following claim gives rise to the Taylor expansion of $\alpha^B(y)$ and $\alpha^S(x)$.

*Claim* 10. *There exists $\delta_1 \in (0,1)$ and $\varepsilon_1 > 0$ such that for all $\delta \in (\delta_1, 1)$ and all $x \in (0, \varepsilon_1)$, $y \in (0, \varepsilon_1)$,*

$$\alpha^B(y) \equiv \alpha_B - \phi_B \sum_{l=1}^{\infty} \gamma_l^B y^l, \tag{33}$$

$$\alpha^S(x) \equiv \alpha_S - \phi_S \sum_{l=1}^{\infty} \gamma_l^S x^l, \tag{34}$$

*where*

$$\alpha_B \equiv \frac{(1-\delta^2)V^B}{\delta(\Delta P + (1-\delta)V^B)}, \gamma_B \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^B} < 0, \phi_B \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^B)} > 0,$$

$$\alpha_S \equiv \frac{(1-\delta^2)V^S}{\delta(\Delta P + (1-\delta)V^S)}, \gamma_S \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^S} < 0, \phi_S \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^S)} > 0,$$

$$\gamma_l^B \equiv \sum_{j=1}^{l} \gamma_B^j \left( \sum_{l_1+\cdots+l_j=l} \frac{d^{l_1} c(s_\infty)/ds^{l_1}}{l_1!} \cdots \frac{d^{l_j} c(s_\infty)/ds^{l_j}}{l_j!} \right),$$

$$\gamma_l^S \equiv \sum_{z=1}^{l} \gamma_S^z \left( \sum_{l_1+\cdots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right),$$

*and* $\gamma_l^S \le |\gamma_S D|(1+|\gamma_S D|)^{l-1}$, $\gamma_l^B \le |\gamma_B D|(1+|\gamma_B D|)^{l-1}$.

*Proof.* As $\delta \to 1$, $\gamma_S$ and $\gamma_B$ converge to zero and so, for $\delta$ sufficiently close to one, $|\gamma_S(v(1)-v(0))| < 1$ and $|\gamma_B(c(1)-c(0))| < 1$. Expanding $\alpha^S(x)$ into the Taylor series, results in

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z (v(b_\infty + x) - v(b_\infty))^z.$$

Since $v$ is a smooth function, expanding it into the Taylor series around $b_\infty$ results in $v(b_\infty + x) - v(b_\infty) = \sum_{l=1}^{\infty} \frac{d^l v(b_\infty)}{db^l} \frac{x^l}{l!}$. By the regularity of $v$, all derivatives $\frac{d^l v(b)/db^l}{l!}$, $l \in \mathbb{N}$ are bounded by $D$ for some $D > 1$. Therefore, the Taylor expansion of $v$ around $b_\infty$ is an absolute convergent series, and by the Merten's theorem the $z$'s power of it equals

$$(v(b_\infty + x) - v(b_\infty))^z = \sum_{l=z}^{\infty} x^l \left( \sum_{l_1+\cdots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right),$$

and so,

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z \sum_{l=z}^{\infty} x^l \left( \sum_{l_1+\cdots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right). \tag{35}$$

Observe that

$$\sum_{z=1}^{\infty} \left| \gamma_S^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\cdots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \le$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\cdots+l_z=l} \left| \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \le$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\cdots+l_z=l} D^z = \sum_{z=1}^{\infty} |\gamma_S|^z D^z \sum_{l=z}^{\infty} x^l \binom{l-1}{z-1} = \sum_{z=1}^{\infty} \frac{(|\gamma_S| D x)^z}{(1-x)^z} < \infty$$

where the first inequality follows from the triangle inequality, the second inequality follows from the regularity of $v$ and the fact that $(l_1 + \ldots + l_z)! \ge l_1! \cdot \ldots \cdot l_z!$, the first equality follows from the fact that a number of compositions of $l$ into exactly $z$ parts is equal to $\binom{l-1}{z-1}$, the second equality results by summing over $l$, and the resulting series is convergent for $x$ sufficiently small (so that $x < (1 + |\gamma_S| D)^{-1}$). Therefore, the series in (35) is absolutely convergent, and by the Fubini's theorem, I could exchange the order of summation in (35) to get expression (33). I have the following upper bound on the absolute values of coefficients $\gamma_l^S$

$$|\gamma_l^S| \leq \sum_{z=1}^{l} |\gamma_S|^z \left( \sum_{l_1+\cdots+l_z=l} \left| \frac{d^{l_1}v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z}v(b_\infty)/db^{l_z}}{l_z!} \right| \right) \leq \sum_{z=1}^{l} |\gamma_S D|^z \binom{l-1}{z-1} = |\gamma_S D|(1+|\gamma_S D|)^{l-1} \tag{36}$$

where the first inequality comes about via the triangle inequality, the second inequality follows from the regularity of $v$, and the equality is obtained by algebraic manipulation. The derivation of the corresponding expression for $\alpha^S(y)$ is analogous. *Q.E.D.*

System (30) has steady states $(z, -z), z \in \mathbb{R}$. By the specification of the problem I am interested only in steady state $(0, 0)$. Around this steady state the linearized system can be written in the matrix form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_B + \alpha_S \alpha_B & -\alpha_B(1-\alpha_S) \\ -\alpha_S & 1 - \alpha_S \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

The matrix has eigenvalues 1 and $\lambda \equiv (1 - \alpha_B)(1 - \alpha_S)$. Since one of eigenvalues is equal to 1, the steady state is unstable, and I cannot conclude that in the neighborhood of the steady state the non-linear system will converge to the steady state. Therefore, I find a particular trajectory that satisfies desired properties.

I conjecture that there exist $\mu_i^x$ and $\mu_i^y$ such that

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{i=1}^{\infty} \lambda^{ik} \begin{pmatrix} \lambda^{i/2}\mu_i^x \\ \mu_i^y \end{pmatrix}, \tag{37}$$

which is the required solution and for all $i \in \mathbb{N}$,

$$|\mu_i^x| \leq u_\delta M^i \text{ and } |\mu_i^y| \leq u_\delta M^i \tag{38}$$

for some positive $M$ and $u_\delta$ such that

$$M < 1 < \frac{1}{\lambda(1 + u_\delta(1 + \max\{|\gamma_S|, |\gamma_B|\}D))}. \tag{39}$$

Given the accuracy of this conjecture, I next derive expressions for coefficients $\mu_i^x$ and $\mu_i^y$, and then verify that for $\delta$ sufficiently close to one, upper bounds on absolute values of coefficients will hold. Series (37) defining $(x_k, y_k)$ are absolutely convergent, as they are dominated by the absolutely convergent series $u_\delta \sum_{i=1}^{\infty} \lambda^{ik} M^i$.

Plugging the solution (37) into system (30), I get

$$\begin{cases}\sum\limits_{i=1}^{\infty}\lambda^{ik}(\mu_i^x-\mu_i^x\lambda^i-\alpha_B(\mu_i^x+\mu_i^y\lambda^{i/2}))=-\phi_B\left(\sum\limits_{l=1}^{\infty}\gamma_l^B\left(\sum\limits_{i=1}^{\infty}\mu_i^y\lambda^{i(k+1)}\right)^l\right)\left(\sum\limits_{i=1}^{\infty}\lambda^{ik}(\mu_i^x+\mu_i^y\lambda^{i/2})\right),\\ \sum\limits_{i=1}^{\infty}\lambda^{ik}(\mu_i^y-\mu_i^y\lambda^i-\alpha_S(\mu_i^x\lambda^{i/2}+\mu_i^y))=-\phi_S\left(\sum\limits_{l=1}^{\infty}\gamma_l^S\left(\sum\limits_{i=1}^{\infty}\mu_i^x\lambda^{ik}\right)^l\right)\left(\sum\limits_{i=1}^{\infty}\lambda^{i(k+1/2)}(\mu_i^x\lambda^{i/2}+\mu_i^y)\right).\end{cases} \tag{40}$$

Consider the first equation in system (40). By the Merten's Theorem, $\left(\sum\limits_{i=1}^{\infty}\mu_i^y\lambda^{i(k+1)}\right)^l = \sum\limits_{i=l}^{\infty}\sum\limits_{i_1+\dots+i_l=i}\mu_{i_1}^y\cdot\dots\cdot\mu_{i_l}^y\lambda^{i(k+1)}$ and

$$\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{ik}\right)^l=\sum_{l=1}^{\infty}\gamma_l^B\sum_{i=l}^{\infty}\sum_{i_1+\dots+i_l=i}\mu_{i_1}^y\cdot\dots\cdot\mu_{i_l}^y\lambda^{i(k+1)}. \tag{41}$$

The series in (41) is absolutely convergent by

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\left|\lambda^{i(k+1)}\gamma_l^B\sum_{i_1+\dots+i_l=i}\mu_{i_1}^y\cdot\dots\cdot\mu_{i_l}^y\right|\leq$$

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)}|\gamma_l^B|\sum_{i_1+\dots+i_l=i}\left|\mu_{i_1}^y\cdot\dots\cdot\mu_{i_l}^y\right|\leq$$

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)}|\gamma_l^B|\sum_{i_1+\dots+i_l=i}u_\delta^l M^i=\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)}|\gamma_l^B|u_\delta^l M^i\binom{i-1}{l-1}\leq$$

$$|\gamma_B D|\sum_{l=1}^{\infty}\sum_{i=l}^{\infty}\lambda^{i(k+1)}(1+|\gamma_B D|)^{l-1}u_\delta^l M^i\binom{i-1}{l-1}=\frac{|\gamma_B D|}{1+|\gamma_B D|}\sum_{l=1}^{\infty}(1+|\gamma_B D|)^l u_\delta^l\left(\frac{\lambda^{k+1}M}{1-\lambda^{k+1}M}\right)^l\leq$$

$$\frac{|\gamma_B D|}{1+|\gamma_B D|}\sum_{l=1}^{\infty}(1+|\gamma_B D|)^l u_\delta^l\left(\frac{\lambda M}{1-\lambda M}\right)^l,$$

where the first inequality arises via the triangle inequality, the second inequality follows from (38), the first equality arises from the fact that the number of compositions of $i$ into exactly $l$ parts is $\binom{i-1}{l-1}$. The third inequality follows from (36), the forth inequality is by $\lambda^{k+1}<\lambda$, and the resulting series is convergent whenever $u_\delta(1+|\gamma_B D|)\frac{\lambda M}{1-\lambda M}<1$, which holds by (39). Therefore, by Fubini's Theorem, exchanging the order of summation in (41) results in

$$\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{ik}\right)^l=\sum_{i=1}^{\infty}\lambda^{i(k+1)}\sum_{l=1}^{i}\sum_{i_1+\dots+i_l=i}\gamma_l^B\mu_{i_1}^y\cdot\dots\cdot\mu_{i_l}^y.$$

By the absolute convergence of both series on the right-hand side of (40), the product on the right-hand side is equal to the Cauchy product, and so I can rewrite system (40) as follows

$$\begin{cases} \sum_{i=1}^{\infty} \lambda^{ik} \left( \mu_i^x - \mu_i^x \lambda^i - \alpha_B(\mu_i^x + \mu_i^y \lambda^{i/2}) + \phi_B \sum\limits_{j=1}^{i-1} (\mu_{i-j}^x \lambda^{j/2} + \mu_{i-j}^y \lambda^{i/2}) \sum\limits_{l=1}^{j} \gamma_l^B \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \right) = 0, \\ \sum_{i=1}^{\infty} \lambda^{ik} \left( \mu_i^y - \mu_i^y \lambda^i - \alpha_S(\mu_i^x \lambda^{i/2} + \mu_i^y) + \phi_S \sum\limits_{j=1}^{i-1} (\mu_{i-j}^x \lambda^{i/2} + \mu_{i-j}^y \lambda^{j/2}) \sum\limits_{l=1}^{j} \gamma_l^S \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \right) = 0. \end{cases}$$

Setting all coefficient at $\lambda^{ik}$ equal to zero results in the system

$$\begin{cases} \mu_i^x - \mu_i^x \lambda^i - \alpha_B(\mu_i^x \lambda^{j/2} + \mu_i^y \lambda^{i/2}) = -\phi_B \sum\limits_{j=1}^{i-1} \left( (\mu_{i-j}^x \lambda^{j/2} + \mu_{i-j}^y \lambda^{i/2}) \sum\limits_{l=1}^{j} \gamma_l^B \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \right), \\ \mu_i^y - \mu_i^y \lambda^i - \alpha_S(\mu_i^x \lambda^{i/2} + \mu_i^y) = -\phi_S \sum\limits_{j=1}^{i-1} \left( (\mu_{i-j}^x \lambda^{i/2} + \mu_{i-j}^y \lambda^{j/2}) \sum\limits_{l=1}^{j} \gamma_l^S \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \right). \end{cases}$$

Using notation $A_i \equiv \begin{pmatrix} 1-\lambda^i-\alpha_B & -\alpha_B\lambda^{i/2} \\ -\alpha_S\lambda^{i/2} & 1-\lambda^i-\alpha_S \end{pmatrix}$, $\mu_i \equiv \begin{pmatrix} \mu_i^x \\ \mu_i^y \end{pmatrix}$, and

$$\varphi_i = \begin{pmatrix} \varphi_i^x \\ \varphi_i^y \end{pmatrix} \equiv \begin{pmatrix} -\phi_B \sum\limits_{j=1}^{i-1} \left( (\mu_{i-j}^x \lambda^{j/2} + \mu_{i-j}^y \lambda^{i/2}) \sum\limits_{l=1}^{j} \gamma_l^B \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y \right) \\ -\phi_S \sum\limits_{j=1}^{i-1} \left( (\mu_{i-j}^x \lambda^{i/2} + \mu_{i-j}^y \lambda^{j/2}) \sum\limits_{l=1}^{j} \gamma_l^S \sum\limits_{j_1+\dots+j_l=j} \mu_{j_1}^x \cdot \ldots \cdot \mu_{j_l}^x \right) \end{pmatrix}, \tag{42}$$

and I can write the system in matrix form as $A_i\mu_i = \varphi_i$. Since $\det(A_i) = (1-\lambda^i)(\lambda-\lambda^i) > 0$, for $i \geq 2$, matrix $A_i$ is invertible, and I can solve for all $\mu_i$ (with the exception of $i = 1$)

$$\mu_i = A_i^{-1}\varphi_i. \tag{43}$$

For $i = 1$, the equations are linearly dependent and the relation between $\mu_1^x$ and $\mu_1^y$ is given by

$$\mu_1^x = \mu_1^y \frac{\alpha_B}{\alpha_S}(1-\alpha_S). \tag{44}$$

Equations (43) and (44) give the desired expressions for $\mu_i$ through the parameters of the model. The next claim verifies that bounds (38) and (39) indeed hold and so, my derivation is justified.

*Claim* 11. *For $M < 1$, there exists $\hat{\delta} \in (0,1)$ such that for any $\delta \in (\hat{\delta}, 1)$ there exist positive $u_\delta$ and $\mu_1^y$ such that* (39) *holds, and for $\mu_i$ defined by (43) and* (44), *bounds* (38) *hold.*

*Proof.* The proof is by induction on $i$. Without loss of generality, I assume that

$$V^S \leq V^B \tag{45}$$

and so, $\alpha_S \leq \alpha_B, |\gamma_S| \geq |\gamma_B|, \phi_S \geq \phi_B$. Let $u_\delta \equiv \frac{u}{2}\min\{|\gamma_S|, |\gamma_B|\}$ where $u = \frac{1}{2}\min\{V^S, V^B\}$.

Let us first show that for our choice of $u_\delta$, $1 < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$ for $\delta$ sufficiently close to one. To see this, observe that for $\delta$ sufficiently close to one, $\max\{|\gamma_B|,|\gamma_S|\}D < 1$ and so, $\frac{1}{\lambda(1+2u_\delta)} < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$. Therefore, it is sufficient to show that $\lambda^{1/2}(1+2u_\delta) < 1$. Then

$$\lambda^{1/2}(1+2u_\delta) = ((1-\alpha_S)(1-\alpha_B))^{1/2}\,(1+u\min\{|\gamma_S|,|\gamma_B|\}) \leq (1-\alpha_S)(1+u|\gamma_S|).$$

Observe

$$(1-\alpha_S)(1+u|\gamma_S|) = \left(1-\frac{(1-\delta^2)V^S}{\delta(\Delta P+(1-\delta)V^S)}\right)\left(1+\frac{(1-\delta)u}{\Delta P+(1-\delta)V^S}\right),$$

and $\lambda^{1/2}(1+2u_\delta) < 1$ is equivalent to

$$\Delta P + (1-\delta)V^S + u(1-\delta) < \frac{\delta(\Delta P+(1-\delta)V^S)(\Delta P+(1-\delta)V^S)}{\Delta P\delta-(1-\delta)V^S},$$

or

$$u < (1+\delta)V^S\frac{\Delta P+(1-\delta)V^S}{\Delta P\delta-(1-\delta)V^S}. \tag{46}$$

As $\delta \to 1$, the right-hand side of (46) converges to $2V^S$. Since $u < V^S$, inequality (46) holds and so $(1-\alpha_S)(1+u|\gamma_S|) < 1$ for sufficiently large $\delta$. Hence, I have proven that (39) holds.

To prove bounds (38), let $\mu_1^x$ and $\mu_1^y$ be defined as follows. If $\frac{\alpha_B}{\alpha_S}(1-\alpha_S) \leq 1$, then let $\mu_1^y = u_\delta M$ and $\mu_1^x = \mu_1^y\frac{\alpha_B(1-\alpha_s)}{\alpha_S} \leq \mu_1^y$, and otherwise let $\mu_1^x = u_\delta M$ and $\mu_1^y = \mu_1^x\frac{\alpha_S}{\alpha_B(1-\alpha_S)} \leq \mu_1^x$. By the definition, $|\mu_1^x|$ and $|\mu_i^y|$ are less than $u_\delta M$, which proves the base of induction.

Suppose that the statement is true for all $j < i$. I show that $|\mu_i^x| < u_\delta M^i$ and $|\mu_i^y| < u_\delta M^i$. I can find closed-form solution to system (43),

$$|\mu_i^x| = \frac{|(1-\lambda^i-\alpha_S)\varphi_i^x+\alpha_B\lambda^{i/2}\varphi_i^y|}{(1-\lambda^i)(\lambda-\lambda^i)} \leq \frac{4\max\{1-\lambda^i,\alpha_S,\alpha_B\}\cdot\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)}$$

and the same upper bound holds for $|\mu_i^y|$. It is sufficient to show that $\frac{4\max\{(1-\lambda^i),\alpha_S,\alpha_B\}\cdot\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)u_\delta M^i} < 1$.

Notice that $\frac{\alpha_S}{1-\lambda^i} < \frac{\alpha_S}{1-\lambda}$ for $i \geq 2$, and by l'Hospital rule $\lim\limits_{\delta\to 1}\frac{\alpha_S}{1-\lambda} = \lim\limits_{\delta\to 1}\frac{\alpha_S}{\alpha_S+\alpha_B-\alpha_S\alpha_B} = \frac{V^S}{V^S+V^B} \leq 1$. Hence, for sufficiently large $\delta$ and all $i \geq 2$, I have $\frac{\alpha_S}{1-\lambda^i} < 1$, and by an analogous argument, $\frac{\alpha_B}{1-\lambda^i} < 1$. Therefore, $\frac{4\max\{1-\lambda^i,\alpha_S,\alpha_B\}}{1-\lambda^i} < 5$ for sufficiently large $\delta$ and it remains to show that $\frac{\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{5}$ for sufficiently large $\delta$.

I next show that $\frac{|\varphi_i^x|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$ (by the symmetric argument $\frac{|\varphi_i^y|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$). From (42) it follows

$$\frac{|\varphi_i^x|}{\phi_B} \le \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^x \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y| + \lambda^{i/2} \sum_{j=1}^{i-1} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^y \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y| \le$$

$$\sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i + \lambda^{i/2} \sum_{j=1}^{i-1} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i \le$$

$$2u_\delta M^i \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| u_\delta^l \binom{j-1}{l-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} u_\delta^l (1 + |\gamma_B D|)^{l-1} \binom{j-1}{l-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} u_\delta \left(1 + u_\delta (1 + |\gamma_B D|)\right)^{j-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} u_\delta (1 + 2u_\delta)^{j-1} =$$

$$2u_\delta M^i |\gamma_B D| \frac{u_\delta \lambda^{1/2} (1 - \lambda^{(i-1)/2} (1 + 2u_\delta)^{i-1})}{1 - \lambda^{1/2} (1 + 2u_\delta))},$$

where the first inequality is due to the triangle inequality, the second inequality arises via the inductive hypothesis, the third inequality makes use of the fact that the number of compositions of $j$ into exactly $l$ parts is $\binom{j-1}{l-1}$ and that $\lambda^j > \lambda^i$ for $j < i$, the forth inequality uses a bound on $|\gamma_l^B|$, the fifth inequality exists by summing over $l$, the sixth inequality is by $|\gamma_B D| < 1$ for sufficiently large $\delta$, the equality is the summation over $j$. It remains to show that

$$2\phi_B |\gamma_B D| \frac{u_\delta \lambda^{1/2} (1 - \lambda^{(i-1)/2} (1 + 2u_\delta)^{i-1})}{(\lambda - \lambda^i)(1 - \lambda^{1/2} (1 + 2u_\delta))} < \frac{1}{5}. \tag{47}$$

Since the denominator in (47) is positive, (47) is equivalent to

$$\lambda - \lambda^i - 10\phi_B |\gamma_B D| \frac{u_\delta \lambda^{1/2} (1 - \lambda^{(i-1)/2} (1 + 2u_\delta)^{i-1})}{1 - \lambda^{1/2} (1 + 2u_\delta)} > 0. \tag{48}$$

The derivative of (48) with respect to $i$ is equal to

$$\lambda^{i/2} \left( -\ln(\lambda) \lambda^{i/2} + 10 \ln(\lambda^{1/2} (1 + 2u_\delta)) \phi_B |\gamma_B D| \frac{u_\delta (1 + 2u_\delta)^{i-1}}{1 - \lambda^{1/2} (1 + 2u_\delta)} \right).$$

Multiplication by $\lambda^{i/2}$ does not affect the sign of the derivative so I focus on the term in brackets.

The positive (first) term in brackets is decreasing in absolute value, while the negative (second) term is increasing in absolute value. Therefore, the minimum of expression (48) is either attained at $i = 2$ or $i \to \infty$. For $i = 2$, (48) is equal to

$$\lambda - \lambda^2 - 10\phi_B|\gamma_B D|u_\delta \lambda^{1/2} > 0, \tag{49}$$

whenever $u_\delta < \frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|}$. By (45), $\frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|} = \frac{\lambda^{1/2}(1-(1-\alpha_B)(1-\alpha_S))}{10\phi_B|\gamma_B D|} \leq \frac{\lambda^{1/2}(1-(1-\alpha_B)^2)}{10\phi_B|\gamma_B D|} \leq \frac{\alpha_B}{\phi_B|\gamma_B|} \to V^B$. Since $u_\delta$ converges to zero as $\delta \to 1$, inequality (49) holds for $\delta$ close to one.

For $i = \infty$, (48) is equal to

$$\lambda\left(1 - 10D\frac{\phi_B|\gamma_B|}{\lambda^{1/2}}\frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)}\right). \tag{50}$$

Observe that $\lim\limits_{\delta\to 1} \frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)} = \frac{u}{V^S+V^B-2u}$. Since $|\gamma_B| \to 0, \lambda \to 1, \phi_B \to 2$ as $\delta \to 1$, I have that (50) is positive for sufficiently large $\delta$. *Q.E.D.*

So far I have constructed the candidate trajectories $(x_k, y_k)$ given by (37). First, notice that by making $k$ sufficiently large the solution approaches zero and so, the Taylor expansion in Claim 10 is justified. Second, observe that $x_k = \sum\limits_{i=1}^{\infty} \lambda^{ik}\lambda^{i/2}\mu_i^x = \lambda^{k+1/2}\left(\mu_1^x + \sum\limits_{i=2}^{\infty}\lambda^{(i-1)k}\lambda^{i/2}\mu_i^x\right)$, and for sufficiently large $k$ , the sign of $x_k$ is determined by $\mu_1^x$ which I can choose to be positive. Analogously, since $\mu_1^y$ has the same sign as $\mu_1^x$ (by the definition), $y_k$ is positive for sufficiently large $k$. By Claim 8, the constructed trajectory $(x_k, y_k)$ is positive.

To show that I can bound the change in $x_k$ and $y_k$ by a term of order $1-\delta$, observe that

$$x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) \leq 2\left(\alpha_B - \phi_B \sum_{z=1}^{\infty} \gamma_B^z (c(s_\infty) - c(s_\infty - y_k))^z\right) \leq$$

$$2\left(\alpha_B - \frac{\phi_B|\gamma_B|\Sigma}{1-|\gamma_B|\Sigma}\right) \sim 1-\delta,$$

and so there exists $C$ such that $x_{k-1} - x_k < (1-\delta)C$ for all $k \in \mathbb{N}$. The analogous bound holds for $y_{k-1} - y_k$. □

**Lemma 12.** *Consider $b_0 \in (0, 1-\eta], s_0 \in [b_0 - \eta, b_0 + \eta) \cap [\eta, 1)$, $P^B$, $P^S$ that satisfy* (9)*. There exist $\bar{\delta} \in (0,1), b_\infty \in (b_{s_0}^\alpha, b_0), s_\infty = b_\infty + \eta$ such that for all $\delta \in (\bar{\delta}, 1)$ there are positive trajectories $x_k$ and $y_k$ that satisfy recursive system* (30)*. Moreover, for all $k \in \mathbb{N}$,* (31) *holds for some constant $C$ that does not depend on $\delta$.*

*Proof.* Fix any $b_\infty \in (b_{s_0}^\alpha, b_0)$ and $s_\infty = b_\infty + \eta$. By Lemma 11, I can construct positive trajectories $x_k$ and $y_k$ that satisfy (30) and(31). Then construct sequences $\hat{b}_n$ and $\hat{s}_n$ by defining $\hat{b}_{2k-1} = \hat{b}_{2k-2} = b_\infty + x_{k-1}$ and $\hat{s}_{2k} = \hat{s}_{2k-1} = s_\infty - y_k$ for $k \in \mathbb{N}$ and letting $\hat{s}_0 = \hat{b}_0 - \eta$. There exist minimal $k_B$ and $k_S$ such that $\hat{b}_{2k_B} < b_0$ and $\hat{s}_{2k_S-1} > s_0$. I define $b_n$ and $s_n$ as subsequences of $\hat{b}_n$ and $\hat{s}_n$ starting from $n_0 = 2\max\{k_B, k_S\}$. Observe that by the construction of $x_k$ and

$y_k$ in (37),(43) and (44), any $x_k$ and $y_k$ are continuous in $b_\infty$ and $s_\infty$. Moreover, for $b_\infty = b_0$, I have that $n_0 = 2k_B$, $b_0 - b_1 = 0$, $s_1 - s_0 = 2\eta$, and at the other extreme, for $b_\infty = s_0 - \eta$, $n_0 = 2k_S$,$b_0 - b_1 = 2\eta$, $s_1 - s_0 = 0$. By continuity, there exists $b_\infty$ (and correspondingly, $s_\infty = b_\infty + \eta$) such that for corresponding $b_n$ and $s_n$ constructed as described above, I have $\max\{|b_0 - b_1|, |s_1 - s_0|\} < (1-\delta)C$. For all $n \geq 1$, $\max\{|b_{n-1} - b_n|, |s_n - s_{n-1}|\} < (1-\delta)C$ follows from the corresponding inequality for $x_k$ and $y_k$. □

*Proof of Lemma 2.* By Lemma 12, I can construct sequences of threshold types $b_n$ and $s_n$ so that corresponding sequences $x_k$ and $y_k$ defined by $x_k = b_{2k} - b_\infty$ and $y_k = s_\infty - s_{2k-1}$ for $k \in \mathbb{N}$ satisfy (30). Since $(x_k, y_k)$ is a positive trajectory and $\alpha^B(y) > 0$ whenever $y > 0$, from (30) it follows that $x_{k+1} - x_k = -\alpha^B(y_{k+1})(x_k + y_{k+1}) < 0$ for all $n \in \mathbb{N}$, and analogously, $y_{k+1} - y_k < 0$. Hence, $b_n$ and $s_n$ are monotone sequences. Since $(x_k, y_k)$ converges to $(0,0)$, the limits of $b_n$ and $s_n$ are $b_\infty$ and $s_\infty$, respectively.

The form of functions $\alpha^B(x)$ and $\alpha^S(y)$ guarantees that equations (25) and (27) hold. Hence, threshold types are indifferent between accepting an opponent's offer in the current round and rejecting it (and accepting in the following round). By Lemma 10, this is sufficient for the optimality of acceptance strategies given by thresholds $b_n$ and $s_n$. Moreover, recursive system (30) guarantees that the probability that threshold types assign to their offer being accepted in the next round is derived from the acceptance policy of the opponent. This completes the construction of the equilibrium strategies on the equilibrium path.

All deviations from acceptance strategies $b_n$ and $s_n$ are ignored. To deter deviations from offers $P^B$ and $P^S$ specify that after deviations from price offers $P^B$ and $P^S$, players switch to the punishing equilibrium of the deviator. By Theorem 8, in such an equilibrium the expected utility of the deviator is uniformly (over all types of the deviator) close to the reservation utility as $\delta$ converges to one. On the other hand, by following the equilibrium strategy any seller type $s \leq s^\omega_{b_0}$ gets at least $P^B - c(s)$, and any buyer type $b \geq b^\alpha_{s_0}$ gets at least $v(b) - P^S$. These utilities are bounded away from the reservation utility by (9). This proves that the constructed thresholds constitute a required continuation CSE whenever recursion (30) has a positive solution. □

#### 8.1.5 Proof of Theorem 2. Sufficiency

Consider a tuple $(b^*_t, s^*_t, q^B_t, q^S_t, T)$ as in the sufficiency part of Theorem 2. For any $\tilde{\varepsilon} > 0$, choose $\tilde{t} \in \mathbb{R}_+$ such that $b^*_{\tilde{t}} < b^*_T + \tilde{\varepsilon}$ and $s^*_{\tilde{t}} > s^*_T - \tilde{\varepsilon}$. Since $b^*_\infty \in (0,1)$ and $s^*_\infty \in (0,1)$, $b^*_T = s^*_T - \eta$ by (2). Therefore, I can choose $\tilde{t}$ sufficiently large so that

$$0 < b^*_{\tilde{t}} < 1 - \eta,\ \eta < s^*_{\tilde{t}} < 1, \text{ and } s^*_{\tilde{t}} \in [b^*_{\tilde{t}} - \eta, b^*_{\tilde{t}} + \eta). \tag{51}$$

By the strict versions of (7) and (8), I have $q^S_0 < \frac{v(1)+c(1)}{2}$ and $q^B_0 > \frac{v(0)+c(0)}{2}$ and so, by the monotonicity of $q^B_t$ and $q^S_t$,

$$\frac{v(0)+c(0)}{2} < q_t^B \leq q_t^S < \frac{v(1)+c(1)}{2}, \tag{52}$$

for all $t \in [0,T]$. For any time $t$, let $N_t^j \equiv \left\lfloor \frac{t}{\Delta_j} \right\rfloor$. There are three cases to consider: 1) $T=\infty$ and $q_T^S > q_T^B$, 2) $T<\infty$ and $q_T^S = q_T^B$, 3) $T=\infty$ and $q_T^S = q_T^B$.

**Case 1) $\mathbf{T}=\infty$ and $\mathbf{q_T^S} > \mathbf{q_T^B}$.** CSEs that I construct to approximate $(b_t^*, s_t^*, q_t^B, q_t^S)$ are in grim-trigger strategies. Players start the game by following the *main path* and continue following it so long as there were no detectable deviations in the past. If one of the sides detects deviation from the main path, then the play switches to the *punishing path of the deviator*.

**Construction of the main path** $(b_n^j, s_n^j, p_n^{Bj}, p_n^{Sj})$**.** Strategies on the main path are constructed separately for times before and after $\tilde{t}$. Since $c(s_T^*) < q_T^B < q_T^S < v(b_T^*)$, I can choose $\tilde{\varepsilon}$ small enough and $\tilde{t}$ large enough so that $v(b_T^* - \tilde{\varepsilon}) > v(b_T^*) - \tilde{\varepsilon}\ell > q_{\tilde{t}}^S$ and $c(s_T^* + \tilde{\varepsilon}) < c(s_T^*) + \tilde{\varepsilon}\ell < q_{\tilde{t}}^B$ where I use the Lipschitz continuity of $v(b)$ and $c(s)$ in the inequalities. Combining this with (52) leads to

$$\min\left\{c(s_T^* + \tilde{\varepsilon}), \frac{v(0)+c(0)}{2}\right\} < q_{\tilde{t}}^B < q_{\tilde{t}}^S < \min\left\{v(b_T^* - \tilde{\varepsilon}), \frac{v(1)+c(1)}{2}\right\}. \tag{53}$$

Let $b_{N_{\tilde{t}}^j} \equiv b_{\tilde{t}}^*$, $s_{N_{\tilde{t}}^j} \equiv s_{\tilde{t}}^*$. By (51) and (53), conditions of Lemma 2 are satisfied and so for $\Delta_j$ sufficiently small, there exists a continuation CSE for $n > N_{\tilde{t}}^j$ such that price paths are constant, $p_n^{Sj} = q_{\tilde{t}}^S$ and $p_n^{Bj} = q_{\tilde{t}}^B$, and $\max\left\{b_{N_{\tilde{t}}^j}^j - b_{N_{\tilde{t}}^j+2}^j, s_{N_{\tilde{t}}^j+2}^j - s_{N_{\tilde{t}}^j}^j\right\} < C\Delta_j$ for some $C>0$ independent of $\Delta_j$.

For $n \leq N_{\tilde{t}}^j - 1$, construct sequences $b_n^j, s_n^j, p_n^{Sj}, p_n^{Bj}$ as follows. For any integer $n \leq N_{\tilde{t}}^j - 1$, I define $b_n^j = b_{n\Delta_j}^*$ for even $n$ and $b_n^j = b_{n-1}^j$ for odd $n$. Analogously, for any integer $n \leq N_{\tilde{t}}^j - 1$, I define $s_n^j = s_{n\Delta_j}^*$ for odd $n$ and $s_n^j = s_{n-1}^j$ for even $n$. For any integer $n \leq N_{\tilde{t}}^j - 1$, I define $\alpha_n^{Sj}$ and $\alpha_n^{Bj}$ by (26) and (28). Given $b_n^j, s_n^j, \alpha_n^{Bj}, \alpha_n^{Sj}$, I construct price paths $p_n^{Bj}$ and $p_n^{Sj}$ starting from round $N_{\tilde{t}}^j - 1$ and proceeding backwards in time so that equations (25) and (27) are satisfied.

**Convergence.** Since $b_t^*$ is continuously differentiable, function $b_t^*$ is Lipschitz continuous with some modulus $\ell_1$ on $[0,\tilde{t}]$, and without loss of generality, let $C < \ell_1$. Hence, the extension $b_t^j$ of $b_n^j$ to continuous domain is Lipschitz-continuous with the same modulus $\ell_1$ on $[0,\tilde{t}]$. Therefore, $b_t^j$ converges to $b_t^*$ uniformly on $[0,\tilde{t}]$ as $\Delta_j \to 0$. Analogously, extension $s_t^j$ of $s_n^j$ to a continuous domain converges uniformly to $s_t^*$ on $[0,\tilde{t}]$.

Writing equation (25) with $n = N_t^j$, subtracting from both sides $e^{-2r\Delta_j}\left(v\left(b_{N_t^j}^j\right) - p_{N_t^j}^{Sj}\right)$

and dividing by $2\Delta_j$, I get

$$\frac{1-e^{-2r\Delta_j}}{2\Delta_j}\left(v\left(b^j_{N^j_t}\right)-p^{Sj}_{N^j_t}\right)=e^{-r\Delta_j}\frac{\alpha^{Sj}_{N^j_t}}{2\Delta_j}\left((1-e^{-r\Delta_j})v\left(b^j_{N^j_t}\right)-p^{Bj}_{N^j_t+1}+e^{-r\Delta_j}p^{Sj}_{N^j_t+2}\right)+e^{-2r\Delta_j}\frac{p^{Sj}_{N^j_t}-p^{Sj}_{N^j_t+2}}{2\Delta_j}. \tag{54}$$

Observe that for $n \leq N^j_{\tilde{t}}-1$,

$$\alpha^{Sj}_n=\max\left\{\frac{s^j_{n+1}-\max\{s^j_{n-1},s^{\alpha}_{b^j_n}\}}{s^{\omega}_{b^j_n}-\max\{s^{\Delta}_{n-1},s^{\alpha}_{b^j_n}\}},0\right\}\leq\frac{2\Delta_j\ell_1}{\tilde{\varepsilon}}, \tag{55}$$

and the same upper bound holds for $\alpha^{Bj}_n$. Therefore, by (54) for all $\Delta_j$ function $p^{Sj}_t$ is Lipschitz continuous with a common (for all $\Delta_j$) modulus of continuity, and hence, over a subsequence $p^{Sj}_n$ converges uniformly on $[0,\tilde{t}]$ to a Lipschitz continuous function $\tilde{q}^S_t$ with the same modulus of continuity. Taking the limit of (54) I get that $\tilde{q}^S_t$ satisfies equation (4). By the Picard-Lindelöf theorem the limit $\tilde{q}^S_t$ coincides with $q^S_t$. Therefore, $p^{Sj}_t$ converges uniformly to $q^S_t$ on $[0,\tilde{t}]$, and by an analogous argument, $p^{Bj}_t$ converges uniformly on $[0,\tilde{t}]$ to $q^B_t$.

*Claim* 12. *For $\hat{T}=\infty$, there exists $\underline{\Delta}>0$ such that for all $\Delta_j<\underline{\Delta}$, $p^{Bj}_n$ and $p^{Sj}_n$ are monotone for $n\leq N^j_{\tilde{t}}$.*

*Proof.* Observe that unless $\tilde{t}>\hat{T}$, in which case price paths are constant, $q^S_t$ is strictly decreasing on $[0,\tilde{t}]$. By the continuous differentiability of $q^S_t$ there exists $\tilde{\gamma}>0$ such that $\dot{q}^S_t<-\tilde{\gamma}$ on $[0,\tilde{t}]$. By the uniform convergence of $b^j_t$, $s^j_t$,$p^{Bj}_t$,$p^{Sj}_t$, (54) implies that $\frac{p^{Sj}_{N^j_t}-p^{Sj}_{N^j_t+2}}{2\Delta_j}$ converges uniformly to $\dot{q}^S_t$ and so $p^S_n$ is decreasing for sufficiently small $\Delta_j$. Analogously, $p^B_n$ is increasing for sufficiently small $\Delta_j$. *Q.E.D.*

Notice that $T_j=T=\infty$ for any CSE constructed. By the definition of $\tilde{t}$, $b^*_{\tilde{t}}\to b^*_T$ and $s^*_{\tilde{t}}\to s^*_T$ as $\tilde{t}\to T$. Therefore taking $\tilde{\varepsilon}$ to zero, and correspondingly $\tilde{t}$ to $T$ results in the desired sequence of approximating CSEs with the smooth limit $(b^*_t,s^*_t,q^B_t,q^S_t)$.

**Construction of the punishing path.** After deviations from the price paths of the main path, players switch to the deviator's punishing equilibrium described in Section 6. If a player deviates from the acceptance strategy, then the deviating side accepts in the next round and the other side ignores such deviation irrespective of whether it detects it or not. The next claim shows that such punishing paths deter deviations from the main path for sufficiently small $\Delta_j$.

*Claim* 13. *For $\Delta_j$ sufficiently small, there are no profitable deviations from the main path.*

There is a difference in the analysis of the incentives to deviate from the main path of buyer types below and above $b^*_T-\tilde{\varepsilon}$. On the one hand, buyer types below $b^*_T-\tilde{\varepsilon}$ expect that with probability one, one of the buyer's offers will be accepted by time $\tilde{t}$. Therefore, the strategies specified after time $\tilde{t}$ do not affect their incentives to deviate. On the other hand, buyer types above $b_T^* - \tilde{\varepsilon}$ could remain in the game after time $\tilde{t}$ and so, their incentives to deviate could be affected in the manner we specified the main path for $t > \tilde{t}$. The following two claims ensure that both groups of types do not have incentives to deviate.

*Claim* 14. *There exists $\hat{\ell}$ such that for any $\Delta_j$, $U_t^{Bj}(b)$ and $\mathcal{U}_t^B(b)$ are Lipschitz continuous in both arguments with modulus $\hat{\ell}$. Moreover, for any $\varepsilon > 0$,*

$$\max_{t\in[0,\tilde{t}],b\in[0,b_T^*-\tilde{\varepsilon}]} |U_t^{Bj}(b) - \mathcal{U}_t^B(b)| < \varepsilon$$

*for sufficiently small $\Delta_j$.*

*Proof.* Let $n_b^j$ be the round during which buyer type $b$ accepts the seller's offer if he follows the strategy $b_n^\Delta$. Consider two buyer types $b$ and $b'$. Observe that

$$\begin{aligned}
U_t^{Bj}(b) = \mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \cap S_{b'}, s \geq s_t^j, n_b^j]\frac{|S_b \cap S_{b'} \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|}+ \\
\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \backslash S_{b'}, s \geq s_t^j, n_b^j]\frac{|(S_b \backslash S_{b'}) \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|} \geq \\
\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \cap S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|S_b \cap S_{b'} \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|}+ \\
\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \backslash S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|(S_b \backslash S_{b'}) \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|} = \\
U_t^{Bj}(b') + \mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-v(b'))|s \in S_b \cap S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|S_b \cap S_{b'} \cap [s_t^j,1]|}{|S_{b'} \cap [s_t^j,1]|}- \\
\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b')-p)|s \in S_{b'} \backslash S_b, s \geq s_t^j, n_{b'}^j]\frac{|S_{b'} \backslash S_b \cap [s_t^j,1]|}{|S_{b'} \cap [s_t^j,1]|}+ \\
\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \backslash S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|S_b \backslash S_{b'} \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|} \geq \\
U_t^{Bj}(b') - \ell|b-b'| - \Sigma|b-b'|,
\end{aligned}$$

where the equalities exist by application of the law of total expectation to $U_t^{Bj}(b)$ and $U_t^{Bj}(b')$, the first inequality arises from the fact that buyer type $b$ prefers to accept in round $n_b^j$ rather than in round $n_{b'}^j$, and the second inequality comes about by the Lipschitz continuity of $v(b)$ and the upper bound on the size of the surplus. Therefore, $U_t^{Bj}(b)$ is Lipschitz continuous in $b$ with modulus $\ell + \Sigma$.

Now for fixed $b$ consider even integers $n < n' < n_b^j$. I have

$$U_n^{Bj}(b) = \sum_{m=n/2+1}^{n'/2-1} e^{-r\Delta_j(2m+1-n)} \frac{s_{2m+1}^j - s_{2m-1}^j}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} \left(v(b) - p_{2m}^{Bj}\right) + e^{-r\Delta_j(n'-n)} \frac{s_b^\omega - \max\{s_b^\alpha, s_{n'}^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} U_{n'}^{Bj}(b). \tag{56}$$

Notice that

$$0 < 1 - e^{-r\Delta_j(n'-n)} \frac{s_b^\omega - \max\{s_b^\alpha, s_{n'}^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} \leq 1 - (1 - r\Delta_j(n'-n)) \left(1 - \frac{\max\{s_b^\alpha, s_{n'}^j\} - \max\{s_b^\alpha, s_n^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}}\right) =$$

$$r\Delta_j(n'-n) + (1 - r\Delta_j(n'-n)) \frac{s_{n'}^j - s_n^j}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} \leq r\Delta_j(n'-n) + \frac{2\Delta_j(n'-n)\ell_1}{\tilde{\varepsilon}}. \tag{57}$$

By (55) and (57), (56) implies

$$|U_n^{Bj}(b) - U_{n'}^{Bj}(b)| \leq \Sigma \left(\frac{4\Delta_j(n'-n)\ell_1}{\tilde{\varepsilon}} + r\Delta_j(n'-n)\right) \equiv \Delta_j(n'-n)\ell_2. \tag{58}$$

Since function $U_t^{Bj}(b)$ is piecewise linear and, by inequality (58) its slope does not exceed $\ell_2$, $U_t^{Bj}(b)$ is Lipschitz continuous in $t$ with modulus $\ell_2$. Hence, $U_t^{Bj}(b)$ is Lipschitz continuous in both arguments with modulus $\hat{\ell} \equiv \ell + \Sigma + \ell_2$. The proof of the Lipschitz continuity of $\mathcal{U}_t^B(b)$ is analogous.

The sequence of functions $U_t^{Bj}(b)$ are Lipschitz continuous for all $j$ with common modulus $\hat{\ell}$. Hence, they converge uniformly to some limit that is Lipschitz continuous with the same modulus $\hat{\ell}$. Moreover, $U_{N_t}^{Bj}(b)$ converges pointwise to $\mathcal{U}_t^B(b)$ for $b \in [0, b_T^* - \tilde{\varepsilon}]$ by construction by the dominated convergence theorem. Hence, $U_t^{Bj}(b)$ converges uniformly to $\mathcal{U}_t^B(b)$ on $t \in [0, \tilde{t}]$ and $b \in [0, b_T^* - \tilde{\varepsilon}]$. *Q.E.D.*

*Claim* 15. *There exists $\underline{\Delta} > 0$ and $u > 0$ such that for all $\Delta_j < \underline{\Delta}$,*

$$\min_{t\in[0,\tilde{t}], b\in(b_T^* - \tilde{\varepsilon}, 1]} U_t^{Bj}(b) - \max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\} > u.$$

*Proof.* The buyer could accept $p_n^{Sj}$ in any even round $n$. Moreover, the buyer could accept seller offer $q_{\tilde{t}}^S$ in round $N_{\tilde{t}}^j$. Therefore,

$$U_n^{Bj}(b) \geq \max\{v(b) - p_n^{Sj}, e^{-r\tilde{t}}(v(b) - q_{\tilde{t}}^S)\}.$$

Denote $u_1 \equiv e^{-r\tilde{t}}(v(b_T^* - \tilde{\varepsilon}) - q_{\tilde{t}}^S)$, and $u_1 > 0$ by (53). For any $b > b_T^* - \tilde{\varepsilon}$, $e^{-r\tilde{t}}(v(b) - q_{\tilde{t}}^S) \geq u_1 > 0$.

I next show that for $u_2 \equiv \frac{1}{4}(\frac{v(1)+c(1)}{2} - q_0^S) > 0$ (by (52)), we have $p_n^{Sj} < \frac{v(1)+c(1)}{2} - u_2$ for $\Delta_j$ sufficiently small. By the convergence of $p_t^{Bj}$ to $q_t^B$ on $[0, \tilde{t}]$, there exists $\underline{\Delta} > 0$ such that for all $\Delta_j < \underline{\Delta}$, $p_0^{Sj} < q_0^S + u_2$, and so $p_n^{Sj} \leq p_0^{Sj} < \frac{v(1)+c(1)}{2} - u_2$. I complete the proof by taking $u \equiv \min\{u_1, u_2\} > 0$. *Q.E.D.*

*Proof of Claim* 13. By Claim 14, continuation utilities of buyer types in $[0, b_T^* - \tilde{\varepsilon}]$ from following the main path converge uniformly (in type and time) to $\mathcal{U}_t^B(b)$. By the strict version of inequality (7), there exists $\varepsilon > 0$ such that $\mathcal{U}_t^B(b) > \max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\} + \varepsilon$ for all $b$ and $t \in [0, \tilde{t}]$. By Claim 15, continuation utilities of buyer types in $(b_T^* - \tilde{\varepsilon}, 1]$ from following the main path are greater than $\max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}$ by at least $u > 0$, for sufficiently small $\Delta$. If the buyer detected that the seller deviated from the acceptance path, then his continuation utility in even round $n$ is $e^{-r\Delta}(v(b) - p_{n+1}^B) \geq e^{-r\Delta}(v(b) - p_T^B) > \max\left\{0, v(b) - \frac{v(1)+c(1)}{2}\right\} + \varepsilon$, for sufficiently small $\Delta$.

By Theorem 6, for any $\varepsilon > 0$, the continuation utility of any type of punished player is at most $\varepsilon$ away from the reservation utility $\max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}$, for sufficiently small $\Delta_j$. Therefore, deviations from the price paths constructed are not profitable for buyer types. By Claim 12, the constructed price paths are monotone and so deviations from the acceptance strategies are not optimal according to Lemma 10. The proof for the seller is symmetric.

*Q.E.D.*

**Case 2) $\mathbf{T < \infty}$ and $\mathbf{q_T^S = q_T^B}$.** Let $\tilde{t} = T, \tilde{\varepsilon} = 0$, and the construction of the main path for case 1 is repeated with the difference that after time $T$ trade stops and there are no types remaining. By the argument analogous to case 1, it can be verified that the constructed main path can be supported by the punishing path and that the strategies describing the main path converge almost everywhere to the corresponding limits on $[0, T]$. Moreover, for all $\Delta_j$, $T_j = T$, and $b_T = b_T^*$, $s_T = s_T^*$ which completes the analysis of case 2.

**Case 3) $\mathbf{T = \infty}$ and $\mathbf{q_T^S = q_T^B}$.** I first construct the following approximation of $(b_t^*, s_t^*, q_t^B, q_t^S)$. For any $\tilde{t} \in \mathbb{R}_+$, let $\hat{b}_t = b_t^* - \frac{t}{\tilde{t}}\left(b_{\tilde{t}}^* - b_\infty^*\right)$ and $\hat{s}_t = s_t^* + \frac{t}{\tilde{t}}\left(s_\infty^* - s_{\tilde{t}}^*\right)$ and construct price offers $\hat{q}_t^S$ and $\hat{q}_t^B$ satisfying (3), (4) and $\hat{q}_t^B = \hat{q}_t^S = q_T^B = q_T^S$. I can proceed as in case 2 to construct an approximating sequence of CSEs of the limit $(\hat{b}_t, \hat{s}_t, \hat{q}_t^B, \hat{q}_t^S, \tilde{t})$. By construction, as $\tilde{t} \to \infty$, $\hat{b}_t$ and $\hat{s}_t$ converge uniformly to $b_t^*$ and $s_t^*$, respectively, as well as their derivatives converging uniformly to the corresponding derivatives of $b_t^*$ and $s_t^*$. By Theorem 1.1 of Freidlin and Wentzell (1984) price offers $\hat{q}_t^S$ and $\hat{q}_t^B$ converge to $q_t^S$ and $q_t^B$, respectively. Moreover, $T_j = \tilde{t}$ converges to $T = \infty$, and $\hat{b}_\infty = \hat{b}_{\tilde{t}} = b_\infty^*$ and $\hat{s}_\infty = \hat{s}_{\tilde{t}} = s_\infty^*$. □

## 8.2 Proofs for Section 4

*Proof of Theorem 3.* I first construct price offers of the seller. Let $q_Z^B = q_Z^S$. By Lemma 2, for sufficiently small $\Delta$, there exists a CSE with constant offers on the equilibrium path $q_{Z-1}^B$ and $q_Z^S$, and acceptance strategies $b_n^{Z-1}$ and $s_n^{Z-1}$ such that $b_\infty^{Z-1} = b^{Z-1}$ and $s_\infty^{Z-1} = s^{Z-1}$. For all

$z = 1, \ldots, Z-2$, let $\hat{s}^z \equiv b^z - \eta$ and $q_z^S$ be such that

$$q_z^S - c(\hat{s}^z) = \delta(q_z^B - c(\hat{s}^z)). \tag{59}$$

Now, equilibrium strategies are described as follows. In the first round, for $z = 2, \ldots, Z-1$ seller types in $[\hat{s}^{z-1}, \hat{s}^z]$ make offer $q_z^S$. In the second round, buyer types in $[\hat{b}^0, \hat{b}^1]$ reject offer $q_2^S$, types in $[\hat{b}^{z-1}, \hat{b}^z], z = 2, \ldots, Z-1$ accept $q_z^S$ and make counter-offer $q_z^B$ to $q_{z+1}^S$, buyer types in $[\hat{b}^{Z-1}, \hat{b}^Z]$ accept $q_Z^S$. After the first two rounds, the remaining types play a corresponding continuation CSE with all subsequent price offers of players equal to their initial price offers. By Lemma 2, I construct a CSE with constant offers on the equilibrium path $q_{z-1}^B$ and $q_z^S$, and acceptance strategies $s_n^z$ and $b_n^z$ such that $b_\infty^z = b^z$ and $s_\infty^z = s^z$. Denote $\hat{b}^0 = \hat{s}^1 = 0$, $\hat{b}^{Z-1} = \hat{s}^{Z-1} = 1$, and $\hat{b}^z = b_2^z$ for $z = 1, \ldots, Z-1$. After any deviation from offers in $Q_z^B$ and $Q_z^S$ players switch to the punishing equilibrium of the deviator. If the seller deviates to a higher price in $Q_Z^S$, then such price is rejected by all remaining buyer types. If the seller deviates to a lower price in $Q_Z^S$, then such price is accepted by all remaining buyer types. If the seller deviated from offering the prescribed equilibrium price to some other offer in $Q_Z^S$ or deviated from the acceptance strategy in the past, then in subsequent rounds, she returns to following equilibrium price offer and acceptance strategy. The buyer strategy after deviations to prices in $Q_Z^B$ and the acceptance strategy is defined analogously.

Observe that if a seller type $\hat{s}^z, z = 2, \ldots, Z-1$ makes a lower offer $q_z^S$, then it is accepted with probability one. Indeed, since $b^z - b^{z-1} > 4\eta$, $\hat{s}^z = b^z - \eta$ and $\hat{b}^{z-1} < b^{z-1} + 2\eta$, we have $\hat{b}^{z-1} < \hat{s}^z - \eta$ and so, all buyer types in $B_{\hat{s}^z}$ accept offer $q_z^S$. By (59), seller type $\hat{s}^z$ is indifferent between offering $q_z^S$ that is accepted for sure and offering $q_{z+1}^S$ that is rejected for sure and accepting the buyer's offer $q_z^B$. By the single-crossing property of the payoffs, seller types above $\hat{s}^z$ strictly prefer the acceptance of offer $q_z^S$ by the buyer in two rounds, and seller type below $\hat{s}^z$ strictly prefer the acceptance of $q_z^S$ in the next round. By the choice of $q_z^S$ and $q_z^B$ and Theorem 6, no player prefers to deviate from the equilibrium price offers to offers outside $Q_Z^S$ and $Q_Z^B$ for $\Delta$ sufficiently small.

The seller does not have incentives to deviate to other offers in $Q_Z^S$ for sufficiently small $\Delta$. Indeed, suppose that the seller type $s$ makes offer $q_z^S$ in the first round. A deviation to a lower offer in $Q_Z^S$ is worse than accepting the buyer price offer for sufficiently small $\Delta$. A deviation to a higher offer in $Q_Z^S$ in one round will not lead to a positive profit. The buyer expects that the seller will decrease price after the round with the deviation, and hence, for sufficiently small $\Delta$, such offer will be rejected by the buyer.

After the first two rounds the game continues only if offers $q_z^S$ and $q_{z-1}^B$ were made for some $z = 2, \ldots, Z$. Then only buyer types are below $\hat{b}^z$ and seller types above $\hat{s}^{z-1}$ remain in the game. Such types are playing a continuation CSE constructed by Lemma 2 with offers $q_z^S$ and $q_{z-1}^B$. Therefore, the probability that the game continues for longer than three rounds is at most $\frac{4\eta^2(Z-1)}{\eta(2-\eta)}$. At the same time, continuation CSEs constructed by Lemma 2 have no almost sure upper bound on the equilibrium delay and so, there is no almost sure upper bound on the equilibrium delay in the constructed segmentation equilibria. $\square$

### 8.3 Proofs for Section 5

*Proof of Theorem 4 .* I apply Theorem 3 with price offers and segments defined as follows. Fix $\varepsilon > 0$ and choose $b^1 = \sqrt{\eta}$, $b^{z+1} = b^z + \sqrt{\eta}$ and $q_z^B = \frac{v(b^{z-1})+c(s^z)}{2}$. Then $Z = 1 + \left\lfloor \frac{1-\eta}{\sqrt{\eta}} \right\rfloor$. I consider only outcomes for types that trade in the first three rounds. As shown in the proof of Theorem 3, the probability of such types is at least $1 - \frac{4\eta^2(Z-1)}{\eta(2-\eta)}$ which converges to one as $\eta \to 0$, since $Z \sim \frac{1}{\sqrt{\eta}}$. Moreover, for such types, $|N\Delta| \leq 2$ and $\left|p - \frac{v(b)+c(s)}{2}\right| \leq \frac{1}{2}|v(b^{z-1}) - v(b)| + \frac{1}{2}|c(s^z) - c(s)| \leq \frac{\ell\sqrt{\eta}}{2} \to 0$ as $\eta \to 0$. This proves, the desired convergence in probability of segmentation equilibria outcomes to the Nash outcome. $\square$

*Proof of Theorem 5.* To prove the first statement, consider a sequence of CSEs indexed by $(\Delta, \eta) \to (0, 0)$. For any $j \in \mathbb{N}$, let $b^j$ be the lowest weak buyer type, and $s^j$ be the highest weak seller type in the CSE of the game with the length of bargaining round $\Delta_j$ and the individual uncertainty parameter $\eta_j$. Denote $w^j \equiv \frac{1}{2}(b^j + s^j) \in [0, 1]$. Then there exists $\omega^* \in [0, 1]$ such that $w^j$ converges to $\omega^*$ over subsequence. Therefore, for any $\varepsilon_2 > 0$, far enough in the sequence all buyer types above $\min\{1, \omega^* + \varepsilon_2\}$ and all seller types below $\max\{0, \omega^* - \varepsilon_2\}$ are weak types. I consider only outcomes for these types, and I cover all the types but type $\omega^*$ by choosing $\varepsilon_2$ sufficiently small.

Any weak type knows at what time and at what price trade will happen, since the probability of the opponent's concession for weak types is zero. In a CSE corresponding to $(\Delta_j, \eta_j)$, for buyer $b > \min\{1, \omega^* + \varepsilon_2\}$, let $t_b^j$ and $p_b^j$ be the time and the price at which such type trades and define analogous quantities $t_s^j$ and $p_s^j$ for sellers $s < \max\{0, \omega^* - \varepsilon_2\}$. By the single crossing property of the payoffs, $t_b^j$ is decreasing and $t_s^j$ is increasing and so, $p_b^j$ is decreasing and $p_s^j$ is increasing. Therefore, a sequence of four monotone functions has a pointwise converging subsequence by Helly's theorem and the limits $(t_b^*, t_s^*, p_b^*, p_s^*)$ exist. For any weak buyer types $b$ and $b'$, buyer type $b$ prefers accepting at time $t_b^j$ to accepting at $t_{b'}^j$, $e^{-rt_b^j}\left(v(b) - p_b^j\right) \geq e^{-rt_{b'}^j}\left(v(b) - p_{b'}^j\right)$. Hence, in the limit $e^{-rt_b^*}(v(b) - p_b^*) \geq e^{-rt_{b'}^*}(v(b) - p_{b'}^*)$, which is condition (12) for buyer and by the same logic condition (14) obtains. Conditions (13) and (15) follow from Lemma 1. Condition 3 follows from the monotonicity of price paths in the definition of the CSE.

To prove the second statement of the theorem, let $\bar{U}^B(\omega) \equiv \bar{P}(\omega)v(\omega) - \bar{X}(\omega)$, $\bar{U}^S(\omega) \equiv \bar{X}(\omega) - \bar{P}(\omega)c(\omega)$, $U^B(b) \equiv P^B(b)v(b) - X^B(b)$, $U^S(s) \equiv X^S(s) - P^S(s)c(s)$. Notice that there is one-to-one mapping between $(\bar{P}^B, \bar{P}^S, \bar{U}^B, \bar{U}^S)$ and $(\bar{P}^B, \bar{X}^B, \bar{P}^S, \bar{X}^S)$. I begin with a preliminary observation, which follows from the argument in Lemma 2 of Myerson (1981).

*Claim* 16. *Condition* (12) *is equivalent to*

$$\bar{P}(\omega) \quad \geq \quad \bar{P}(\omega') > 0, \tag{60}$$

$$\bar{U}^B(\omega) \quad = \quad \bar{U}^B(\omega') + \int_{\omega'}^{\omega} \bar{P}(w)dv(w), \tag{61}$$

*for any* $\omega \geq \omega' > \omega^*$.

Now, consider $\mathcal{O} = (\bar{P}^B, \bar{P}^S, \bar{U}^B, \bar{U}^S)$ satisfying conditions of theorem. , I construct a sequence of CSEs indexed by $j \in \mathbb{N}$ with $\mathcal{O}^j = (P^{Bj}, P^{Sj}, U^{Bj}, U^{Sj})$ such that $\mathcal{O}^j$ converges a.e. to $\mathcal{O}$, and $(\Delta_j, \eta_j) \to (0,0)$.

Consider a monotone sequence $\eta_j \to 0$. Suppose that $\omega^* \in (0,1)$, and without loss of generality, suppose that

$$\eta_j < \min\left\{\frac{\omega^*}{2}, \frac{1-\omega^*}{2}\right\}$$

for all $j \in \mathbb{N}$. When $\omega^*$ equals 0 or 1, the argument below is first carried for $\tilde{\omega}^*$ equal to $\tilde{\varepsilon}$ or $1-\tilde{\varepsilon}$, respectively, for some $\tilde{\varepsilon} > 0$, and then I take $\tilde{\varepsilon} \to 0$.

Define $t_b^* \equiv -\frac{1}{r}\ln \bar{P}(b)$ for $b > \omega^*$ and $t_s^* \equiv -\frac{1}{r}\ln \bar{P}(s)$ for $s < \omega^*$. By (60) in Claim 16, function $\bar{P}(b)$ is increasing in type $b$ for $b > \omega^*$ and so, $t_b^*$ is decreasing in $b$. Analogously, $t_s^*$ is increasing in $s$. Consider inverse functions $b_t^* \equiv \inf\{b \in [0,1] : t_b^* \leq t\}$ and $s_t^* \equiv \sup\{s \in [0,1] : t_s^* \leq t\}$. Since $\bar{P}(\omega^* + 0) = \bar{P}(\omega^* - 0)$ and $\bar{P}(\omega) > 0$ for all $\omega \neq \omega^*$ (by condition (16)), I can choose $\tau_j < \infty$ to be the minimal $\tau_j$ such that $b_{\tau_j}^* - s_{\tau_j}^* \leq \eta_j$ and, in particular,

$$0 < b_{\tau_j}^* < \omega^* + \eta_j \leq 1 - \eta_j \text{ and } 1 > s_{\tau_j}^* > \omega^* - \eta_j \geq \eta_j. \tag{62}$$

Let $T \equiv t_{\omega^*}^*$ and observe that $\tau_j \to T$ as $j \to \infty$.

**Construction of CSE strategies.** I construct a CSE by the same scheme as in the proof of Theorem 2. Since $v(b)$ and $c(s)$ are continuous and $v(\omega^*) - c(\omega^*) \geq \xi > 0$, $v(\omega^* - \eta_j) > c(\omega^* + \eta_j)$ for sufficiently small $\eta_j$. If $\bar{X}(b_{\tau_j}^*) = \bar{X}(s_{\tau_j}^*)$, specify that at time $\tau_j$ all remaining types trade at price $\bar{X}(b_{\tau_j}^*)/\bar{P}(b_{\tau_j}^*)$ and the construction is carried as in case 2 in the proof of Theorem 2. If $\bar{X}(b_{\tau_j}^*) > \bar{X}(s_{\tau_j}^*)$, then I proceed as in case 1 in the proof of Theorem 2. I define $P^{Sj} \equiv \bar{X}(b_{\tau_j}^*)/\bar{P}(b_{\tau_j}^*) - \varepsilon_j$, $P^{Bj} \equiv \bar{X}(s_{\tau_j}^*)/\bar{P}(s_{\tau_j}^*) + \varepsilon_j$ where $\varepsilon_j \in [0, 2^{-j}]$ is small enough so that condition (9) in Lemma 2 is satisfied. By (62), for times after time $\tau_j$, the continuation equilibrium can be constructed by Lemma 2. For the rounds before time $\tau_j$, the acceptance functions $b_t^j$ and $s_t^j$, and price offers $p_t^{Bj}$ and $p_t^{Sj}$ are constructed as in the proof of the sufficiency part of Theorem 2. I choose $\Delta_j$ sufficiently small so that for given $\eta_j$, the constructed main path can be supported by the punishing path in the construction in the proof of Theorem 2.

**Convergence.** By the construction, for $b \in (b^*_{\tau_j}, 1]$ the difference between the type $b$'s acceptance time and $t^*_b$ is at most $2\Delta_j$ and so, as $j \to \infty$, $P^{Bj}(b)$ converges uniformly to $\bar{P}(b)$ for such types. By the argument analogous to Claim 16, for weak types $b \in (b^*_{\tau_j - 2\Delta_j}, 1]$,

$$U^{Bj}(b) = U^{Bj}(b^*_{\tau_j - 2\Delta_j}) + \int\limits_{b^*_{\tau_j - 2\Delta_j}}^{b} P^{Bj}(b) dv(b).$$

As $j \to \infty$, $U^{Bj}(b^*_{\tau_j - 2\Delta_j}) = e^{-r\tau_j}(v(b^*_{\tau_j - 2\Delta_j}) - P^{Bj})$ converges to $\bar{U}^B(b^*_\tau)$ and so, by the dominated convergence theorem, $U^{Bj}(b)$ converges to $\bar{U}^B(b)$ for $b \in (b^*_{\tau_j}, 1]$. By the integral formula, $U^{Bj}(b)$ and $\bar{U}^B(b)$ are Lipschitz continuous with modulus one and so, $U^{Bj}(b)$ converges uniformly to $\bar{U}^B(b)$ on $(b^*_{\tau_j}, 1]$.

Now for seller types $s \in [\omega^* + 2\eta_j, 1]$,

$$P^{Sj}(s) = \frac{1}{|B_s|} \int\limits_{B_s} P^{Bj}(b) db$$

and

$$U^{Sj}(s) = \frac{1}{|B_s|} \int\limits_{B_s} (P^{Bj}(b)(v(b) - c(s)) - U^{Bj}(b)) db.$$

By the monotonicity of $P^{Bj}(b)$, for $s \in [\omega^* + 2\eta_j, 1]$, $P^{Bj}(b^\alpha_s) \leq P^{Sj}(s) \leq P^{Bj}(b^\omega_s)$ and so, by the uniform convergence of $P^{Bj}(b)$ on $(b^*_\tau, 1]$, $\bar{P}^B(s - \eta_j) - \eta_j \leq P^{Sj}(s) \leq \bar{P}^B(s + \eta_j) + \eta_j$ for $\Delta_j$ sufficiently small. As $\eta_j \to 0$, $P^{Sj}(s)$ converges to $\bar{P}^S(s)$ for a.e seller type above $\omega^*$.[56] Further,

$$P^{Bj}(b^\alpha_s)(v(b^\alpha_s) - c(s)) - \frac{1}{|B_s|} \int\limits_{B_s} U^{Bj}(b) db \leq U^{Sj}(s) \leq P^{Bj}(b^\omega_s)(v(b^\omega_s) - c(s)) - \frac{1}{|B_s|} \int\limits_{B_s} U^{Bj}(b) db,$$

and by the uniform convergence of $P^{Bj}(b)$ and $U^{Bj}(b)$ on $(b^*_\tau, 1]$, for $\Delta_j$ sufficiently small,

$$\bar{P}^B(s - \eta_j)(v(s - \eta_j) - c(s)) - \frac{1}{|B_s|} \int\limits_{B_s} \bar{U}^B(b) db - \eta_j \leq U^{Sj}(s),$$

$$\bar{P}^B(s + \eta_j)(v(s + \eta_j) - c(s)) - \frac{1}{|B_s|} \int\limits_{B_s} \bar{U}^B(b) db + \eta_j \geq U^{Sj}(s),$$

for all $s \in [\omega^* + 2\eta_j, 1]$. As $\eta_j \to 0$, $\frac{1}{|B_s|} \int\limits_{B_s} \bar{U}^B(b) db \to \bar{U}^B(s)$ for a.e. seller type in $[\omega^* + 2\eta_j, 1]$ and so, $U^S(s)$ converges to $\bar{U}^S(s)$ for such types. The argument for types below $\omega^*$ is symmetric. Therefore, I constructed the required sequence of CSEs. $\square$

[56] The convergence is not guaranteed only at discontinuity points of $\bar{P}^S(s)$. By Claim 16, $\bar{P}^S(s)$ is monotone on $(\omega^*, 1]$ and the set of its discontinuity points is at most countable.

## 8.4 Proofs for Section 6

**Lemma 13.** *Suppose that for some $\underline{b} \in [0,1]$ beliefs of buyer types above $\underline{b}$ and seller types above $s^\alpha_{\underline{b}}$ are described by (17) and (20). Then the following strategies are the equilibrium strategies for such buyer and seller types. After any history, buyer type $b$ in the interval $(\underline{b}, 1]$ accepts price offer less than or equal to $\check{P}^B(b)$. Otherwise, such type makes counter-offer $\check{A}^B(b)$. After any history seller type $s$ in the interval $(s^\alpha_{\underline{b}}, 0]$ accepts price offer greater than or equal to $\check{P}^S(s)$. Otherwise, such type makes counter-offer $\check{A}^S(s)$. Functions $\check{P}^B(b)$, $\check{P}^S(s)$, $\check{A}^B(b)$, $\check{A}^S(s)$are given by*

$$\check{P}^B(b) = \begin{cases} (1-e^{-r\Delta})v(b) + e^{-r\Delta}\check{P}^S(0) \\ \frac{v(b)+e^{-r\Delta}c(b-\eta)}{1+e^{-r\Delta}} \end{cases} \quad \check{A}^B(b) = \begin{cases} \check{P}^S(0), & \text{for } b \in [0,\eta), \\ \frac{c(b-\eta)+e^{-r\Delta}v(b)}{1+e^{-r\Delta}}, & \text{for } b \in [\eta, 1]; \end{cases}$$

$$\check{P}^S(s) = \begin{cases} \frac{c(s)+e^{-r\Delta}v(s+\eta)}{1+e^{-r\Delta}} \\ (1-e^{-r\Delta})c(s) + e^{-r\Delta}\check{P}^B(1) \end{cases} \quad \check{A}^S(s) = \begin{cases} \frac{v(s+\eta)+e^{-r\Delta}c(s)}{1+e^{-r\Delta}}, & \text{for } s \in [0, 1-\eta], \\ \check{P}^B(1), & \text{for } s \in (1-\eta, 1]. \end{cases}$$

*Proof of Lemma 13.* Consider buyer types in the interval $[\underline{b}, 1] \cap [\eta, 1]$. Buyer type $b$ in such interval puts probability one on seller type $s^\alpha_b$ by (17), while seller type $s^\alpha_b$ puts probability one on type $b$ by (20). By Rubinstein (1982) strategies of these two types given in Lemma 13 constitute the subgame perfect equilibrium of the complete information game with valuation $v(b)$ and cost $c(s^\alpha_b)$.

Now consider seller types $s \in [s^\alpha_{\underline{b}}, 1] \cap (1-\eta, 1]$ that put probability one on buyer type 1. Buyer type 1, in turn, puts probability one on seller $1-\eta$ is willing to pay $\check{P}^B(1)$. Since $\check{P}^B(1) > c(1)$, seller types $s \in (1-\eta, 1]$ make price offer $\check{P}^B(1)$. Moreover, they are willing to pay up to $\check{P}^S(s)$ given by $\check{P}^S(s) - c(s) = \delta\left(\check{P}^B(1) - c(s)\right)$. The argument for buyer types $b \in [\underline{b}, 1] \cap [0, \eta)$ is symmetric. □

### 8.4.1 Existence of the Punishing Equilibrium

**Lemma 14.**

*Proof of Lemma 3.* The analysis of this subgame is standard, and I only sketch the argument. I start by constructing a PBE in a game between seller type 0 and buyer types in $[0, \eta]$, in which the buyer is restricted to either accept the last seller price offer or make counter-offer $\frac{\delta v(0)+c(0)}{1+\delta}$. I use the analysis of Fudenberg, Levine, and Tirole (1985) to construct a PBE in such game described by two functions $P^0(b)$ and $t_{\beta,p}$ and $\bar{\beta} \in [0, \eta]$ such that

1. buyer type $b$ accepts any price offer below $P^0(b)$ and makes counter-offer $\frac{\delta v(0)+c(0)}{1+\delta}$ otherwise;

2. given the highest buyer type $\beta > \bar{\beta}$ and previous price offer $p$, seller type 0 randomized

between the lowest types of the buyer to whom she allocates in the current round according to $t_{\beta,p} \in \Delta(\mathbb{R})$;

3. for $\beta \leq \bar{\beta}$, seller type 0 accepts offer $\frac{\delta v(0)+c(0)}{1+\delta}$;

4. $P^0(b)$ is strictly increasing and left-continuous.

The argument in Fudenberg, Levine, and Tirole (1985) should be slightly modified to incorporate the possibility that all buyer types pool on a particular price offer that could be accepted by seller type 0. I start by showing that for $\beta$ smaller than some $\bar{\beta}$ the seller prefers to accept $\frac{\delta v(0)+c(0)}{1+\delta}$ rather than continue screening. This implies that there is a finite date after which bargaining ends with probability one by the argument analogous to Lemma 3 in Fudenberg, Levine, and Tirole (1985). I follow the steps in their proof of Proposition 1 to construct equilibrium strategies by backward induction on beliefs starting from beliefs supported by $[0,\beta], \beta < \bar{\beta}$ with the only difference that instead of asking price $v(0)$, the seller accepts price offer $\frac{\delta v(0)+c(0)}{1+\delta}$ for such beliefs. This gives the desired equilibrium in the game with restricted buyer price offers. Note that by the argument from the Theorem 3 in Gul, Sonnenschein and Wilson (1986) the Coase Conjecture holds for such game, and for any $\varepsilon > 0$, after any history the first price offer of the seller does not exceed $\frac{v(0)+c(0)}{2} + \varepsilon$ for $\delta$ sufficiently close to one.

To support the constructed equilibrium as an equilibrium in the game with unrestricted buyer price offers specify the following punishment for detectable deviations of the buyer. If the buyer deviates and makes an offer different from $\frac{\delta v(0)+c(0)}{1+\delta}$, then the seller puts probability one on the buyer type $\eta$ and the game proceeds as in the unique subgame perfect equilibrium of the game with complete information with the seller cost equal $c(0)$ and the buyer valuation equal $v(\eta)$. Then trade happens immediately at a price that is close to $\frac{v(\eta)+c(0)}{2}$ for $\delta$ close to one. By the Coase Conjecture the first seller price offer is close to $\frac{v(0)+c(0)}{2}$ for $\delta$ close to one, making the deviation of the buyer non-profitable. $\square$

**Lemma 15.** *Suppose $t_\beta(s)$ is a best-reply to willingness to pay $P(b)$. Then $R_\beta(s)$ is non-decreasing in $\beta$ , satisfying: for $0 \leq \beta'' < \beta' \leq 1$ we have $0 < R_{\beta'}(s) - R_{\beta''}(s) \leq \Sigma(\beta' - \beta'')$ whenever $R_{\beta'}(s) > 0$, and $R_{\beta'}(s) = R_{\beta''}(s) = 0$ whenever $R_{\beta'}(s) = 0$. Moreover, $R_\beta(s)$ is Lipschitz-continuous in both $\beta$ and $s$ of modulus $\ell_R \equiv \ell + \Sigma$.*

*Proof.* The first part of Lemma 15 follows from Lemma A.2 in Ausubel, Deneckere (1989). To show that $R_\beta(s)$ is Lipschitz continuous consider two seller types $s$ and $s'$. Let $R_\beta(s,s')$ be the value function of seller type $s$ from following $t_\beta(s')$. Since seller type $s$ prefers policy $t_\beta(s)$ to $t_\beta(s')$, $R_\beta(s) \geq R_\beta(s,s')$. Let $p_s^{s'}$ and $q_s^{s'}$, respectively, be discounted transfer and probability of allocation, respectively, when seller type $s$ follows optimal policy of seller type $s'$ (and we write $p_s$ for $p_s^s$ and $q_s$ for $q_s^s$). Then

$$R_\beta(s,s') = p_s^{s'} - q_s^{s'} c(s) \geq p_s^{s'} - q_s^{s'} c(s') - |c(s) - c(s')| \geq$$

$$p_{s'} - q_{s'}c(s') - (\ell + \Sigma)|s - s'| = R_\beta(s') - (\ell + \Sigma)|s - s'|.$$

The first inequality is by $q_s^{s'} \in [0,1]$. To see the second inequality consider two cases. When $s > s'$, by using $t_{s'}(\beta)$ seller type $s$ gets the same profit from buyer types in $[b_s^\alpha, b_{s'}^\omega]$ as seller type $s'$, but looses at most $\Sigma$ from buyer types in $[b_{s'}^\alpha, b_s^\alpha]$. When $s < s'$, by using $t_{s'}(\beta)$ seller type $s$ gets the same profit from buyer types in $[b_s^\alpha, b_s^\omega]$ as seller type $s'$, but looses at most $\Sigma$ from buyer types in $[b_s^\omega, b_{s'}^\omega]$. Hence, $|R_\beta(s) - R_\beta(s')| \leq (\ell + \Sigma)|s - s'|$. $\square$

**Lemma 16.** *Suppose that $t_\beta(s)$ is a best-reply to willingness to pay $P_b$. Then $t_\beta(s)$ is non-decreasing in $s$ and $\beta$. Moreover, for any $\beta$, $T_\beta(s)$ has a closed graph, and in particular, $t(s)$ is left-continuous in $s$.*

*Proof.* Denote current profit function of seller type $s$ by $\pi_\beta(s,b) = (\beta - b)(P(b) - c(s))$ and constraint is $b \in B_s \cap [0,\beta]$. Since $\frac{\partial}{\partial \beta}\pi_\beta(s,b) = P(b)$ is increasing in $b$, function $\pi_\beta(s,b)$ is supermodular in $(\beta, b)$. Since $\frac{\partial}{\partial s}\pi_\beta(s,b) = -c'(s)(\beta - b)$ is increasing in $b$, function $\pi_\beta(s,b)$ has increasing differences in $b$ and $s$. Further, consider $b \geq b'$, $\beta \geq \beta'$, $s \geq s'$ and suppose $b' \in B_s \cap [0,\beta]$ and $b \in B_{s'} \cap [0,\beta']$. Then $b \leq \beta' \leq \beta$, $b \leq s' + \eta \leq s + \eta$, $b \geq b' \geq s - \eta$ and, therefore, $b \in B_s \cap [0,\beta]$. Analogously, we could show that $b' \in B_{s'} \cap [0,\beta']$. Hence, the constraint sets are ascending in the terminology of Hopenhayn and Prescott (1992).[57] By Proposition 2 in Hopenhayn and Prescott (1992) value function $R_\beta(s)$ has increasing differences in $\beta$ and $s$ and solution $t_\beta(s)$ is non-decreasing in $s$ and $\beta$. By the generalization of Theorem of the Maximum in Ausubel and Deneckere (1988), for any $\beta$, $T_\beta(s)$ has a closed graph and so, $t(s)$ is left-continuous in $s$. $\square$

**Lemma 17.** *For all $b$ we have $\pi^i(b) \geq c(s_b^\alpha) + (1-\delta^2)\xi$ and for all $s \in [-1, -\eta]$, $\Pi^i(s) > C(\eta, \delta) > 0$ with $C(\eta,\delta) \sim (1-\delta)^2$ where $\Pi^i(s)$ is the expected profit of seller $s$ that faces demand $\pi^i(b)$.*

*Proof.* For all buyers $b$, $\pi^i(b) = (1-\delta^2)v(b) + \delta^2\hat{\pi}^{i-1}(\tau^i(s_b^\alpha)) \geq (1-\delta^2)v(b) + \delta^2 c(s_b^\alpha) \geq c(s_b^\alpha) + (1-\delta^2)\xi$. The first inequality follows from the fact that seller types in $[0, s_{i+1})$ get positive profit when best-replying to static demand given by $\pi^{i-1}(b)$ and the second inequality follows from $v(b) - c(s_b^\alpha) \geq \xi$.

To derive the lower bound on the profit, suppose seller type $s \in [0, 1-\eta]$ makes price offer $c(s) + (1-\delta^2)\frac{\xi}{2}$. By the lower bound on willingness to pay $P(b)$ derived above, buyer types with $\pi^i(b) > c(s) + (1-\delta^2)\frac{\xi}{2\ell}$ accept such price offer. The mass of buyer types who accept such price offer and are in the support of beliefs $B_s$ is at least $\min\{2\eta, (1-\delta^2)\frac{\xi}{2\ell}\}$ and seller type $s$ is guaranteed to get profit $\min\{2\eta, (1-\delta^2)\frac{\xi}{2\ell}\}(1-\delta^2)\frac{\xi}{2\ell} \equiv C(\eta,\delta)$. This minimal profit is equal to $(1-\delta^2)^2\frac{\xi^2}{4\ell}$ for $\delta$ close to one and, hence, $C(\eta,\delta) \sim (1-\delta)^2$. $\square$

[57] The case when $b' \geq b$ is checked trivially.

**Lemma 18.** *For all $s \in (-1, -\eta]$, $b_s^\omega - t(s) > c(\eta, \delta)$. Moreover, $c(\eta, \delta) \sim (1-\delta)^3$ as $\delta$ goes to one.*

*Proof.* I make change of variable $x = b_s^\omega - b$ in the seller's problem (18). Then $\Pi^i(s) = x(\pi^i(b_s^\omega - x) - c(s)) + \delta^2 \Pi^i_{b_s^\omega - x}(s) \le x(\pi^i(b_s^\omega - x) - c(s)) + \delta^2(\Pi^i(s) + \ell_R x)$ where the inequality follows from the Lipschitz continuity of $\Pi^i(s)$ (by Lemma 15). Therefore, I get $x \ge \frac{\Pi^i(s)(1-\delta^2)}{\pi^i(b_s^\omega) - c(s) + \delta^2 \ell_R} \ge \frac{C(\eta,\delta)(1-\delta^2)}{\Sigma + \ell_R}$ where I used the lower bound on $R(s)$ from Lemma 17. □

**Lemma 19.** *On each step of the iterative algorithm, function $\pi^i(b)$ is left-continuous and strictly increasing.*

*Proof.* The proof is by induction on the step of the algorithm. For $i = 0$, the strict monotonicity of $\pi^0(b)$ follows from the strict monotonicity of $P^0(b)$ and $v(b)$, and the fact that $P^0(\eta) \le v(\eta)$. The left-continuity of $\pi^0(b)$ follows from the left-continuity of $P^0(b)$ and the continuity of $v(b)$.

Suppose by the inductive hypothesis that $\pi^{i-1}(b)$ is left-continuous and strictly increasing. For $b \in (\eta + ic(\eta, \Delta), 1]$, $\pi^i(b) = v(b)$ is strictly increasing and left-continuous. For $b \in [0, \eta + ic(\eta, \Delta)]$, $\pi^i(b)$ is a convex combination of strictly increasing $v(b)$ and $\hat{\pi}^{i-1}(\tau^i(s_b^\alpha))$. Function $\hat{\pi}^{i-1}(\tau^i(s_b^\alpha))$ is increasing, as $\hat{\pi}^{i-1}$ is increasing by the inductive hypothesis and $\tau^i(s_b^\alpha)$ is increasing by Lemma 16. Therefore, $\pi^i(b)$ is strictly increasing on $[0, \eta + ic(\eta, \Delta)]$. Moreover, $\pi^i(\eta + ic(\eta, \Delta)) \le v(\eta + ic(\eta, \Delta))$, which completes the proof of the strict monotonicity of $\pi^i(b)$.

I next show that $\hat{\pi}^{i-1}(\tau^i(s_b^\alpha))$ is left-continuous. This would imply that $\pi^i(b)$ is left-continuous on $[0, \eta + ic(\eta, \Delta)]$ as a convex combination of left-continuous functions. Suppose to contradiction that there exist $\hat{b}$ and an increasing sequence $b_j \to \hat{b}$ such that $\lim\limits_{j\to\infty} \hat{\pi}^{i-1}(\tau^i(s_{b_j}^\alpha)) < \hat{\pi}^{i-1}(\tau^i(s_{\hat{b}}^\alpha))$. Denote $s_j = s_{b_j}^\alpha$ for all $j \in \mathbb{N}$ and $\hat{s} = s_{\hat{b}}^\alpha$. By Lemma 16,

$$\lim_{j\to\infty} \tau^i(s_j) = \tau^i(s_j). \tag{63}$$

If $\hat{\pi}^{i-1}(b)$ is continuous at $\tau^i(\hat{s})$, then $\lim\limits_{j\to\infty} \hat{\pi}^{i-1}(\tau^i(s_j)) = \hat{\pi}^{i-1}(\tau^i(\hat{s}))$, which is a contradiction. If $\hat{\pi}^{i-1}(b)$ is discontinuous at $\tau^i(\hat{s})$, then the first price offer of all seller type $s_j$ is below $\hat{\pi}^{i-1}(\tau^i(\hat{s})) - \varepsilon$ for some $\varepsilon > 0$, while the first price offer of seller type $\hat{s}$ is equal to $\hat{\pi}^{i-1}(\tau^i(\hat{s}))$. Therefore,

$$\Pi^i(\hat{s}) = (\hat{\pi}^{i-1}(\tau^i(\hat{s})) - c(\hat{s}))(\hat{b} - \tau^i(\hat{s})) + \delta^2 \Pi^i_{\tau^i(\hat{s})}(\hat{s}) >$$

$$(\varepsilon + \hat{\pi}^{i-1}(\tau^i(s_j)) - c(\hat{s}))(\hat{b} - \tau^i(\hat{s})) + \delta^2 \Pi^i_{\tau^i(\hat{s})}(\hat{s}) = \varepsilon(\hat{b} - \tau^i(\hat{s})) + \lim_{j\to 1} \Pi^i(s_j),$$

where the equality follows from the continuity of $c(s)$ and $\Pi^i_\beta(s)$ (by Lemma 15) and (63). This contradicts the continuity of $\Pi^i(s)$ (again by Lemma 15) and so, $\hat{\pi}^{i-1}(\tau^i(s_b^\alpha))$ is left-continuous. For $b \in [0, \eta + ic(\eta, \Delta)]$, $\pi^i(b)$ is a convex combination of continuous $v(b)$ and left-continuous $\hat{\pi}^{i-1}(\tau^i(s_b^\alpha))$ and so, is left-continuous itself completing the proof of the inductive step. □

**Lemma 20.** *Suppose $P(b)$ and $t_\beta(s)$ satisfy equations (18) and (19). Then for $\delta$ sufficiently close to one, in the (seller) punishing equilibrium on-path strategies given by $P(b)$ and $t_\beta(s)$ are optimal for the seller and the buyer.*

*Proof.* From the design of the algorithm the screening strategy $t_\beta(s)$ is optimal for the seller who faces the static demand given by $P(b)$. I next show that the buyer does not have incentives to deviate either from the acceptance strategy $P(b)$ or from pooling on the price offer $\frac{\delta v(0)+c(0)}{1+\delta}$.

If the highest remaining buyer type exceeds $b$, then buyer type $b$ interprets the previous seller's offers as seller's deviations. In this case, buyer type $b$ expects the seller to restart screening. From equation (19) it follows that any price offer above $P(b)$ would be rejected by buyer $b$. To complete the verification of optimality of the threshold strategy, we next show that prices below $P(b)$ are accepted by buyer $b$.

Suppose to contradiction that the seller makes price offer $p$ which is accepted by buyer $b'$ and rejected by buyer type $b$ and $b > b'$. First, observe that if $b \le \bar{\beta}$, then both types $b$ and $b'$ put probability one on seller type 0, and the result follows from the single crossing property of the payoffs

Next, suppose that $b' > \bar{\beta}$. Define buyer $b'' = \inf\{b : P(b) \ge p\}$. If the buyer rejects price offer $p$, then the highest buyer type remaining in the game is $b''$. Each seller type $s$ uses screening policy $t_{b''}(s)$ after rejection. Then for all $k \in \mathbb{N}$,

$$v(b') - p \ge \delta^{2k} \left( v(b') - \hat{P}(t^{(k)}_{s^\alpha_{b'}}(b'')) \right) \tag{64}$$

and

$$v(b) - p < \delta^{2K} \left( v(b) - \hat{P}(t^{(K)}_{s^\alpha_{b}}(b'')) \right) \tag{65}$$

for some $K$.[58] That is, buyer type $b'$ accepts price offer $p$, and buyer type $b$ rejects such price offer and expects to accept price offer $P(t^K_{s^\alpha_b}(b''))$ from seller type $s^\alpha_b$. Subtracting inequality (64) (with $k = K$) from (65), I get

$$v(b) - v(b') < \delta^{2K} \left( v(b) - v(b') - \hat{P}(t^{(K)}_{s^\alpha_b}(b'')) + \hat{P}(t^{(K)}_{s^\alpha_{b'}}(b'')) \right)$$

or

$$\left(1 - \delta^{2K}\right) (v(b) - v(b')) < -\delta^{2K} \left( \hat{P}(t^{(K)}_{s^\alpha_b}(b'')) - \hat{P}(t^{(K)}_{s^\alpha_{b'}}(b'')) \right). \tag{66}$$

The left-hand side of (66) is greater than zero, as $b > b'$. By Lemma 16, $t^{(K)}_{b''}(s^\alpha_b) \ge t^{(K)}_{b''}(s^\alpha_{b'})$, and moreover, $P(b)$ is increasing. Hence, the right-hand side of (66) is less than zero, which gives a contradiction.

Finally, if $b' \le \bar{\beta} < b$, then the only difference with the previous case is that now buyer $b'$

[58]Notation $f^{(k)}(x)$ stands for $k$-superposition of function $f$, i.e. let $f^{(0)}(x) \equiv f(x)$ and for $k \ge 1$ $f^{(k)}(x) \equiv f(f^{(k-1)}(x))$.

could prefer price $p$ not only to all the future price offers of the seller, but also to the seller's acceptance of offer $\frac{\delta v(0)+c(0)}{1+\delta}$. That is, it is possible that

$$v(b') - p \geq \delta \left( v(b') - \frac{\delta v(0) + c(0)}{1+\delta} \right)$$

or more weakly

$$v(b') - p \geq \delta^{2K} \left( v(b') - \frac{\delta v(0) + c(0)}{1+\delta} \right).$$

Combining this inequality with the same argument as before I get contradiction again.

The fact that buyers are better off pooling on $\frac{\delta v(0)+c(0)}{1+\delta}$ is the following claim and follows from the Contagious Coasian Property proven in the next section.

*Claim* 17. *For sufficiently large $\delta$, in the seller punishing equilibrium no buyer type prefers to deviate from pooling on offer $\frac{\delta v(0)+c(0)}{1+\delta}$.*

*Proof.* By Theorem 8 any buyer type $b$ above $\eta$ expect to get the good in the next round buyer is active at price uniformly close to $P^*(b)$. By Lemma 13 if such buyer type deviates he trades with the seller at price close to $\frac{v(b)+c(s_b^\alpha)}{2} > P^*(b)$, hence, the deviation is not profitable for such buyer types for sufficiently large $\delta$. Now buyer types below $\eta$ expect the first price offer of the seller to be close to $\frac{v(0)+c(0)}{2}$ which is preferred to immediate trade at $\frac{v(\eta)+c(0)}{2}$, making the deviation unprofitable for such types. *Q.E.D.* ☐

*Proof of Theorem 7.* The iterative algorithm converges in a finite number of steps by Lemma 18 and the resulting strategies are optimal by Lemma 20. ☐

### 8.4.2 Proof of the Contagious Coasian Property

Let $Q_\beta(s) \equiv \min\{\beta, b_s^\omega\} - \min\{\beta, b_s^\alpha\}$ be the mass of remaining buyer types in the support of beliefs of seller type $s$ when $\beta$ is the highest remaining buyer type. Consider a sequence of discount factors $\delta_j \to 1$. In the punishing equilibrium of the game with discount factor $\delta_j$, I denote by $A^j(s)$ the first price offer of seller $s$.

**Lemma 21.** *There exist a limit point of sequences $P^j(b), t^j(s), A^j(s), R^j_\beta(s)$.*

*Proof.* By Lemma 15, function $R^j_\beta(s)$ is Lipschitz continuous in $s$ and $\beta$ with Lipschitz constants not exceeding 3. Hence, for $(s,\beta)$ and $(s',\beta')$ such that $|s-s'|+|\beta-\beta'| < \varepsilon$, $|R^j_\beta(s) - R_{\beta'}(s')| \leq |R^j_\beta(s) - R^j_\beta(s')| + |R^j_\beta(s') - R^j_{\beta'}(s')| \leq \ell_R(|s-s'|+|\beta-\beta'|) < \ell_R\varepsilon$. Hence, family of continuous functions $R^j_\beta(s)$ is equicontinuous and so, by the Arzela-Ascoli theorem, $R^j_\beta(s)$ converges (over subsequence) to some continuous function $R^*_\beta(s)$. Moreover, $R_\beta(s)$ converges uniformly to $R^*_\beta(s)$ as a sequence of continuous functions on a compact set that converges to a continuous function. Consider now sequences of non-decreasing functions $P^j(b)$, $t^j(s)$, $A^j(s)$. By Helly's theorem there is a subsequence along which the sequence converges to a non-decreasing limit $P^*(b)$, $t^*(s)$, $A^*(s)$ pointwise. ☐

*Proof of Lemma 4.* Suppose to contradiction that there exists $\hat{b} \in (0,1)$ with $P^*(\hat{b}) > c(s^{\alpha}_{\hat{b}})$, and for any $\phi > 0$, $P^*(\hat{b} - \phi) \leq P^*(\hat{b}) < P^*(\hat{b} + \phi)$. Let $\varepsilon \equiv \frac{P^*(\hat{b}) - c(s^{\alpha}_{\hat{b}})}{2}$. Consider some seller type $\hat{s} > s^{\alpha}_{\hat{b}} + \frac{\varepsilon}{4\ell}$. By the left-continuity of $P^*(b)$, we choose $\phi$ small enough so that $P^*(\hat{b} - \phi) > P^*(\hat{b}) - \frac{\varepsilon}{4}$ and $(\hat{b} - \phi, \hat{b} + \phi) \subset B_{\hat{s}}$. By the pointwise convergence of the sequence $P^j(b)$, for $\delta_j$ sufficiently large, I have $P^j(\hat{b} - \phi) > P^*(\hat{b} - \phi) - \frac{\varepsilon}{4} > P^*(\hat{b}) - \frac{\varepsilon}{2} > c(s^{\alpha}_{\hat{b}}) + \frac{\varepsilon}{2} > c(\hat{s}) + \frac{\varepsilon}{4}$. There are two cases to consider: $A^*(\hat{s}) > P^*(\hat{b})$ and $A^*(\hat{s}) \leq P^*(\hat{b})$.

**Case 1) $\mathbf{A^*(\hat{s}) > P^*(\hat{b})}$ .** In the proof of case 1, I restrict that $\delta_j$ is sufficiently large so that $A^j(\hat{s}) > \frac{1}{3}P^*(\hat{b}) + \frac{2}{3}A^*(\hat{s})$ and $P^j(\hat{b}) < \frac{2}{3}P^*(\hat{b}) + \frac{1}{3}A^*(\hat{s})$ (by pointwise convergence of $A^j(s)$ and $P^j(b)$) and so,

$$A^j(\hat{s}) > P^j(\hat{b}) + \frac{1}{3}(A^*(\hat{s}) - P^*(\hat{b})). \tag{67}$$

I show that seller type $\hat{s}$ prefers to deviate from the equilibrium strategy by speeding up screening of buyer types above $\hat{b}$ which gives a contradiction. Observe that for all $\delta_j$ sufficiently large, $R^j_{\hat{b}}(\hat{s}) \geq 2\phi(P^j(\hat{b} - \phi) - c(\hat{s})) \geq \frac{\phi\varepsilon}{2} > 0$.

Let $K_j \leq \infty$ be the round of screening when price offer of the seller type $\hat{s}$ drops below $P^j(\hat{b})$. Buyer type $b^{\omega}_{\hat{s}}$ prefers to purchase immediately rather than wait until price drops below $P^j(\hat{b})$ and so, $v(b^{\omega}_{\hat{s}}) - A^j(\hat{s}) \geq \delta_j^{2K_j}(v(b^{\omega}_{\hat{s}}) - P^j(\hat{b}))$ or by (67)

$$\delta_j^{2K_j} \leq \frac{v(b^{\omega}_{\hat{s}}) - A^j(\hat{s})}{v(b^{\omega}_{\hat{s}}) - P^j(\hat{b})} < \frac{v(b^{\omega}_{\hat{s}}) - P^j(\hat{s}) - \frac{1}{3}(A^*(\hat{s}) - P^*(\hat{b}))}{v(b^{\omega}_{\hat{s}}) - P^j(\hat{b})} = 1 - \frac{1}{3}\frac{A^*(\hat{s}) - P^*(\hat{b})}{v(b^{\omega}_{\hat{s}}) - P^j(\hat{b})}. \tag{68}$$

The right-hand side of (68) converges to a limit that is strictly less than 1 and so, $\lim\limits_{j\to\infty} \delta_j^{2K_j} < 1$.

Observe that the profit of seller type $\hat{s}$ in the equilibrium satisfies $R^j(\hat{s}) \leq \int_{\hat{b}}^{b^{\omega}_{\hat{s}}} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R^j_{\hat{b}}(\hat{s})$. Consider an alternative screening policy in which for integer $M_j$ seller type $\hat{s}$ posts price sequence $\{A_m\}_{m=1}^{M_j}$ such that $A_m = v(b^{\omega}_{\hat{s}}) + \frac{m}{M_j}(c(\hat{s}) - v(b^{\omega}_{\hat{s}}))$ and sell with probability one in $M_j$ rounds. Moreover, the loss in profit from each sale is at most $\frac{\Sigma}{M_j}$. By the optimality of the seller's equilibrium strategy, $R^j(\hat{s}) \geq \delta_j^{2M_j}\left(\int_{\underline{b}}^{b^{\omega}_{\hat{s}}} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j}\right)$ where $\underline{b} = \inf\{b \in B : P^j(b) > c(\hat{s})\}$. Therefore,

$$\delta_j^{2M_j}\left(\int_{\underline{b}}^{b^{\omega}_{\hat{s}}} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j}\right) \leq \int_{\hat{b}}^{b^{\omega}_{\hat{s}}} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R^j_{\hat{b}}(\hat{s})$$

or after rearranging terms

$$\delta_j^{2M_j}\left(\int_{\underline{b}}^{\hat{b}} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j}\right) \leq \left(1 - \delta_j^{2M_j}\right)\int_{\hat{b}}^{b^{\omega}_{\hat{s}}} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R^j_{\hat{b}}(\hat{s}).$$

Since $R^j_{\hat{b}}(\hat{s}) \leq \int\limits_{\underline{b}}^{\hat{b}} (P^j(b) - c(\hat{s}))db$,

$$\delta_j^{2M_j}\left(R^j_{\hat{b}}(\hat{s}) - \frac{\Sigma}{M_j}\right) \leq \left(1-\delta_j^{2M_j}\right)\int_{\hat{b}}^{b^\omega_{\hat{s}}}(P^j(b) - c(\hat{s}))db + \delta_j^{2K_j}R^j_{\hat{b}}(\hat{s}) \leq \Sigma(b^\omega_{\hat{s}} - \hat{b})\left(1-\delta_j^{2M_j}\right) + \delta_j^{2K_j}R^j_{\hat{b}}(\hat{s}),$$

where the last inequality follows from the fact that values are bounded. Since $R^j_{\hat{b}}(\hat{s}) \geq \frac{\phi\varepsilon}{2} > 0$,

$$\delta_j^{2K_j} \geq \delta_j^{2M_j} - \frac{1}{R^j_{\hat{b}}(\hat{s})}\left(\frac{\delta_j^{2M_j}}{M_j} + \Sigma(b^\omega_{\hat{s}} - \hat{b})\left(1-\delta_j^{2M_j}\right)\right) \geq \delta_j^{2M_j} - \frac{2}{\phi\varepsilon}\left(\frac{\delta_j^{2M_j}}{M_j} + \Sigma(b^\omega_{\hat{s}} - \hat{b})\left(1-\delta_j^{2M_j}\right)\right)$$

For each $\delta_j$, I can choose $M_j$ such that $\delta_j^{2M_j}$ converges to one, as $\delta_j \to 1$. Hence, from the last inequality it follows that $\delta_j^{2K_j}$ is arbitrarily close to one which contradicts (68).

**Case 2) $\mathbf{A}^*(\hat{\mathbf{s}}) \leq \mathbf{P}^*(\hat{\mathbf{b}})$ .** Consider an alternative screening policy, in which seller type $\hat{s}$ posts price $P^j(\hat{b}+\phi)$ in the first round, then makes offer $A^j(\hat{s})$ and proceeds with the screening policy as in the equilibrium. From the optimality of the equilibrium strategy, it follows

$$(b^\omega_{\hat{s}} - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^2 R^j_{t^j(\hat{s})}(\hat{s}) \geq$$

$$(b^\omega_{\hat{s}} - \hat{b} - \phi)(P^j(\hat{b}+\phi) - c(\hat{s})) + \delta_j^2(\hat{b}+\phi - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^4 R_{t(\hat{s})}(\hat{s})$$

or

$$(1-\delta_j^2)\left((b^\omega_{\hat{s}} - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^2 R^j_{t^j(\hat{s})}(\hat{s})\right) \geq (b^\omega_{\hat{s}} - \hat{b} - \phi)\left(P^j(\hat{b}+\phi) - \delta_j^2 A^j(\hat{s}) - (1-\delta_j^2)c(\hat{s})\right) \tag{69}$$

The left-hand side of (69) goes to zero as $\delta_j \to 1$ and the right hand side of (69) converges to $(b^\omega_{\hat{s}} - \hat{b} - \phi)(P^*(\hat{b}+\phi) - A^*(\hat{s})) > 0$ which is a contradiction. □

**Corollary 1.** *For any $b < b^\omega_{s+}$, $P^*(b) = \frac{v(0)+c(0)}{2}$.*

*Proof of Corollary 1.* For any buyer type $b \in [0, b^\omega_{s+})$, $P^j(b) \geq \frac{v(0)+\delta c(0)}{1+\delta} > \frac{v(0)+c(0)}{2} > c(s^\alpha_b)$ and so, $P^*(b) > c(s^\alpha_b)$ for $b \in [\eta, b^\omega_{s+})$. Therefore, by Lemma 4, function $P^*(b)$ is constant on this interval. Since $P^*(b) = \frac{v(0)+c(0)}{2}$ for $b \in [0,\eta]$, we have $P^*(b) = \frac{v(0)+c(0)}{2}$ on $[0, b^\omega_{s+})$. □

**Definition 8.1.** A monotone function $f(x)$ on $[0,1]$ is $\varepsilon$-continuous if for any open interval $I \subset [f(0), f(1)]$ of length at least $\varepsilon$ we have $f([0,1]) \cap I \neq \emptyset$.

**Lemma 22.** *For any $\varepsilon > 0$ there exists $\bar{\delta} \in (0,1)$ such that for all $\delta_j > \bar{\delta}$, function $P^j(b)$ is $\varepsilon$-continuous, and for any seller type $s \in [0,1]$ and buyer type $\beta \in B_s$,*

$$\hat{P}^j(\beta) - \hat{P}^j(t^j_\beta(s)) \leq \varepsilon. \tag{70}$$

*Proof.* Suppose to contradiction that there exist $\varepsilon > 0$, $\underline{P}$, and $\overline{P} > \underline{P} + \varepsilon$ such that for any $b \in [0,1]$ and infinitely many $j$s, either $P^j(b) \geq \overline{P}$ or $P^j(b) \leq \underline{P}$. Without loss of generality, take $\underline{P}$ and $\overline{P}$ such that $\overline{P} - \underline{P}$ is maximal. For any $j$, consider $b_j \equiv \sup\{b : P^j(b) < \underline{P}\}$. By equation (19), for any $b \in [0,1]$,

$$P^j(b) - \hat{P}^j(t(s^\alpha_b)) = (1 - \delta_j^2)(v(b) - \hat{P}^j(t^j(s^\alpha_b)) \leq (1 - \delta_j^2)\Sigma < \frac{\varepsilon}{2} \tag{71}$$

for $\delta_j$ sufficiently close to one. Consider buyer type $\hat{b}_j \equiv b_j + \frac{c(\eta,\delta_j)}{2}$ and $\check{b}_j \equiv b_j + \frac{c(\eta,\delta_j)}{2}$. Then

$$P^j(\hat{b}_j) - \hat{P}^j(t(s^\alpha_{\check{b}_j})) > P^j(\hat{b}_j) - \hat{P}^j(\check{b}_j) > \varepsilon,$$

which gives a contradiction to (71).

To prove (70), observe that by Lemma 16, for any $j \in \mathbb{N}$,

$$P^j(\beta) - \hat{P}^j(t^j_\beta(s)) \leq P^j(\beta) - \hat{P}^j(t^j_\beta(s^\alpha_\beta)), \tag{72}$$

for all $b \in [0,1]$. For any $\varepsilon > 0$, choose $\delta_j$ sufficiently large so that $P^j(b)$ is $\frac{\varepsilon}{2}$-continuous. This implies that the right-hand side of (72) is less than $\frac{\varepsilon}{2}$, and moreover, there exists $\beta_j > \beta$ such that $P^j(\beta_j) - P^j(\beta) < \frac{\varepsilon}{2}$. Together with (72), this gives

$$\hat{P}^j(\beta) - \hat{P}^j(t^j_\beta(s)) \leq P^j(\beta_j) - P^j(\beta) + P^j(\beta) - \hat{P}^j(t^j_\beta(s)) < \varepsilon,$$

which proves (70). $\square$

**Lemma 23.** *For any $\delta_j$, let two converging sequences of buyer types $\{b_j\}_{j=1}^\infty$ and $\{b'_j\}_{j=1}^\infty$ be such that $P^j(b_j) - \hat{P}^j(b'_j)$ and $v(b_j) - \hat{P}^j(b'_j)$ are uniformly bounded away from zero. Then there exist a function $\gamma(\delta_j) \sim (1 - \delta_j)^2$ and an integer $J$ such that $b_j - b'_j \geq \gamma(\delta_j)$ for all $j \geq J$.*

*Proof.* Define sequence $t^j_l$, $l = 0, \ldots, L_j + 1$ as follows. Let $t^j_0 = b_j$ and $t^j_l = t^j(s^\alpha_{t^j_{l-1}})$ for $l = 1, \ldots, L_j + 1$ where $L_j$ is the largest integer such that $t^j_{L_j} \geq b'_j$. By (19), I have

$$P^j(b_j) = (1 - \delta_j^2) \sum_{l=0}^{L_j} \delta^{2l} v(t^j_l) + \delta_j^{2(L_j+1)} \hat{P}^j(t^j_{L_j+1}).$$

Since $\hat{P}^j(b)$ is increasing in $b$ and $b'_j \in [t^j_{L_j+1}, t^j_{L_j}]$,

$$P^j(b_j) - \hat{P}^j(b'_j) \leq (1 - \delta_j^2) \sum_{l=0}^{L_j} \delta^{2l} v(t^j_l) - (1 - \delta_j^{2(L_j+1)}) \hat{P}^j(b'_j) \leq (1 - \delta_j^{2(L_j+1)})(v(b_j) - \hat{P}^j(b'_j)).$$

Since $P^j(b_j) - \hat{P}^j(b'_j)$ and $v(b_j) - \hat{P}^j(b'_j)$ are uniformly bounded away from zero, $1 - \delta_j^{2(L_j+1)}$ is uniformly bounded away from zero. Hence, the exists $C_1 > 0$ and an integer $J_1$ such that

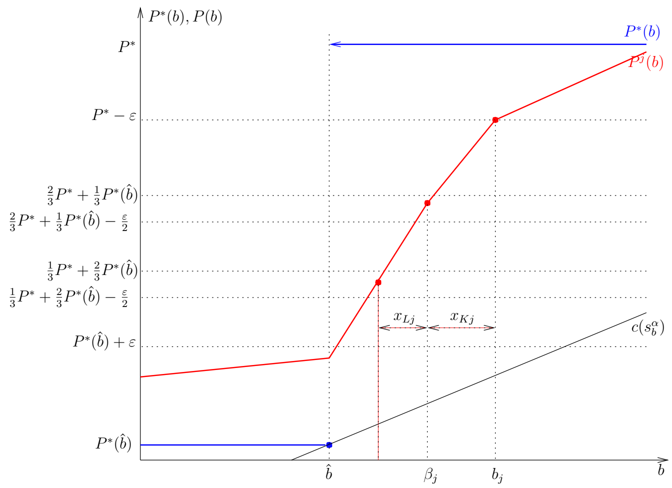



Figure 7: Illustration of the proof of Lemma 5

$L_j \geq -C_1/\ln\delta_j$ for all $j \geq J_1$.

By Lemma 18, there exists $C_2 > 0$ and an integer $J_2$ such that $t^j_{l-1} - t^j_l > C_2(1-\delta_j)^3$ for all $l \in 1, \ldots, L_j$ and all $j \geq J_2$. Hence, $b_j - b'_j = \sum_{l=1}^{L_j}(t^j_{l-1} - t^j_l) + t^j_{L_j} - b'_j \geq C_2(1-\delta_j)^3 L_j \geq -C_1C_2(1-\delta_j)^3/\ln\delta_j \sim (1-\delta_j)^2$ for $j \geq J \equiv \max\{J_1, J_2\}$. The function $\gamma(\delta_j) = -C_1C_2(1-\delta_j)^3/\ln\delta_j$ satisfies the desired properties. $\square$

*Proof of Lemma 5.* Suppose to contradiction that there exists $\hat{b}$ such that $P^* \equiv P^*(\hat{b}+0) > P^*(\hat{b})$ (see Figure 4 for the illustration of the proof). By Corollary 1, $\hat{b} \geq s^+ + \eta$, and by Lemma 4, $P^*(\hat{b}) = c(s^\alpha_{\hat{b}})$. Fix $\varepsilon > 0$ small enough so that $P^* - P^*(\hat{b}) > \frac{9}{2}\varepsilon$, which ensures that $P^* - \varepsilon > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} > P^*(\hat{b}) + \varepsilon$. Let $b_j \equiv \inf\{b : P^j(b) \in (P^* - \varepsilon, P^*)\}$ and $s_j \equiv s^\alpha_{b_j}$. Let $K_j \leq \infty$ be the first round of screening, in which seller type $s_j$ makes a price offer below $\frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b})$ and allocates to all buyer types above some $\beta_j$. In the proof, I restrict that $\delta_j$ is sufficiently close to one so that the conclusions of the following claim obtain.

*Claim* 18. *It holds*

$$\lim_{j\to\infty} P^j(b_j) = P^* - \varepsilon, \tag{73}$$

*and for $\delta_j$ sufficiently large,*

$$\frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) > \hat{P}^j(\beta_j) > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}, \tag{74}$$

$$b_j < \hat{b} + \frac{\varepsilon}{2\ell} \text{ and } c(s_j) \leq P^*(\hat{b}) + \frac{\varepsilon}{2}. \tag{75}$$

*Proof.* By Lemma 22 for any $\varepsilon$ there exists $J(\varepsilon)$ such that $P^j(b)$ is $\varepsilon$-continuous for $j \geq J(\varepsilon)$

and so, (73) obtains. Inequality (74) follows from the definition of $\beta_j$ and (70) in Lemma 22. By the pointwise convergence of $P^j(b)$, $\lim_{j\to\infty} b_j = \hat{b}$ and so, $b_j < \hat{b} + \frac{\varepsilon}{2\ell}$ for $\delta_j$ sufficiently large. This, in turn, implies $c(s_j) < c(s^{\alpha}_{\hat{b}} + \frac{\varepsilon}{2\ell}) < c(s^{\alpha}_{\hat{b}}) + \frac{\varepsilon}{2} = P^*(\hat{b}) + \frac{\varepsilon}{2}$ where the second inequality is by Lipschitz continuity of $c(s)$. *Q.E.D.*

**Optimality of strategy of type $s_j$.** In the first $K_j$ rounds of screening, seller type $s_j$ allocates to the mass of buyer types $x_{Kj} \equiv b_j - \beta_j$. Since buyer type $b_j$ prefers to buy at price $P^j(b_j)$ rather than wait until price drops to $\hat{P}^j(\beta)$,

$$\frac{v(b_j) - P^j(b_j)}{v(b_j) - \hat{P}^j(\beta_j)} \geq \delta_j^{2K_j}. \tag{76}$$

By (73) and (74), the upper bound on $\delta_j^{2K_j}$ in (76) converges to at most $\frac{v(\hat{b})-P^*+\varepsilon}{v(\hat{b})-\frac{2}{3}P^*-\frac{1}{3}P^*(\hat{b})} < 1$. Therefore, $\delta_j^{2K_j}$ converges to some limit $\lambda_K < 1$ as $\delta_j \to 1$ and so, $\lim_{j\to\infty}(1-\delta_j^2)K_j = -\ln\lambda_K > 0$.

For any integer $M_{Kj}$, consider an alternative screening strategy, in which seller type $s_j$ speeds up screening in the first $\lfloor K_j/M_{Kj}\rfloor$ rounds. Let $A_k$ be the price offer that seller type $s_j$ makes in round $k$. Define $q_k = P^j(b_j) + \frac{kM_{Kj}}{K_j}\left(A_{K_j-1} - P^j(b_j)\right)$, $k = 1, 2, .., \lfloor K_j/M_{Kj}\rfloor$. In the alternative strategy, seller type $s_j$ makes price offer $p_k \equiv \min\{q_k, A_k\}$ in rounds $k \leq \lfloor K_j/M_{Kj}\rfloor$, makes offer $A_{K_j}$ in round $\lfloor K_j/M_{Kj}\rfloor + 1$ and continues following equilibrium strategy from then on. The total loss from using the alternative strategy is at most $M_{Kj}x_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right)/K_j$. Indeed, in each round the loss of seller type $s_j$ compared to the maximum surplus that could be extracted is at most $\frac{P^j(b_j)-\hat{P}^j(\beta_j)}{K_j/M_{Kj}} \leq M_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right)/K_j$ where the inequality follows from (73) and (74). Moreover, there is no loss due to discounting, as the allocation to all buyer types happens sooner under the alternative strategy than under the equilibrium strategy.

At the same time, by speeding up the screening seller type $s_j$ gains at least $\left(\delta_j^{2K_j/M_{Kj}} - \delta_j^{2K_j}\right)V_{Kj}$, where $V_{Kj}$ is the continuation utility of seller type $s_j$ after she makes price offer $A_{K_j}$ and follows the equilibrium strategy further. By the optimality of strategy of seller type $s_j$,

$$\frac{M_{Kj}}{K_j}x_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right) \geq \left(\delta_j^{2K_j/M_K} - \delta_j^{2K_j}\right)V_{Kj} \tag{77}$$

**Optimality of strategy of type $\sigma_j$.** Consider seller type $\sigma_j \equiv s^{\alpha}_{\beta_j}$ and let $L_j$ be the first round of screening, in which seller type $\sigma_j$ makes a price offer below $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b})$. By the analogous argument as with $K_j$ and seller type $s_j$, I have $\delta_j^{2L_j}$ converges to the limit $\lambda_L < 1$ (correspondingly, $(1-\delta_j^2)L_j \to -\ln\lambda_L > 0$), and for the optimality of strategy of seller type $\sigma_j$ it is necessary that

$$\frac{M_{Lj}}{L_j}x_{Lj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right) \geq \left(\delta_j^{2L_j/M_{Lj}} - \delta_j^{2L_j}\right)V_{Lj}, \tag{78}$$

for any integer $M_{Lj}$. In inequality (78), $x_{Lj}$ denotes the mass of buyer types to whom seller type $\sigma_j$ allocates in the first $L_j$ rounds, and $V_{Lj}$ denotes the continuation utility of seller type $\sigma_j$ after price offer in round $L_j$ and follows the equilibrium strategy further.

**Lower bound on $V_{Kj}$ .** Observe that seller type $s_j$ could post price $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}$ after price offer $A_{K_j}$. The mass of buyer types that accept such price is $x_{Lj}$, and the profit from each such buyer is $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} - c(s_j) \geq \frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b}) - \varepsilon$ by (75). Hence,

$$V_{Kj} \geq x_{Lj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b}) - \varepsilon\right). \tag{79}$$

**Lower bound on $V_{Lj}$ .** Suppose that the seller allocated in previous rounds to all buyer types with $P^j(b) > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}$. If the seller posts price $P^*(\hat{b}) + \varepsilon$ after such history, then by Lemma 23, the mass of buyer types who accept such price is at least $\gamma(\delta_j) > 0$. The profit of seller type $\sigma_j$ from such buyer types is $P^*(\hat{b}) + \varepsilon - c(\sigma_j) \geq P^*(\hat{b}) + \varepsilon - (P^*(\hat{b}) + \frac{\varepsilon}{2}) = \frac{\varepsilon}{2}$ (by (75) and $\sigma_j < s_j$). Hence,

$$V_{Lj} \geq \gamma(\delta_j)\frac{\varepsilon}{2} \tag{80}$$

**Lower bound on $x_{Kj}$.** Combining inequalities (77), (78), (79), (80) I get

$$Cx_{Kj} \geq \frac{K_j(1-\delta_j)}{M_{Kj}}\frac{L_j(1-\delta_j)}{M_{Lj}}\left(\delta_j^{2K_j/M_{Kj}} - \delta_j^{2K_j}\right)\left(\delta_j^{2L_j/M_{Lj}} - \delta_j^{2L_j}\right)\frac{\gamma(\delta_j)}{(1-\delta)_j^2}\frac{\varepsilon}{2}. \tag{81}$$

where I collect all the constants into a positive constant $C$. Since $\frac{K_j(1-\delta_j)}{M_{Kj}} \sim \frac{K_j \ln(\delta_j)}{M_{Kj}}$, $\frac{L_j(1-\delta_j)}{M_{Lj}} \sim \frac{L_j \ln(\delta_j)}{M_{Lj}}$ and $\gamma(\delta_j) \sim (1-\delta_j)^2$, I can find $M_{Kj}$ and $M_{Lj}$ (in general dependent on $\delta_j$) such that right-hand side of inequality (81) converges to a positive number. On the other hand, $x_{Kj} \leq b_j - \hat{b} \leq \frac{\varepsilon}{2\ell}$ by (75). This contradicts the fact that $\varepsilon$ was chosen arbitrary. $\square$

**Corollary 2.** *For any $b \geq b^{\omega}_{s+}$, $P^*(b) = c(s^{\alpha}_b)$.*

*Proof of Corollary 2.* Suppose to contradiction that there exists some $\tilde{b} \geq b^{\omega}_{s+}$ such that $P^*(\tilde{b}) > c(s^{\alpha}_{\tilde{b}})$. For $b < b^{\omega}_{s+}$, $c(s^{\alpha}_b) > P*(b)$ and combined with Lemma 4, this implies that there is $\hat{b} \geq b^{\omega}_{s+}$ such that $P^*(b)$ is discontinuous at $\hat{b}$ which contradicts Lemma 5. $\square$

**Lemma 24.** *Sequence $P^j(b)$ converges uniformly to $P^*(b)$ on $[0,1]$.*

*Proof.* I show that the function $f^j(b) = \hat{P}^j(b) - P^*(b)$ converges uniformly to zero on $[0,1]$, which would imply the desired uniform convergence of $P(b)$ by the following claim.

*Claim* 19. *For any $b \in [0,1]$, $0 < P^j(b) - P^*(b) \leq f^j(b)$.*

*Proof.* First, for all $b \in [0,1]$, $P^j(b) \leq \hat{P}^j(b)$ by the definition and so, $P^j(b) - P^*(b) \leq f^j(b)$. Second, by Lemma 1, $P^j(b) \geq \frac{v(0)+\delta c(0)}{1+\delta} > \frac{v(0)+c(0)}{2}$ for all $b \in [0,1]$. Moreover, by Lemma 17, $P^j(b) > c(s^{\alpha}_b)$ for all $b \in [0,1]$. Therefore, $0 < P^j(b) - P^*(b)$ for all $b \in [0,1]$. *Q.E.D.*

*Claim* 20. *The function $f^j(b)$ is upper-semicontinuous, and for any $\varepsilon > 0$, $f^j(b+\frac{\varepsilon}{\ell}) \geq f^j(b) - \varepsilon$.*

*Proof.* To show that $f^j(b)$ is upper-semicontinuous, consider a sequence $\{b_i\}_{i=1}^{\infty}$ converging to some $b \in [0,1]$. Then by continuity of $P^*(b)$ and right-continuity of $\hat{P}^j(b)$, $\limsup_{i\to\infty}(\hat{P}(b_i) - P^*(b_i)) = \limsup_{i\to\infty} \hat{P}(b_i) - P^*(b) \leq \hat{P}(b) - P^*(b)$.

Next, choose any $\varepsilon > 0$. Since $\hat{P}^j(b)$ is increasing, $\hat{P}^j\left(b + \frac{\varepsilon}{\ell}\right) \geq \hat{P}^j(b)$. Moreover, $P^*(b) = \max\left\{\frac{v(0)+c(0)}{2}, c(s_b^{\alpha})\right\}$ and the derivative of $c(s)$ is bounded above by $\ell$ and so, $-P^*\left(b + \frac{\varepsilon}{\ell}\right) \geq -P^*(b) - \varepsilon$. Therefore, $f^j(b + \frac{\varepsilon}{\ell}) \geq f^j(b) - \varepsilon$. *Q.E.D.*

*Claim* 21. *Function $f^j(b)$ converges uniformly to 0 on* $[0,1]$.

*Proof.* Function $f^j(b)$ converges poinwise to 0 on $[0,1]$. Since $f^j(b)$ is upper-semicontinuous function on a compact set by Claim 20, $f^j(b)$ achieves its maximum at some $b_j \in [0,1]$.

I next show that $f^j(b_j)$ converges to 0 as $j \to \infty$. Suppose to contradiction that for all $j \in \mathbb{N}$ there exists $\varepsilon > 0$ so that $f^j(b_j) > \varepsilon$. By Claim 20, $f^j(b) > \frac{\varepsilon}{2}$ for all $b \in \left[b_j, b_j + \frac{\varepsilon}{2\ell}\right]$. By compactness of $[0,1]$, sequence $b_j$ converges (over subsequence) to some $b^* \in [0,1]$ as $j \to \infty$ and so, there exists $J$ such that for all $j \geq J$, $b_j \in \left[b^* - \frac{\varepsilon}{8}, b^* + \frac{\varepsilon}{8}\right]$. Hence, for $b \in \left[b^* + \frac{\varepsilon}{8}, b^* + \frac{3\varepsilon}{8}\right]$, $f^j(b_j) > \frac{\varepsilon}{2}$ for all $j \geq J$. This contradicts the pointwise convergence of $f^j(b)$ to 0. *Q.E.D.* □

*Proof of Theorem 8.* The result follows from Corollaries 1 and 2 and Lemma 24. □

*Proof of Theorem 6.* Observe that continuation utility of seller type $s$ in the seller punishing equilibrium is bounded above by $P^j(b_s^{\omega}) - c(s)$. By Theorem 8, $\sup_{s\in[0,1]} |P^j(b_s^{\omega}) - c(s) - \max\{\frac{v(0)+c(0)}{2} - c(s), 0\}|$ converges to zero as $\delta_j \to 0$ which gives the desired conclusion. □

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