Evidence and Skepticism
in Verifiable Disclosure Games[*]

Daniel Rappoport[†]
Columbia University

January 4, 2018
Job Market Paper

Abstract

A key feature of communication with evidence is *skepticism*: to the extent possible, a receiver will attribute any incomplete disclosure to the sender concealing unfavorable evidence. The degree of skepticism depends on how much evidence the sender is expected to possess. I characterize when a change in the prior distribution of evidence induces *more skepticism*, i.e. induces any receiver to take an equilibrium action that is less favorable to the sender following every message. I formalize an increase in the sender's (ex-ante) amount of evidence and show that this is equivalent to inducing more skepticism. My analysis provides a method to solve general verifiable disclosure games, including an expression for equilibrium actions. I apply these results to a dynamic disclosure problem in which the sender obtains and discloses evidence over time. I identify the necessary and sufficient condition on the evidence structure such that the receiver can benefit from early inspections.

Keywords: Verifiable Disclosure, Hard Information, Monotone Likelihood Ratio Property, Comparative Statics

1. Introduction

A fundamental question in communication with misaligned interests is "what are you not telling me?" Auditors will be suspicious about missing line items, and juries are less likely to convict on purely circumstantial cases. This *skepticism* in response to incomplete disclosures is a shared feature of communication with evidence. Seen from the perspective of prosecutors and firms, jurors and corporate auditors who hold less skeptical beliefs will be preferred. Identifying beliefs that induce more skepticism is therefore important in understanding how jurors or corporate auditors are selected, or more generally, any situation in which the "sender" can affect or select the beliefs of the "receiver".

The degree of skepticism depends on beliefs about the availability of evidence: a circumstantial case provoked less suspicion thirty years ago than it does currently. The criminal justice literature terms this the "CSI effect": Shelton *et al.* (2009) find that jurors who are more informed about forensics expect more hard evidence to be available to the prosecutor, and as a consequence are less likely to convict for every evidence profile. This suggests that prosecutors will try to avoid jurors who expect more evidence to be available. However, in criminal trials, or corporate disclosures, there are potentially multiple dimensions over which one can be informed. In these general evidence environments, what does it mean for there to be more available evidence? And, to what extent is this associated with inducing more skepticism?

Model and Main Results   I address these questions in a general verifiable disclosure framework. An informed sender communicates with an uninformed receiver in order to influence his action choice. While the receiver's preferences over actions depend on the private information or "type" of the sender, the sender always prefers higher actions.[1] Following Hart *et al.* (2017), and Ben-Porath *et al.* (2017), I model the structure of hard evidence as a partial order: type t dominates type s according to the "disclosure order", or $t \succeq_d s$, if type t can mimic type s. For example, a suspect with an alibi dominates a suspect without one, as the former can simply conceal his alibi. Importantly, there is no assumed relationship between whether a type is "high value" (commands a favorable best response from the receiver) and whether that type is dominant according to the disclosure order (has a large feasible message set).

The main goal of this paper is to characterize changes in the prior distribution of evidence (or types) that induce greater skepticism. One prior distribution induces *more skepticism* than another if equilibrium actions are lower following any message, regardless of the receiver's preferences.[2] To characterize this equilibrium notion I formalize an increase in the amount of evidence: one prior distribution f has *more evidence* than another prior distribution g if whenever $t \succeq_d t'$, the likelihood ratio $\frac{f}{g}$ is greater at t than at t'. That is, whenever type t can mimic t', type t is relatively more likely than t' under the distribution with more evidence.

---

[1] The receiver chooses the action from a subset of $\mathbb{R}$.

[2] There are multiple equilibria. I focus on the receiver optimal equilibrium.

The main result, Theorem 1, shows that the more skepticism and more evidence orders are equivalent. That is, (i) if the sender has more evidence, then any receiver takes a lower action for every type (in fact, after every message) and (ii) if the sender does not have more evidence then there exists a receiver that will strictly increase his action following some message. The key to establishing Theorem 1 is characterizing equilibrium: Theorem 3 provides an explicit expression for the associated mapping from types to actions.

The characterization of more skepticism is important for understanding which prior beliefs the sender wants to induce in the receiver. The sender's ex-ante payoff depends on both his and the receiver's prior beliefs over evidence. Theorem 1 identifies the more evidence order as the preference over receiver prior beliefs that is common to all senders. That is, a sender with any ex-ante distribution over evidence prefers to induce one prior belief over another if and only if it has less evidence. This illustrates how the "CSI effect" can shape jury selection: the prosecutor will try to avoid jurors who believe he has more evidence.

Why More Evidence Induces More Skepticism   Any equilibrium partitions the sender types into *pooled sets* that obtain the same equilibrium action. The central insight to Theorem 1 is that in the receiver optimal equilibrium, the value of these pooled sets decreases under a more evidence shift. The analysis proceeds by first characterizing these pooled sets, and then using this and novel comparative statics techniques to establish the above claim.

Intuitively, pooled sets "cannot be separated" because they involve low value types that are more dominant in the disclosure order mimicking high value types that are less dominant. For example, the set of job applicants that present no references include experienced applicants with bad references mimicking higher quality fresh applicants who actually have no references. Definition 5 formalizes this intuition. Pooled sets are those over which the receiver's best response is *downward biased*: any subset of types that cannot mimic their complement (i.e. a lower contour subset according to $\succeq_d$) has higher value than the set as a whole.[3] The receiver optimal equilibrium is uniquely characterized by the receiver's best response being downward biased on each pooled set.[4]

This observation underlies two novel solution methods for receiver optimal equilibria: (i) an algorithm to find the equilibrium partition; and (ii) an explicit expression for the equilibrium actions. The pooled set containing type t forms through the following two step process: first, t chooses to mimic a set of types with higher value, and second, some set of types chooses to mimic t if t has higher value. The former serves to maximize the action for type t while the latter serves to minimize it. The result is the familiar "minmax" form of the expression for equilibrium actions in Theorem 3.

I next analyze how the value of pooled sets changes under a more evidence shift. Proposition 3 establishes that the value of a set decreases under any more evidence shift if and only if the down-

---

[3] A lower contour subset S of a partially ordered set $(X, \geq)$ is all the elements dominated by elements in S, i.e. $\{s \in X : \exists s' \in S, s' \geq s\}$.

[4] Some additional constraints on the partition are required and made precise in Proposition 1.

ward biased property holds. This means that the downward biased property characterizes both equilibrium pooled sets and monotone comparative statics (MCS) under any more evidence shift.

This connection between MCS under any more evidence shift and pooled sets is not implied by known results concerning monotone likelihood ratio shifts in the distribution. For instance, the well known result that a monotone likelihood ratio increase lowers the expectation of a decreasing function cannot be directly applied. The reason is that the downward biased condition does not imply that the value is decreasing in the disclosure order. That is, within a pooled set, it is not necessarily true that if t can mimic t', the value of t is lower than that of t'. This means that the effect of a more evidence change on the value of a pooled set, which shifts probability to mimicking types, is unclear.

Section 5 develops novel results to deal with downward biased sets. The main contribution is Algorithm 2 which iteratively pools larger and larger subsets based on incentives to mimic. At each stage, one subset only mimics another if the value of the latter subset is higher. This means that at every stage the value of each "currently pooled subset" is lower under the more evidence distribution. Forming the equilibrium partition through this process shows that the value of a pooled set decreases under a more evidence shift. Subsection 1.1 illustrates this algorithm in a specific example.

Application to Dynamic Disclosure   The relationship between more skepticism and the sender's ex-ante preferences over receiver prior beliefs also has applications to dynamic disclosure. In a pre-play communication stage the sender will never signal to the receiver that he expects more evidence.

I apply this observation to a game in which the sender obtains and discloses evidence over two periods. I examine whether the receiver can benefit from "early inspections" relative to the game in which communication only occurs in period 2. The sender's distribution over period 2 evidence depends on his period 1 evidence, so there is potential for informative signaling. However, because evidence accumulates over time, period 1 disclosures can reveal to the receiver that the sender expects more evidence in period 2, and thereby induce more skepticism in the receiver.

I show that the receiver does not benefit from early inspections regardless of his preferences or prior beliefs if and only if the evidence structure satisfies the "Unique Evidence Path Property" (UEPP). The UEPP holds, if for any two types that cannot mimic each other in period 1, the types that they can possibly become in period 2 also cannot mimic each other. Broadly, This ensures that any "potential for separation" created in period 1 is preserved in period 2. Without this property, the receiver can benefit from early inspections by separating period 1 evidence realizations that could otherwise lead to inseparable realizations in period 2. In contrast, under the UEPP any "informative signaling" in period 1 will violate sender incentive compatibility. More specifically, the receiver's beliefs following any two different period 1 disclosures will be ordered by the more evidence relation from the perspective of some type.

Implications and Extensions Theorem 1 unifies some existing results from the verifiable disclosure literature. Seemingly different changes in the distribution considered in Jung & Kwon (1988), Guttman *et al.* (2014), and Dziuda (2011) imply decreases in equilibrium actions for specific receiver preferences and evidence structures. This paper identifies the more evidence relation as the common thread between these changes. While examples like Guttman *et al.* (2014), who consider adding evidence types to the Dye model, are consistent with the interpretation of the main result, Theorem 1 also applies to seemingly less related changes in the distribution of evidence. I generalize the result from Dziuda (2011) that decreasing the probability of honest types (types who must fully reveal themselves) decreases the equilibrium actions for all messages by showing that this change corresponds to a more evidence shift.

The full characterization of receiver optimal equilibrium also allows for new insights when the assumptions on the sender's preferences are relaxed. First, I consider introducing some probability of senders who are "unbiased", i.e. have the same preferences as the receiver. I show that unlike in cheap talk games (Kim & Pogach (2014)) the receiver optimal equilibrium is equivalent to that in a game with the same probability of honest senders. The actions in this equilibrium are the same as one without unbiased senders, but where the receiver has a higher best response to any subset of evidence types. Moreover, I show that a decrease in the probability of unbiased types can be seen as a more evidence change, and thereby induces more skepticism.

Second, I consider a game in which the receiver does not know whether the sender prefers higher or lower actions. I construct a disclosure game in which the sender has known preferences toward higher actions that has the same receiver optimal equilibrium. In this equilibrium there exists exactly one message that does not credibly convey the sender's direction of bias. I use Theorem 1 to show that the sender always wants to convince the receiver that his bias is the opposite of his own.

Layout The paper proceeds as follows. Subsection 1.1 previews the model, characterization approach, and comparative statics result in a simple example. Subsection 1.2 discusses the related literature. Section 2 lays out the model and lists examples that fit my framework. Section 3 defines the more skepticism and more evidence orders and states the main result that they are equivalent. The analysis proceeds by first characterizing the receiver optimal equilibrium in Section 4, and then introducing and using the comparative statics techniques in Section 5. I then move on to two applications. Section 6 considers two extensions based on the sender's preferences, and Section 7 discusses a dynamic disclosure framework. Section 8 concludes. Unless noted otherwise, all proofs are in the appendix.

1.1. An Example of the Model and a More Evidence Shift

Consider an entrepreneur (the sender) who instructs his engineers to run a beta test for a new software. There are four different outcomes. The software could perform **above expectations** garnering positive reviews from its users. The reviews could also reveal that software is **inaccessible** to non-scientific users. Having realized that the software is inaccessible, the engineers could partially

4

salvage the problem by adding a useful **tutorial**. Lastly, the beta test could yield **no evidence**, perhaps because the users were not a representative sample or because there was a bug in the software.

The entrepreneur reports to his investor (the receiver), and attempts to extract the most funding possible. However, communication is not "cheap talk" and some of the above outcomes can be certified. If the product performs above expectations, or if the software is inaccessible, the reviews can confirm this. In addition, if the software is inaccessible and a tutorial is developed the entrepreneur can credibly present this new tutorial. Although, in this case it will be apparent that the software was inaccessible to begin with. Finally, the entrepreneur can always claim that the test results were unusable.

The entrepreneur can be one of four "types"- no evidence (NE), above expectations (AE), inaccessible (I), and Tutorial (T). The problem is illustrated in Figure 1. The directed graph illustrates the disclosure order: each vertex represents a type, and the available messages to each type are the set of vertices accessible via a directed path. For example, T can declare $\{T, I, NE\}$ but not $\{AE\}$. The investor's type dependent value for the product is displayed above each vertex. Suppose that the investor's prior over entrepreneur types is uniform and the investor chooses an amount of funding equal to the expected value of the product.[5]

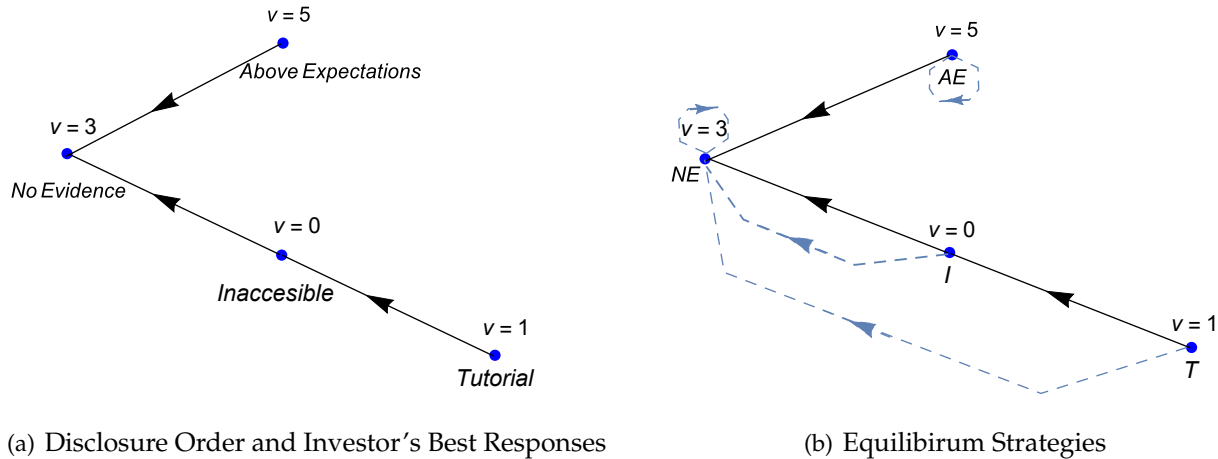(a) Disclosure Order and Investor's Best Responses   (b) Equilibirum Strategies

Figure 1: Investor with Uniform Prior

The unique equilibrium involves the pure strategies represented by the dotted arrows in the right panel of Figure 1. Types in $\{NE, I, T\}$ all claim to have no evidence, and obtain funding equal to $a(NE) = 4/3$. The AE type truthfully reveals and obtains funding $a(AE) = 5$.[6] The interpretation is that an entrepreneur will only reveal positive test results, and will claim the test

---

[5] Formally, let the investors utility over actions and types be given by $-(a - v(t))^2$ where $v(t)$ is the value above each vertex.

[6] There are many ways to set the actions for the off path declarations I and T. The analysis in the main text does this according to the truth leaning refinement by HKP, which in this case dictates that $a(I) = 0$ and $a(T) = 1$.

was faulty otherwise. The investor anticipates this, and is skeptical upon receiving NE, i.e. he forms a lower expectation of the value of the product than if he were certain that the test were faulty.

The above equilibrium can also be seen as a partition of the types into sets who obtain the same equilibrium action, or *pooled sets*. This partition is (P_1, P_2) where $P_1 = \{NE, I, T\}$ and $P_2 = \{AE\}$. The most important feature of this partition is that the investor's expected value is downward biased on each pooled set: the lower contour subsets of P_1, i.e. subsets who cannot mimic their complement, induce a higher investor expected value than P_1 as a whole. To verify, note that the expected values of NE and $\{NE, I\}$, (the two (strict) lower contour subsets) are 3 and 3/2 respectively, which are both greater than 4/3. In Proposition 1, I show that the downward biased property characterizes pooled sets in the receiver optimal equilibrium.

Suppose that the investor learns that the beta test and engineers are of higher quality, and therefore provide the entrepreneur with more evidence about his product. Specifically, the investor now believes there is a $\frac{1}{12}$ probability of NE, a $\frac{1}{6}$ probability of I, a $\frac{1}{4}$ probability of T, and a $\frac{1}{2}$ probability of AE. This means the entrepreneur is both more likely to get a test result and more likely to develop a tutorial. This distribution and the original uniform distribution are compared by the more evidence relation: the likelihood ratio between any type and some type that can be mimicked has increased. For example, the T type can mimic the I type, and the likelihood ratio between T and I has increased from 1 to 3/2. Does the entrepreneur benefit or suffer from this change in the investor's beliefs (assuming that his true distribution over evidence remains fixed)?

The receiver optimal equilibrium involves the same pooled sets as under the uniform prior. Thus, the answer to the above question depends on how the investor's value for the set $\{NE, I, T\}$ changes when the entrepreneur is believed to have more evidence (the funding for AE remains at 5). While in this case, one can simply calculate that it decreases, $a(\{NE, I, T\}) = 6/5$, it is not apparent whether this is general for more evidence shifts in the investor's prior. Probability is shifted from NE to I which tends to decrease the investor's value, but probability is also shifted from I to T which increases the investor's value.[7] Theorem 1 shows that despite this ambiguity, the fact that these types pool together ensures that the investor's value decreases under any more evidence shift. Furthermore, the decrease in equilibrium funding would hold regardless of how the investor values different realizations of the beta test. In the language of this paper, the investor's new prior induces *more skepticism* than the uniform prior.

To see why this is true consider pooling the $\{NE, I, T\}$ set iteratively as follows. Notice that whatever the prior, the I type will pool with the NE type because $v(I) < v(NE)$. This means that one can treat the $\{NE, I\}$ set as a single type with an expected value (under the uniform distribution) of 3/2. Therefore, type T will also pool with $\{NE, I\}$ because $v(T) < 3/2$. The key observation is that any more evidence shift can be decomposed in a similar way. Probability is shifted first from

---

[7] One could also see this change as moving probability from both NE and I to T. The same comment applies because the former change decreases the investor's value while the latter increases it.

NE to I, and second from the set $\{NE, I\}$ to T. Since each shift decreases the investors value, one can conclude that the overall effect is negative. Section 5 generalizes this argument to show that the downward biased property characterizes monotone comparative statics under any more evidence shift.

1.2. Related Literature

The first verifiable disclosure models were introduced by Milgrom (1981), Grossman (1981), and Grossman & Hart (1980). The sender, who knows the state and is biased towards higher actions, can be vague but not lie, i.e. he can declare any subset of states that contains the true state. The main finding is the "unraveling" result that in any equilibrium the sender fully reveals his information. There are multiple ways in which unraveling can fail: if the sender's direction of bias depends on his type (e.g. Seidmann & Winter (1997)), if the sender pays a cost to disclose information (e.g. Verrecchia (1983)), or if it is unknown whether the sender knows the state (e.g. Dye (1985) and Jung & Kwon (1988)).[8],[9]

The Dye (1985) model in which the sender has the possibility to be uninformed has been extended to incorporate more complex evidence structures. Shin (2003) and Dziuda (2011) consider multidimensional versions of this model in which the sender obtains potentially multiple pieces of either good or bad evidence and can disclose any subset. Dziuda (2011) and Einhorn (2007) also consider uncertainty over the preferences of the sender and over whether he is honest or strategic.

There have also been a number of dynamic extensions. Guttman *et al.* (2014) and Acharya *et al.* (2011) consider dynamic frameworks in which the receiver is also uncertain about *when* the sender has obtained evidence. Einhorn & Ziv (2008) consider a repeated dye model in which the sender wants to achieve a reputation of having less evidence. These dynamic models develop or use seemingly different changes in the distribution that induce lower equilibrium actions. I show that these changes are examples of more evidence shifts.

Another strand of literature shows that the receiver's utility in some equilibrium of the verifiable disclosure game is the same as that in which the receiver can commit to a best response before learning the sender's message. This equivalence was first introduced in Glazer & Rubinstein (2004) and further explored by Sher (2011), and Ben-Porath *et al.* (2017). Hart *et al.* (2017) identifies the equilibrium that achieves this equivalence through the "truth leaning refinement". I focus on this receiver optimal equilibrium and my model is the same as that in Hart *et al.* (2017) and Sher (2011).

In addition to the above equivalence results, Glazer & Rubinstein (2004) and Sher (2014) derive methods to find the receiver optimal equilibrium. However, their models involve a binary action choice and only two types of senders - acceptable and unacceptable. My algorithm for solving for equilibrium bears similarities to that in Bertomeu & Cianciaruso (2016) which characterizes equilibrium in disclosure games when pure strategy equilibria exist. I focus on equilibrium outcomes

---

[8] Hagenbach *et al.* (2014) and Mathis (2008) provide necessary and sufficient conditions for unraveling in a general framework.

[9] For surveys of the verifiable disclosure literature see Milgrom (2008) and Dranove & Jin (2010).

which also allows for a tractable characterization in games with only mixed strategy equilibria such as that in Dziuda (2011).

To my knowledge there is no other study that examines monotone likelihood ratio shifts over a partially ordered set. The more evidence shift involves the monotone likelihood ratio order because my methodology requires that first order stochastic dominance (FOSD) hold on any subset of types. Milgrom (1981) showed that such a change in the distribution must be a monotone likelihood ratio shift. I iteratively apply the well known result that an FOSD shift in the distribution lowers the expectation of a decreasing function.

2. Model

The setting involves a single sender and a single receiver. The sender observes his type $t \in T$, where $|T| = n$, and sends a message from his feasible set. The receiver observes this message and then chooses an action $a \in A$, where A is a compact convex subset of $\mathbb{R}$.

The receiver has a prior belief $h \in \Delta T$ over the sender's type. In order to investigate the sender's preferences over inducing different receiver prior beliefs, I allow the sender to have a potentially different prior, $\eta \in \Delta T$. However, the set of equilibria does not depend on the sender's prior because the sender chooses his message after realizing his type. Unless otherwise noted, I assume that η and h have common and full support over T. Appendix G extends the results to general distributions.

2.1. Preferences

The receiver's utility, $U^R : A \times T \to \mathbb{R}$, depends on both the action and the sender's private information. The sender always prefers higher actions and so it is without loss of generality to let $U^S(a, t) = a$. I assume that U^R is strictly concave and differentiable in a. Denote the set of all such receiver utilities Υ. The strict concavity ensures that the receiver has a unique optimal action for all distributions over sender types.[10]

The preference assumptions ensure the "in-betweenness" property from Hart *et al.* (2017).

Lemma 1. *For any distribution over types $q \in \Delta T$, define $a^*(q) \equiv \arg\max_a \mathbb{E}[U^R(a, t)|t \sim q]$. Consider two distributions $q_1, q_2 \in \Delta T$ such that $a^*(q_1) < a^*(q_2)$. For any $\lambda \in (0, 1)$,*

$$a^*(q_1) < a^*(\lambda q_1 + (1 - \lambda)q_2) < a^*(q_2).$$

This says that the optimal action for the mixture of two distributions is between the optimal actions in response to each individual distribution.[11] Lemma 1 ensures that whenever two types pool together (declare the same message), one type would like to credibly reveal himself to the

---

[10] Making the same weaker assumptions as in Hart *et al.* (2017) would not change the results, i.e. $\mathbb{E}[U^R(a, t)|t \sim q]$ is strictly quasi-concave in a, $\forall q \in \Delta T$.

[11] One could abstract from the action choice, and alternatively model the sender and receiver preferences over induced

receiver, while the other type would like to remain pooled. Thus in order for the former type to separate in equilibrium he must have access to a message that the latter type does not.

Denote the receiver's unique best response to type t by $v(t) \equiv \arg\max_a U^R(a,t)$. Similarly, define $V_h(S) \equiv \arg\max_a \mathbb{E}[U^R(a,t)|t \in S, t \sim h]$ to be the receiver's best response conditional on the sender's type being in S and distributed according to h. I refer to sets of types with relatively high (low) optimal actions, as "high (low) value".

The leading example for the receiver's utility will be quadratic loss defined by $U^R(a,t) = -(a - v(t))^2$ for any function $v : T \to \mathbb{R}$. In this case $V_h(S) = \mathbb{E}[v(t)|t \in S, t \sim h]$ is the expectation of the receiver's value for each type.

2.2. Messaging Technology

I assume that the message space is the type space and interpret type t sending message t' as type t "mimicking" t'. In addition, there is a partial order $\succeq_d$ over T, such that $t \succeq_d t'$ means that t can mimic t'. The set of available messages to each type t is given by $M(t) \equiv \{s : t \succeq_d s\}$. The partial order assumption imposes reflexivity and transitivity. That is, (i) every type can truthfully reveal (send their corresponding message) and (ii) if t can mimic t' and t' can mimic t'', then t can mimic t''.[12] I refer to $\succeq_d$ as the "disclosure order".

Remark 1. One can add additional messages, by adding zero probability types.

The verifiable disclosure literature sometimes uses a more general messaging structure. There is an arbitrary message space C and each type has access to some subset $E(t) \subset C$. However, these papers often assume a normality (Bull & Watson (2004)) or nested range condition (Green & Laffont (1986)) which make the two sets of assumptions "equivalent". That is, given a message space C and message sets $E : T \to 2^C$, there exists a $(T, \succeq_d)$ such that the set of equilibria are the same under both messaging structures.[13,14]

Two generalities of the model are worth emphasis. First, the disclosure order is arbitrary. And second, there is no assumed relationship between the disclosure order and the receiver's preferences. Common examples that fit the framework are described below. I represent $(T, \succeq_d)$ as a directed graph with vertices representing types and directed paths between types representing dominance in the disclosure order.

---

receiver beliefs given by $U^S, U^R : \Delta T \to \mathbb{R}$. In this framework Lemma 1 is a necessary assumption for my results. The two following assumptions would also be required: (i) $\forall \mu, \mu' \in \Delta T$, $U^S(\mu) = U^S(\mu') \implies U^R(\mu) = U^R(\mu')$, and (ii) $U^S(\mu)$ is continuous in μ.

[12] The partial order also imposes antisymmetry but this is without loss of generality. Types within an equivalence class cannot induce different equilibrium actions, because the sender always prefers higher actions.

[13] A message structure is normal if $\forall t \in T$, there exists $e_t \subset E(t)$, such that $\forall t, t' \in T$, $e_t \in E(t') \implies E(t) \subset E(t')$. The following disclosure ordered type space $(T, \succeq_d)$ has the same equilibrium set. $T \equiv M$ with $pr(e_t) \equiv pr(t)$ and $pr(m) = 0$ otherwise, and $e_t \succeq_d m$ if $m \in E(t)$.

[14] For examples of disclosure games without normality, see Sher (2014) and Rubinstein & Glazer (2006).

2.3. Examples

Dye Evidence The type space is $T = \{t_1, ..., t_{n-1}, t_\emptyset\}$. With probability $(1-p)$ the sender is "uninformed", $t_\emptyset$, and with probability p the sender draws an "evidence type" from $h \in \Delta\{t_1, ..., t_{n-1}\}$. Denote the total probability distribution $h_p \in \Delta T$. The disclosure order is given by $M(t_i) = \{t_i, t_\emptyset\}$, $\forall i < n$, and $M(t_\emptyset) = \{t_\emptyset\}$. The interpretation is that the evidence types can certify their type or pretend to be uninformed, while the uninformed type cannot certify his lack of evidence. This model was first introduced by Dye (1985), and Jung & Kwon (1988), and has been widely used in the verifiable disclosure literature, e.g. by Grubb (2011), Acharya *et al.* (2011), and Bhattacharya & Mukherjee (2013).

Vagueness The sender learns the state $x \in X$ drawn from $h \in \Delta X$. The type space is all non-empty subsets of X: $T = 2^X \setminus \emptyset$. The disclosure order is given by $M(t) = \{t' \in T, t \subset t'\}$, $\forall t \in T$. The interpretation is that each type t can credibly reveal himself or be "vague". For example, in Figure 2, $X = \{0, 1\}$, and type $\{0\}$ can either truthfully report $\{0\}$, or be vague and report $\{0, 1\}$, however she cannot "lie" and report $\{1\}$. This description uses zero probability types to model additional messaging options for positive probability types. Any non-singleton type t is zero probability, but represents a feasible message for types $t' \subset t$. These message structures were first introduced by Grossman (1981) and Milgrom (1981).

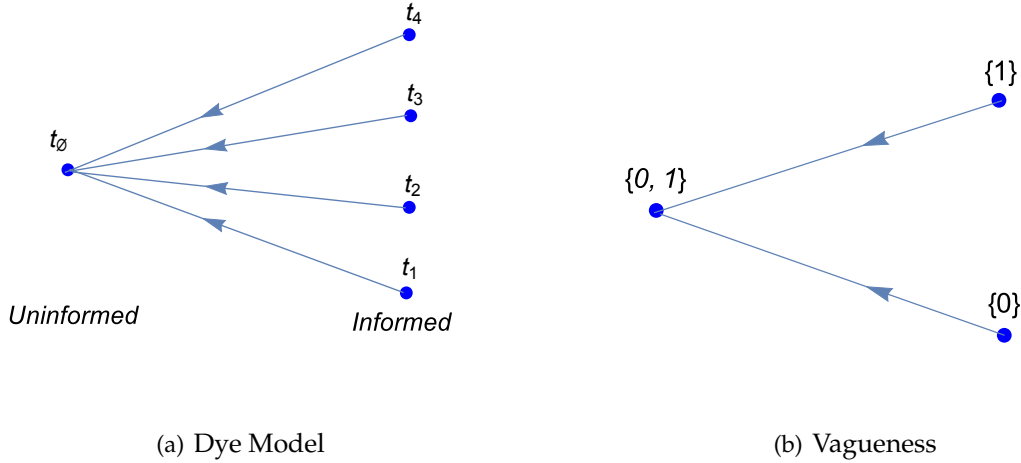(a) Dye Model   (b) Vagueness

Figure 2: Common Examples

Multidimensional Evidence The agent draws an integer m from some $g \in \Delta\{0, 1, ..., k\}$. The agent then draws a sample of size m from some distribution $h \in \Delta X$ where X is a finite set. Each type is a sample of size m, i.e. $t = \{x_1, .., x_m\}$, and the type space is $T = \{t \in 2^X : |t| \leq k\}$. The disclosure order is given by $M(t) = \{t' : t' \subset t\}$ $\forall t \in T$. The interpretation is that each type can report any combination of pieces of evidence in his possession. Examples of multidimensional

evidence models include Guttman *et al.* (2014), Dziuda (2011), and Shin (2003). Figure 3 illustrates a multidimensional evidence model where $k = 2$ and $X = \{0, 1\}$.
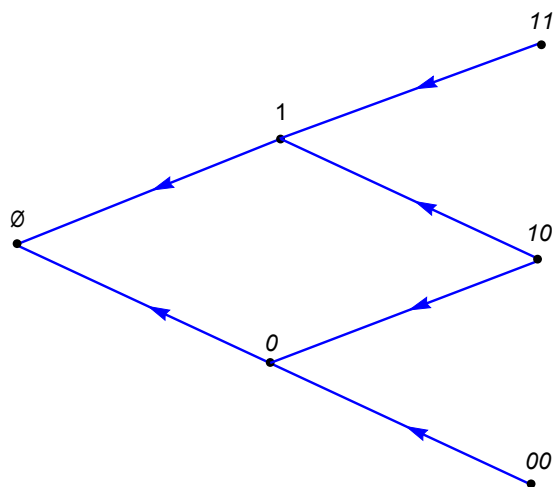

Figure 3: Mutlidimensional Evidence

Honest Types In addition to obtaining evidence from some T', the sender can either be strategic, S, with probability p, or honest, H, with probability $1 - p$. Strategic types can disclose evidence according to some disclosure order $\succeq'_d$, while honest types must truthfully reveal. The total type space and disclosure order are given by $(T, \succeq_d)$ defined as follows: $T = T' \times \{S, H\}$, with $M(t, S) = M_{\succeq'_d} \times \{S, H\}$ and $M(t, H) = (t, H)$. Figure 4 displays the multidimensional example from Figure 3 with the addition of honest types.
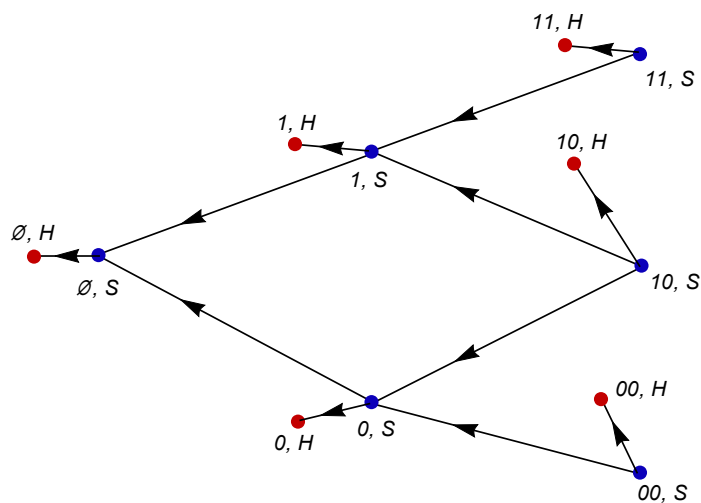

Figure 4: Honest Types

Complete Order and Empty Order   The cases in which the disclosure order is complete or empty serve as illustrative examples. A completely disclosure ordered type space is given by $(T, \succeq_d)$, with $T = \{t_1, ..., t_n\}$, and $i \geq j \iff t_i \succeq_d t_j$. That is, types with higher indices can report all types with lower indices. An empty disclosure ordered type space $(T, \succeq_d)$ is given by $t \succeq_d t' \iff t = t'$. Since no type dominates any other in the disclosure order, each type is forced to truthfully reveal.

2.4. Strategies and Equilibrium

A strategy for the sender is $\sigma : T \to \Delta T$ where $\mathrm{Supp}(\sigma_t) \subset M(t)$, $\forall t$. $\sigma_t(s)$ refers to the probability that type t declares type s. Because the receiver's utility is strictly concave, it is without loss to restrict the receiver to use a pure strategy, $a : T \to A$, which specifies an action choice in response to each message. A perfect Bayes equilibrium (PBE) is a pair of strategies for the sender and receiver such that $\sigma_t(s) > 0 \implies s \in \arg\max_{s' \in M(t)} a(s')$, $a(s) = \arg\max_a \mathbb{E}[U^R(a, t)|\sigma, s]$, $\forall s \in \mathrm{Supp}(\sigma)$, and $a(s) = a^*(q)$ for some $q \in \Delta\{t : t \succeq_d s\}$, $\forall s \in T$.

I focus on the receiver optimal PBE. A number of studies have provided justifications for this selection. Hart *et al.* (2017) has shown that the truth leaning refinement, in which the receiver interprets each off path message credulously, selects the receiver optimal equilibrium. Relatedly, Bertomeu & Cianciaruso (2016) shows that the receiver optimal equilibrium is also the unique equilibrium without "self signaling sets".

My methodology focuses on the equilibrium *outcomes*, i.e. the equilibrium mapping from types to actions. Denote $\pi_h(t|U^R)$ as the receiver optimal equilibrium action for type t facing a receiver with preferences U^R and prior beliefs $h \in \Delta T$.

3. Characterizing Increased Skepticism

The main goal is to explore how the receiver optimal equilibrium actions, $\pi_h(t|U^R)$, depend on the receiver's prior distribution, h.

3.1. The More Skepticism Order

Definition 1. Let $f, g \in \Delta T$. f induces more skepticism than g, also expressed as $f \geq_{MS} g$, if

$$\pi_f(t|U^R) \leq \pi_g(t|U^R), \ \forall t \in T, \ \forall U^R \in \Upsilon. \tag{1}$$

One prior belief induces more skepticism than another if it leads *any* receiver to take a lower equilibrium action for every type.

Remark 2. Definition 1 identifies the preference order over receiver prior beliefs that is common to any sender. That is, ex-ante, the sender would like to induce the receiver to hold less skeptical beliefs regardless of his actual distribution over evidence, η, or the receiver's preferences, U^R. Also, any distribution that does not induce more skepticism is preferred by the sender for some η. This

means that if the sender can signal to the receiver prior to realizing his type, he will never send signals that make the receiver more skeptical.[15]

One basic question concerns whether $\geq_{MS}$ is empty. Alternatively, does one's level of skepticism always depend on their preferences? To see that it does not, consider the following simple example.

Example 1. Let the type space be $T \equiv \{t_1, t_2\}$ with $t_2 \succeq_d t_1$. The prior distribution is given by $p \equiv pr(t_2)$. The equilibrium structure has two cases based on U^R: (i) $v(t_1) \geq v(t_2)$, t_1 and t_2 both declare t_1, and obtain the "average action" $V_p(T)$; and (ii) $v(t_1) < v(t_2)$, t_1 and t_2 separate, and obtain actions $v(t_1)$ and $v(t_2)$ respectively.

Now consider two such prior distributions $p' > p''$. In case (ii), the equilibrium actions do not depend on the prior, so $\pi_{p'}(\cdot|U^R) = \pi_{p''}(\cdot|U^R)$. In case (i), p' shifts probability to the lower value type t_2 relative to p'', and so $V_{p'}(T) \leq V_{p''}(T)$. This means that for all receiver preferences $\pi_{p'}(\cdot|U^R) \leq \pi_{p''}(\cdot|U^R)$, i.e. $p' \geq_{MS} p''$. △

Example 1 shows that shifting probability to types that are more dominant in the disclosure order can induce more skepticism. More specifically, the relative probability of the mimicking type-t_2, to the mimicked type-t_1, is higher under the more skeptical prior. The next definition extends this notion to more complex disclosure orders.

3.2. The More Evidence Order

Definition 2. Let $f, g \in \Delta T$. f has more evidence than g with respect to $\succeq_d$, also expressed as $f \geq_{ME} g$, if

$$\forall t, t' \in T, \quad t \succeq_d t' \implies \frac{f(t)}{g(t)} \geq \frac{f(t')}{g(t')}. \tag{2}$$

The more evidence relation depends only on the disclosure order (a dependence I often omit). For any type t that can mimic t', t is relatively more likely than t' under a prior distribution with more evidence. If $\succeq_d$ were a complete order, then $f \geq_{ME} g$ would be equivalent to f monotone likelihood ratio (MLR) dominates g on $(T, \succeq_d)$. Definition 2 is an extension of MLR dominance to a partially ordered set that only imposes the likelihood ratio inequality on comparable pairs of types.

Natural changes in the prior distribution are more evidence shifts. To preview now, recall the examples from Subsection 2.3. In the multidimensional evidence model, performing an MLR increase in the distribution g (the distribution over the number of pieces of evidence the sender obtains), while leaving h (the distribution of each piece of evidence) constant is a more evidence shift. In the honest types model, Increasing the probability of strategic types, p, is a more evidence shift.

---

[15] This statement will not be true if the prior belief of the sender and receiver are identical and one considers changing both simultaneously. That is, it is possible that despite the decrease in utility "type by type", the sender that induces more skepticism has a higher probability of being higher valued types, and therefore a higher ex-ante utility.

While Definition 2 concerns changes in the distribution, by allowing for zero probability types, more evidence changes can also capture changes in the disclosure order or "evidence structure". For distributions without full support, $f \geq_{ME} g$ with respect to $\succeq_d$ if

$$\forall t, t' \in T, \quad t \succeq_d t' \implies f(t)g(t') \geq f(t')g(t). \tag{3}$$

Thus, adding a "no evidence type" which cannot mimic any other type is a less evidence change. Similarly, adding a type which cannot be mimicked is a more evidence change.

3.3. The Equivalence Result

Theorem 1. *Let $f, g \in \Delta T$. The more skepticism and more evidence relations are equivalent, that is*

$$f \geq_{ME} g \iff f \geq_{MS} g.$$

The result says that If f has more evidence than g, all types obtain a lower action under f than g for any receiver preferences. The converse also holds: if f does not have more evidence than g, then there will exist a receiver that treats some type strictly more favorably under f than under g. In light of Remark 2, one can restate Theorem 1 as follows: the sender prefers to induce one belief over another in any receiver, regardless of his ex-ante distribution over evidence, if and only if it has less evidence.

The inequality in (1) is weak (vs. strict) partly because changes in the distribution do not affect outcomes for types that completely separate in equilibrium. However, Appendix I provides a strict counterpart to Theorem 1: if the likelihood ratio inequality in Definition 2 is strict, then every type that is "pooled" will experience a strict decrease in their equilibrium action under a more evidence shift.

The fact that more skepticism implies more evidence is relatively straightforward. If the likelihood ratio inequality in (2) does not hold for some pair of ordered types, $t \succeq_d t'$, one can consider receiver preferences such that these types pool together, while all other types separate. Since t only pools with t' if it has lower value, the receiver's best response to this set is higher under f than g.

The broad intuition for why more evidence implies more skepticism is as follows. Any equilibrium is a partition of the type space into sets of types that "pool together" or obtain the same equilibrium action. These *pooled sets* involve types with higher value that are less dominant in the disclosure order being mimicked by types with lower value that are more dominant in the disclosure order. A more evidence change shifts probability "up the disclosure order" on each of these pooled sets to the types with relatively lower value. This tends to reduce the receiver's best response to each pooled set, i.e. induce more skepticism.

The intuition above echoes Example 1 in which the more dominant type t_2 pools with the less dominant type t_1 because it has lower value. This case is simple because the receiver's best response is decreasing in the disclosure order (which is complete) on each pooled set. If this were true in

general then Theorem 1 would follow from: (i) the result in Milgrom (1981) that MLR dominance induces FOSD on subsets, and (ii) that the expected value of a decreasing function is lower under an FOSD shift. However, Figure 1 exemplifies a pooled set over which the receiver's best response is non-monotonic: the NE, I, and T types pool together in equilibrium, but $T \succeq_d I \succeq_d NE$, while $v(NE) > v(T) > v(I)$. In addition, pooled sets are not always completely ordered by $\succeq_d$.

The analysis deals with these issues in two steps. First, Section 4 characterizes pooled sets in terms of the disclosure order and the receiver's best response. Second, Section 5 shows that this condition is equivalent to the expected value decreasing under a more evidence shift. Before moving onto these two steps, I briefly illustrate Theorem 1, and discuss its extensions and limitations.

I next show how Theorem 1 can reconcile different comparative statics results in the Dye evidence model from Subsection 2.3. For two distributions in $f, g \in \Delta(t_1, ..., t_{n-1}, t_\emptyset)$ in the Dye model, $f \geq_{ME} g \iff$

$$\frac{f(t_\emptyset)}{g(t_\emptyset)} \leq \min_{i<n} \frac{f(t_i)}{g(t_i)}. \tag{4}$$

Jung & Kwon (1988) consider increasing p, the probability of evidence types while holding constant, h, the distribution over evidence types. They find that the non-disclosure action decreases. In this case, the inequality in (4) reduces to $\frac{1-p_2}{1-p_1} \leq \frac{p_2}{p_1}$, which holds for $p_2 > p_1$. For any probability of evidence p, let $h_p \in \Delta T$ denote the receiver's prior belief over the sender's type.

Corollary 1. *In the Dye Model, if $p_2 > p_1$, then $h_{p_2} \geq_{ME} h_{p_1}$, and $h_{p_2} \geq_{MS} h_{p_1}$.*

Guttman *et al.* (2014) consider a different change in the prior distribution but also find that the non-disclosure action decreases. Their change involves augmenting the Dye model with additional evidence types. For any two nested subsets of evidence types $S \subset S' \subset \{t_1, ..., t_{n-1}\}$, they find that the non-disclosure action is lower when the distribution is conditioned on the larger set S'. For an original prior $h \in \Delta T$, define the restricted prior h_S by

$$h_S(t) \equiv \begin{cases} \frac{h(t)}{H(S \cup \{t_\emptyset\})} & if \ t \in S \cup \{t_\emptyset\} \\ 0 & otherwise \end{cases}.$$

To see that $h_{S'} \geq_{ME} h_S$ note that (4) is an equality for $t_i \in S$, and the RHS is "infinite" for $t_i \in S' \backslash S$.[16]

Corollary 2. *In the Dye Model, if $S \subset S'$, then $h_{S'} \geq_{ME} h_S$, and $h_{S'} \geq_{MS} h_S$.*

3.3.1. Skepticism with Restricted Receiver Preferences

A more skeptical distribution induces lower equilibrium actions for all types, regardless of the receiver's preferences. The latter condition may be overly demanding in scenarios in which it is known that the receiver will value certain types higher than others. For example, an entrepreneur

---

[16] More formally, one would use the definition of more evidence in Equation 3.

with positive customer reviews will have a higher value according to any investor than one with negative customer reviews. This section generalizes Theorem 1 to characterize a version of more skepticism in which the inequality in (1) need only hold for certain receiver utilities.

Consider an arbitrary partial order $\succeq_v$ on T. Define the restricted set of receiver utilities,

$$\Upsilon_{\succeq_v} \equiv \{U^R \in \Upsilon : \ t \succeq_v t', \ t \neq t' \implies v(t) \geq v(t')\}.$$

Definition 3. A prior distribution f is more skeptical than g for receivers who agree on $\succeq_v$ if,

$$\pi_f(t|U^R) \leq \pi_g(t|U^R), \ \forall t \in T, \ \forall U^R \in \Upsilon_{\succeq_v}.$$

f is more skeptical than g for receivers who agree on $\succeq_v$ if it induces all receivers who share rankings according to $\succeq_v$ to take a lower equilibrium action for all types. Thus, Definition 1 corresponds to the above definition with an empty $\succeq_v$.

Define the *limited disclosure order*, $\succeq_{d,v}$, over T as the transitive closure of the relation given by $t \succeq_d t'$ and $t \not\succeq_v t'$.[17] The interpretation is that $t \succeq_{d,v} t'$, if t not only has the ability to mimic t', but also the incentive, for some receiver preferences in $\Upsilon_{\succeq_v}$.

Theorem 2. f is more skeptical than g for receivers who agree on $\succeq_v$ if and only if $f \geq_{ME} g$ with respect to $\succeq_{d,v}$.

The interpretation of the result is as follows. In order to ensure that equilibrium actions decrease for every type, the likelihood ratio inequality must only hold for the mimicking relationships "that will be used" in equilibrium. The broad intuition is that if $t \succeq_d t'$ and $v(t) \geq v(t')$ there exist receiver optimal equilibrium strategies in which t does not mimic t'. Thus, shifting probability between t and t' does not affect the receiver's best response.

Theorem 2 is useful in comparing different information structures when the evidence is a signal about an unknown payoff relevant state. Let the state space be X with prior distribution $h \in \Delta X$. The receiver has utility given by $U^R : A \times X \to \mathbb{R}$. The sender obtains evidence in T' drawn according to the *information structure*, $pr(t|x) \equiv q_x(t)$, which can be disclosed according to $\succeq'_d$.

This information structure framework fits into the basic model by defining $U^R(a,t) \equiv \frac{1}{\sum_{x \in X} q_x(t)} \sum_{x \in X} U^R(a,x) q_x(t)$. However, this means that a change in the prior distribution over T' does not necessarily correspond to a change in the sender's information structure and vice-versa. A change in the prior distribution over T assumes that the value of each type or signal stays constant which excludes certain changes in the information structure. Conversely, a change in the prior distribution over T can change the "expected value" of the sender's evidence which cannot occur under a change in the information structure.

By incorporating the state space into the type space and applying Theorem 2, one can characterize the changes in the information structure that induce more skepticism. Label $X = (x_1, ..., x_m)$

---

[17] the transitive closure of a binary relation, is the coarsest transitive refinement.

so that the receiver prefers higher actions in higher states, i.e. $v(x_1) < ... < v(x_m)$. Let the new type space be $T \times X$. A prior distribution, $f \in \Delta(T \times X)$ is an information structure if the total probability of each state accords with the prior, i.e. $\sum_t f((t,x)) = h(x), \; \forall x \in X$. Since the sender cannot credibly reveal the state, messaging is given by the preorder, $\succeq_d$, defined by $(t,x) \succeq_d (t',x')$ if $t \succeq'_d t'$. However, the receiver's best response to states will not change with the information structure, so define $\succeq_v$ as $(t,x_i) \succeq_v (t',x_j)$ if $i \geq j$. Thus if f and g are two information structures, f induces lower equilibrium actions than g if and only if $f \geq_{ME} g$ with respect to $\succeq_{d,v}$.

This is illustrated in the the Dye evidence model with a "good" and "bad" state, i.e. $X = \{G, B\}$, in Figure 5. The receiver's value is higher for the good state than the bad state and does not depend on the evidence, so $(t,G) \succeq_v (t',B), \; \forall t, t' \in T$. Equivalently, one can eliminate the possibility for G types to mimic B types because this option will never be used in equilibrium.
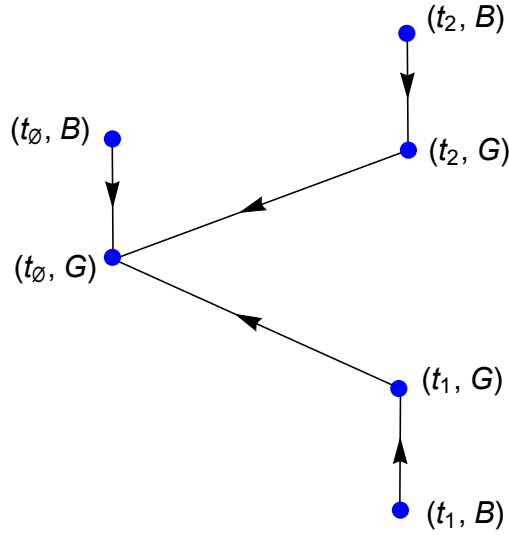

Figure 5: $\succeq_{d,v}$ with a Good and Bad state

4. Equilibrium Characterization

The equivalence between more skepticism and more evidence relies on the structure of the receiver optimal equilibrium. This section characterizes this structure and provides two ways to find the corresponding equilibrium actions.

The following notation for upper and lower contour sets will be useful. For $S \subset T$, let $W(S) \equiv \{s \in T : \exists t \in S, t \succeq_d s\}$ be the lower contour set of S according to $\succeq_d$. These are the types that are **worse** than some type in S by $\succeq_d$, or equivalently, the set of types that can be mimicked by some type in S. Similarly, for any subset $S \subset T$, let $B(S) \equiv \{s \in T : \exists t \in S, s \succeq_d t\}$ be the upper contour set of S according to $\succeq_d$. This is the set of types that are **better** than some type in S by $\succeq_d$, or equivalently, the set of types that can mimic some type in S. This notation omits the dependence on

the disclosure order. When dealing with other ordered sets– $(X, \geq)$, I refer to the lower and upper contour sets as $W_\geq$ and $B_\geq$ respectively.

4.1. Equilibria as Partitions

Because the sender's payoff is strictly increasing in the action, in equilibrium no type can mix over declarations that induce different actions. Thus, any equilibrium is associated with a partition of the type space, $P = \{P_1, ..., P_m\}$,[18] into *pooled sets* that obtain the same action, where each type only mimics other types in his associated pooled set. I call this the *equilibrium partition*.

Every equilibrium has an associated partition, but when does an arbitrary partition P represent the payoff equivalence classes of an equilibrium? Receiver incentive compatibility imposes that the action for each part P_i must be $V_h(P_i)$. That is, the receiver best responds to the prior belief conditioned on $t \in P_i$. I conventionalize that higher indices correspond to parts with higher actions, i.e. $V_h(P_i)$ is increasing in i. Thus, sender incentive compatibility implies that no type $t \in P_j$ should be able to declare any type $s \in P_k$ when $j < k$. Otherwise, t would deviate to the strategy of s and obtain a strictly higher action. I call a partition that satisfies this property an *interval partition* which is formalized below in the context of arbitrary partial orders.

Definition 4. Let $(X, \geq)$ be a partially ordered set. An interval partition of $(X, \geq)$ is $P = (P_1, ..., P_m)$ such that $W_\geq(P_k) \cap P_j = \emptyset, \ \forall j > k$.[19]

Since higher indices correspond to higher actions, P being an interval partition means that the disclosure order is "consistent" with the order on the receiver's best responses. The interval partition condition roughly characterizes when a partition represents the payoff equivalence classes of an equilibrium.[20]

In the receiver optimal equilibrium, pooled sets have the additional property that they cannot be further "separated". Intuitively, this means that within a pooled set, types that are dominant in the disclosure order must have lower value than types that are less dominant. Otherwise, the receiver would prefer an equilibrium that separated these more dominant types by giving them a higher action. The next definition formalizes this property which turns out to be necessary and sufficient for the pooled sets of the receiver optimal equilibrium.

Definition 5. The set function $H : 2^X \to \mathbb{R}$ is downward biased on an ordered subset $(X, \geq)$ if

$$H(W_\geq(\tilde{X}) \cap X) \geq H(X), \ \forall \tilde{X} \subset X. \tag{5}$$

---

[18] I omit the dependence of P on the prior h and receiver utility U^R.

[19] The term "interval partition" is used because for any interval partition $P = (P_1, ..., P_m)$ of the reals $(\mathbb{R}, \geq)$, each part is an interval, i.e. $\overline{P_i} = [a, b], \ \forall i$, for some (extended) real numbers $a \leq b$.

[20] One other technical condition is required: for each P_i, there exists a feasible sender strategy $\sigma : P_i \to \Delta P_i$, such that the best response of the receiver is to choose the same action for all on path declarations in P_i. This property is characterized in terms of the primitives of the model in Appendix J.

Proposition 1. *Let P be a partition of T, where $V_h(P_i)$ is increasing in i. P is the receiver optimal equilibrium partition $\iff$*

$$V_h \text{ is downward biased on } (P_i, \succeq_d), \; \forall i, \tag{6}$$

$$(P_1, ..., P_m) \text{ is an interval partition of } (T, \succeq_d). \tag{7}$$

The receiver's best response is downward biased on S if all lower contour subsets, subsets that cannot mimic their complement in S, have higher value than the set as a whole. Note that due to [Lemma 1](), an equivalent version of the receiver's best response being downward biased is

$$V_h(B(\tilde{S}) \cap S) \leq V_h(S), \; \forall \tilde{S} \subset S. \tag{8}$$

That is, all upper contour subsets of S, subsets of S that cannot be mimicked by their complement, have lower value than the set as a whole. As a shorthand I also refer to sets over which V_h is downward biased as downward biased sets. The next two examples illustrate the downward biased condition by showing its implications for empty and complete disclosure orders.

Example 2. Consider that the disclosure order is the empty order as in [Subsection 2.3](). In this case, any subset of S is a lower contour subset. Because $W(\{s\}) = \{s\}$, $\forall s \in S$, V_h is downward biased on $(S, \succeq_d)$ implies that $v(s) \leq V_h(S)$, $\forall s \in S$. By [Lemma 1](), this implies that $v(s) = V_h(S)$, $\forall s \in S$. That is, when no type can mimic another, all types pool together only if the receiver's best response is constant across types. $\triangle$

Example 3. Let $(S, \succeq_d)$ be completely ordered as in [Subsection 2.3](). If V_h is downward biased on $(S, \succeq_d)$, then $\forall i = 1, ..., m$, $V_h(\{s_1, ..., s_i\}) \geq V_h(S)$. A specific example is illustrated in [Figure 6](). In this case the receiver's utility is quadratic loss, $S \equiv (s_1, ..., s_8)$, and h is the uniform distribution. The left panel shows the receiver's best response to each type and the right panel shows the receiver's best response to lower contour subsets. This demonstrates that on a completely ordered set, V_h being downward biased on $(S, \succeq_d)$ is weaker than v being decreasing on $(S, \succeq_d)$. $\triangle$

Remark 3. These examples illustrate that refining the disclosure order makes the condition that V_h is downward biased on $(S, \succeq_d)$ less restrictive. Consider two disclosure orders, $\succeq_d$ and $\succeq'_d$ on S, such that $\succeq'_d$ is a refinement of $\succeq_d$.[21] Any lower contour subset of $(S, \succeq'_d)$ is also a lower contour subset of $(S, \succeq_d)$. Therefore V_h is downward biased on $(S, \succeq_d)$ implies V_h is also downward biased on $(S, \succeq'_d)$.

4.2. Solving for Equilibrium

This section uses [Proposition 1]() to develop two methods to find receiver optimal equilibria.

Lemma 2. *For any subset $S \subset T$, let $J \subset \arg\min_{\tilde{S} \subset S} V_h(W(\tilde{S}) \cap S)$. $\cup_{\hat{S} \in J} W(\hat{S}) \cap S$ is a downward biased set.*

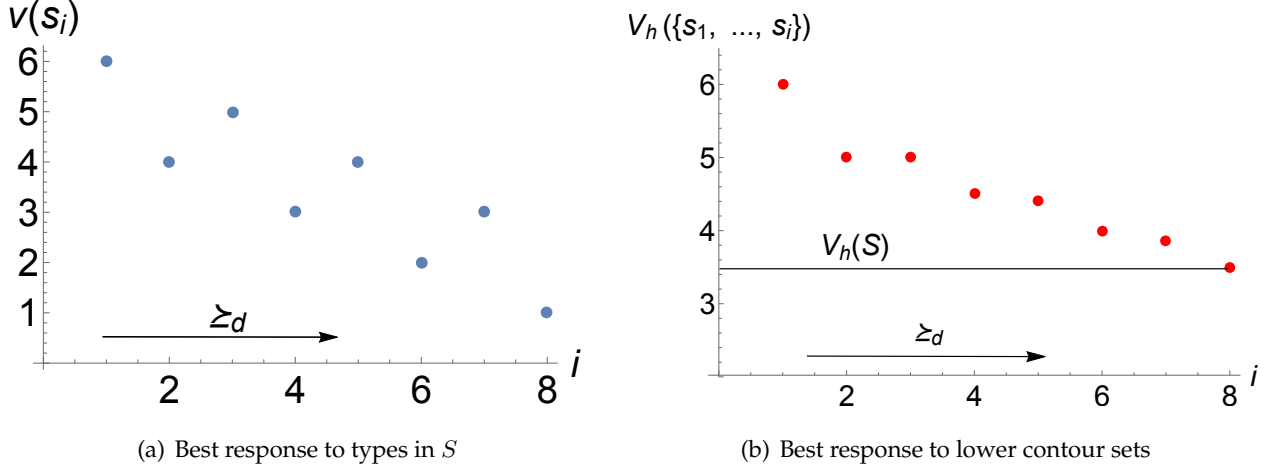[21] $\succeq'_d$ is a refinement of $\succeq_d$ if $\forall t, t' \in S$, $t \succeq_d t' \implies t \succeq'_d t'$.

(a) Best response to types in S (b) Best response to lower contour sets

Figure 6: V_h is downward biased on $(S, \succeq_d)$

ALGORITHM 1: Partition into Pooled Sets

Input: $(T, \succeq_d)$

Output: Equilibrium partition

$i = 1; S_1 = T;$

while $S_i \neq \emptyset$ **do**

$\quad \overline{P}_i = \arg\min_{\tilde{S}_i \subset S_i} V_h(W(\tilde{S}_i) \cap S_i);$

$\quad P_i = \cup_{S \in \overline{P}_i} S;$

$\quad i = i + 1;$

$\quad S_i = S_{i-1} \setminus P_{i-1};$

end

Lemma 2 says that V_h is downward biased on the minimal valued lower contour subset. Symmetrically, V_h is downward biased on the maximal valued upper contour subset.

The ability to find downward biased sets is useful in finding the equilibrium partition. Consider applying the above result as follows. Begin with the entire type set T and use Lemma 2 to find a downward biased P_1. Next remove P_1 and apply Lemma 2 to $T \setminus P_1$ to find another downward biased set P_2. Repeat this process, until the type space is exhausted. This algorithm, which I call *partition into pooled sets*, generates the receiver optimal equilibrium partition.

Proposition 2. *The output of "Partition into Pooled Sets" $(P_1, ..., P_m)$ is the receiver optimal equilibrium partition with $t \in P_i \implies \pi_h(t|U^R) = V_h(P_i)$.*

The algorithm constructs the equilibrium partition from the "bottom-up", i.e. starting with the lowest payoff part. If the minimization were replaced with a maximization, the algorithm would construct the same equilibrium partition from the "top-down", i.e. starting from the highest payoff part. This means that the payoff of very dominant types will correspond to that for a maximal

valued upper contour subset, and the payoff for non-dominant types will correspond to that for a minimal valued lower contour subset. The next result extends this description showing that the payoff for any type can be expressed as a "minmax" expression.

Theorem 3. *Let $\pi_h : T \to \mathbb{R}$ be the equilibrium payoff vector.*

$$\pi_h(t|U^R) = \min_{\{S_a : t \in S_a\}} \max_{\{S_b : t \in S_b\}} V_h(W(S_a) \cap B(S_b)). \tag{9}$$

The expression in (9) corresponds to the equilibrium utility (and thereby obtained action) of the sender of type t.[22] To elucidate the result, consider that $(T, \succeq_d)$ is completely ordered as in Subsection 2.3, and fix any type t_i. For any feasible S_a, $W(S_a) = \{t_1, ..., t_i, ..., t_a\}$ for some $a \geq i$. Similarly, for any feasible S_b, $B(S_b) = \{t_b, ..., t_i, ..., t_n\}$ for some $b \leq k$. Thus $B(S_b) \cap W(S_a) = \{t_b, ..., t_i, ..., t_a\}$. In this case, the above problem reduces to choosing sets of types $\{t_i, ..., t_a\}$ and $\{t_b, ..., t_i\}$ that will pool with type t_i from "above" and "below" respectively. This means (9) can be rewritten as,

$$\pi_h(t_i|U^R) = \min_{a \geq i} \max_{b \leq i} V_h(\{t_b, ..., t_a\}). \tag{10}$$

The problem in (10) suggests that each pooled set forms through the following process. Given a set of types that pool with t_i from above, t_i will choose to pool with a set of lower types in order to maximize his value. Similarly, types that pool from above will only do so if it increases their value. This second process serves to minimize the expression in (10).

The next section uses these characterization results to show that the equilibrium actions decrease under a more evidence shift, i.e. that more evidence implies more skepticism.

5. Why More Evidence implies More Skepticism

Consider two distributions $f, g \in \Delta T$, such that $f \geq_{ME} g$, and where f is only a "small perturbation" from g.[23] Generically, the equilibrium partition will be the same under f and g. To prove Theorem 1 in this case, it suffices to prove that the value of each pooled subset decreases when the sender has more evidence. In light of Proposition 1, I prove the following.

Proposition 3. *Fix $f \in \Delta T$. $V_f(S) \leq V_g(S)$, $\forall g \in \Delta T : f \geq_{ME} g \iff V_f$ is downward biased on $(S, \succeq_d)$.*

The result says that the condition that characterizes pooled sets in the receiver optimal equilibrium also characterizes monotone comparative statics (MCS) under any more evidence shift. The fact that MCS under any more evidence shift implies that the set is downward biased is relatively straightforward. If V_h is not downward biased on $(S, \succeq_d)$, there is a lower contour subset with lower

---

[22] For solutions S_a and S_b to (9), $W(S_a) \cap B(S_b)$ is the part of the equilibrium partition that contains t

[23] For example, consider $\tilde{f} \geq_{ME} g$ and let $f(t) \equiv \alpha \tilde{f}(t) + (1 - \alpha)g(t)$; $\forall t$, for some small $\alpha \in (0, 1]$.

ALGORITHM 2: Monotonic Coarsening

    Input: Distributions $f, g \in \Delta S$.

    Output: An interval partition of $(S, \geq_{f/g})$.

    $i = 1; m(1) = m; P^1 = (\{t_1\}, ..., \{t_m\});$

    while $i = 1$ *or* $P^i \neq P^{i-1}$ **do**

        $I_0 = 0;$

        $k = 0;$

        while $I_k < m(i)$ **do**

            $k = k + 1;$

            $I_k = \max\{j > I_{k-1} : \ V_f(P^i_{I_{k-1}+1}) \geq ... \geq V_f(P^i_j)\}$

        end

        $i = i + 1;$

        $P^i = (\cup_{I_0 < j \leq I_1} P^{i-1}_j, ..., \cup_{I_{k-1} < j \leq I_k} P^{i-1}_j);$

        $m(i) = k;$

    end

---

value than S. Moving probability from this subset to its complement is a more evidence shift and increases the value of S.

As mentioned, the reverse direction in Proposition 3 is complicated by the fact that the downward biased condition is weaker than the receiver's best response being decreasing in the disclosure order. This section introduces an algorithm that deals with this issue by iteratively pooling larger and larger subsets based on incentives to mimic.

5.1. Iteratively Pooling Subsets

For any two distributions $f, g \in \Delta S$, define the induced $f - g$ likelihood ratio order, $\geq_{f/g}$, as

$$x \geq_{f/g} x' \iff \frac{f(x)}{g(x)} \geq \frac{f(x')}{g(x')}.$$

$(S, \geq_{f/g})$ is a completely ordered set of types, in which more dominant types are relatively more likely under f than under g. Notice that the definition of more evidence can be restated as the $f - g$ likelihood ratio order is a refinement of the disclosure order. This means that if V_f is downward biased on $(S, \succeq_d)$, then by Remark 3, V_f is also downward biased on $(S, \geq_{f/g})$.

To ease exposition, I focus on the quadratic loss case in which V_f is a conditional expectation and leave the case of general receiver utilities to the appendix. The following algorithm is used to establish that if V_f is downward biased on $(S, \geq_{f/g})$, then $V_f(S) \leq V_g(S)$.

Description of Algorithm 2 The algorithm begins with the complete interval partition of $(S, \geq_{f/g})$, $P^1 = (\{t_1\}, \{t_2\}, ..., \{t_m\})$. Beginning with t_1, the algorithm repeatedly forms the largest

22

sequence of elements such that $v(t_j)$ is decreasing in j. That is, the first sequence is $\{t_1, t_2, ..., t_{I_1}\}$ such that $v(t_1) \geq ... \geq v(t_{I_1})$ and $v(t_{I_1}) < v(t_{I_1+1})$, the second sequence is $\{t_{I_1+1}, ..., t_{I_2}\}$ such that $v(t_{I_1+1}) \geq ... \geq v(t_{I_2})$ and $v(t_{I_2}) < v(t_{I_2+1})$, and so on until all types in S are exhausted. Next, a coarser interval partition P^2 is formed by pooling all the elements of each decreasing sequence into an associated single part. That is, $P_1^2 \equiv \{t_1, ..., t_{I_1}\}$, $P_2^2 = \{t_{I_1+1}, ..., t_{I_2}\}$, and so on. This process is repeated: at each stage, P^i is coarsened into P^{i+1} where each part of P^{i+1} pools a consecutive sequence of P_j^i over which $V_f(P_j^i)$ is decreasing in j. The algorithm concludes when $P^T = P^{T+1}$.

The interpretation of this algorithm is that each stage pools two subsets P_1 and P_2 only if they would mimic each other for any distribution that preserves their value. That is, only if P_2 is "adjacently more dominant" in the disclosure order and has lower value than P_1.

Implications for Downward Biased Sets   There are two key features of this algorithm. First, if V_f is downward biased on $(S, \geq_{f/g})$, then the algorithm concludes with the trivial partition, i.e. $P^T = (S)$. To see this, note that $P^T = P^{T+1}$ only if $V_f(P_i^T)$ is strictly increasing in i. If P^T were non-trivial, then by iterated expectations, $V_f(P_1^T) < V_f(S)$. But this contradicts the downward biased condition as P_1^T is a lower contour subset. The second key feature uses the following well known result.

Remark 4. Let $v : S \to \mathbb{R}$, and $f, g \in \Delta S$. If v is decreasing on $(S, \geq_{f/g})$, then $V_f(S) \leq V_g(S)$.

Each P_j^i is composed of a sequence of parts from P^{i-1} over which V_f is decreasing. Thus, one can apply Remark 4 to obtain,

$$\frac{1}{G(P_j^i)} \sum_{P_l^{i-1} \subset P_j^i} V_f(P_l^{i-1}) G(P_l^{i-1}) \geq \frac{1}{F(P_j^i)} \sum_{P_l^{i-1} \subset P_j^i} V_f(P_l^{i-1}) F(P_l^{i-1}) = V_f(P_j^i), \ \forall P_j^i. \quad (11)$$

Since the process ends with the trivial partition, using a sequence of these inequalities gives $V_f(S) \leq V_g(S)$. While the details are left to the appendix, I illustrate the algorithm in the following example.

Example 4. Recall the example in Figure 6 in which V_f is downward biased on $(S, \succeq_d)$. Specifically, the disclosure order is complete on $S = (s_1, ..., s_8)$, f is the uniform distribution on S, and $(v(s_1), ..., v(s_8)) = (6, 4, 5, 3, 4, 2, 3, 1)$. Let $g \in \Delta S$ such that $f \geq_{ME} g$. Since the disclosure order is complete, $\geq_{f/g} = \succeq_d$. The goal is to show that $V_f(S) \leq V_g(S)$ establishing MCS under any more evidence shift in this case. However as v is not decreasing, one cannot directly apply Remark 4. Instead Algorithm 2 pools types iteratively such that at each stage the value of the currently pooled subset is lower under the more evidence distribution.

Since $v(s_1) > v(s_2)$, s_2 will pool with s_1 regardless of the distribution. Thus, the algorithm pools $\{s_1, s_2\}$ into a "single type" with value given by $V_f(\{s_1, s_2\})$. Similar logic also pools $\{s_3, s_4\}$, $\{s_5, s_6\}$, and $\{s_7, s_8\}$. The result is the partition P_2 illustrated in the left panel of Figure 7. Notice that since $v(s)$ is decreasing on each P_i^2, Remark 4 implies that $V_f(P_i^2) \leq V_g(P_i^2)$, $\forall i = 1, ..., 4$.

Next, note that since $V_f(P_i)$ is decreasing in i, these sets will all pool together in equilibrium.

Thus, the algorithm pools $\{P_1^2, ... P_4^2\}$ into a single pooled set with value $V_f(S)$. The result is a coarser (trivial) partition P^3 illustrated in the right panel of Figure 7. As before, since $V_f(P_i^2)$ is decreasing in i, Remark 4 implies that

$$V_f(S) = \sum_{i=1}^{4} V_f(P_i^2)F(P_i^2) \leq \sum_{i=1}^{4} V_f(P_i^2)G(P_i^2).$$

Using the inequalities derived at each of the two stages gives the desired result, i.e. $V_f(S) \leq V_g(S)$.
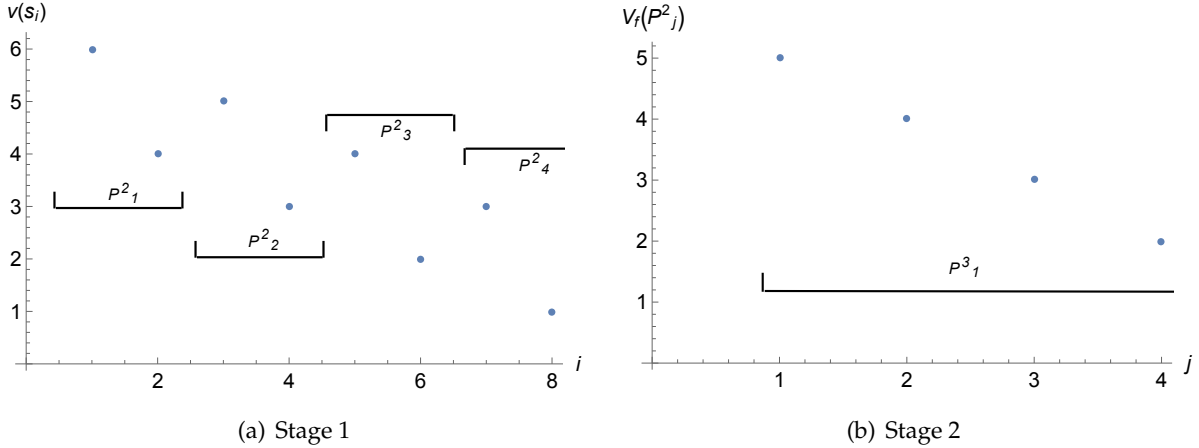

(a) Stage 1

(b) Stage 2

Figure 7: Applying Algorithm 2

$\triangle$

5.2. Changes in the Equilibrium Partition

The preceding analysis is only sufficient for Theorem 1 in the case when the equilibrium partition is constant across distributions f and g. This section provides some intuition for how this result extends to cases in which the equilibrium partition changes when the sender has more evidence.

Consider that $f \geq_{ME} g$. For simplicity let $P^g = (P_1^g, P_2^g)$ have two elements, while $P^f = (P_1^f)$, has only one. Define the combination distribution, $h_\alpha \equiv \alpha f + (1 - \alpha)g$ with corresponding equilibrium partition $P^\alpha \equiv (P_1^\alpha, ..., P_{m_\alpha}^\alpha)$. First, since the equilibrium action is increasing in the disclosure order, $V_g(P_1^g) < V_g(P_2^g)$, and second, since the receiver's best response is downward biased on each part, $V_f(P_1^g) \geq V_f(P_2^g)$. Thus, because the receiver's best response to these subsets is continuous in α, there must exist some α^* after which the equilibrium partition changes from P^g to P^f and $V_{h_{\alpha^*}}(P_1^g) = V_{h_{\alpha^*}}(P_2^g)$. For simplicity, suppose that this is the only change in the equilibrium partition as α increases from 0 to 1. Figure 8 illustrates this example.

Notice that $f \geq_{ME} h_{\alpha^*} \geq_{ME} g$, $\forall \alpha \in [0, 1]$. Thus, Proposition 3 completes the argument in this case. The idea is that each equilibrium part, P_1^g, P_2^g decreases in value as α increases until equalizing at $\alpha = \alpha^*$. For $\alpha > \alpha^*$, all types pool together and so again by Proposition 3, the value of $P_f^1 = S$ decreases in α. Figure 9 illustrates the equilibrium utilities as a function of α.
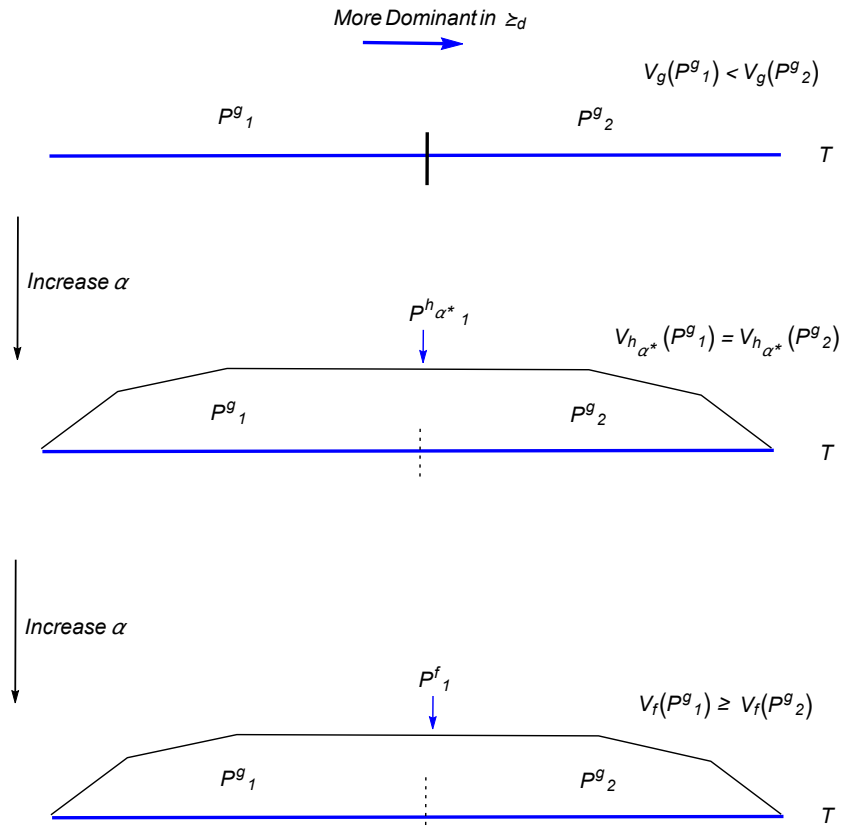
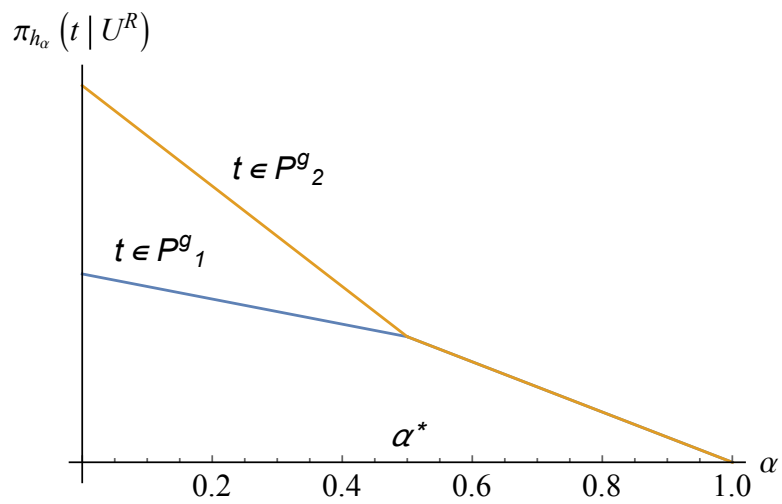Figure 8: More Evidence Changes the Equilibrium Partition

Figure 9: Equilibrium Actions under Changes in the Equilibrium Partition

6. Application 1: Extensions of the Sender's Preferences

This section uses the characterization results from Section 4 to extend the model beyond one with sender's who always prefer higher actions. I solve for equilibrium in games with unbiased senders, and senders who have unknown bias. Then I use Theorem 1 to obtain new comparative statics predictions.

6.1. Unbiased vs. Honest Senders

In many communication environments the sender does not always attempt to induce the highest action from the receiver. For example, criminal investigations are sometimes carried out by "good cops" whose goal is to find out the truth rather than to always convict the suspect. Does this goodness originate from a compulsion to be honest or from preferences that are aligned with the court, and do these two explanations lead to different outcomes? More specifically, does a game with some probability of honest senders have different equilibria than one with the same probability of unbiased senders? Kim & Pogach (2014) show that the answer to this question is yes in a cheap talk setting: depending on the specification the receiver can prefer either honest or unbiased senders. I show that in the disclosure setting, unbiased and honest senders lead to the same receiver optimal equilibrium outcomes.

Consider two games, $\tilde{H}$, and $\tilde{UB}$, defined as follows. In both games there is probability p that the sender prefers higher actions (S type). These senders have evidence from T' distributed according to $f \in \Delta T'$, and can disclose according to $\succeq_d$. In both $\tilde{H}$ and $\tilde{UB}$, there is probability $1 - p$ of "non-strategic types" (NS type), who has evidence from T' distributed according to $g \in \Delta T'$. In $\tilde{H}$, the non-strategic sender is honest (H type) in that he can only declare truthfully. In $\tilde{UB}$, the non-strategic sender can disclose according to $\succeq_d$, but is unbiased (UB type) in that he has the same utility as the receiver. In both games, the receiver's utility is dependent only on the evidence: $U^R : T' \times A \to \mathbb{R}$.

$\tilde{H}$ fits the basic framework (it is an example from Subsection 2.3), while $\tilde{UB}$ does not. The total type space for $\tilde{UB}$ is $T' \times \{S, UB\}$ with messaging given by the disclosure order $\succeq_d^{UB}$ defined as follows:

$$(t, UB) \sim_d^{UB} (t, S), \ \forall t \in T',$$
$$(t, S) \succeq_d^{UB} (t, S) \ if \ t \succeq_d t'.$$

The definition of $\succeq_d^{UB}$ captures that both UB and S types have the same disclosure opportunities, and can make cheap talk declarations about their preference type.

6.1.1. Characterization

Since $\tilde{H}$ is in the form of the basic model,[24] its receiver optimal equilibrium is given by Theorem 3. The following definition further refines this characterization in honest type games.

For any $R \subset T'$, define

$$\tilde{V}(R) \equiv \max_{R' \subset R} \ \arg\max_{a \in A} \ (1-p) \sum_{t \in R} U^R(a,t)f(t) + p \sum_{t \in R'} U^R(a,t)g(t).$$

This is the receiver's best response to the biased senders in R and the honest or unbiased senders in some subset $R' \subset R$. R' is the set of types in R who have higher value than $\tilde{V}(R)$, i.e. $R' \equiv \{s \in R : v(s) \geq \tilde{V}(R)\}$. Mechanically, $\tilde{V}(R) \geq V_f(R)$, $\forall R \subset T'$.

Proposition 4. *Let the receiver optimal equilibrium allocation (actions to types) in $\tilde{H}$ and $\tilde{UB}$ be given by π^H, π^{UB} respectively. The receiver optimal equilibrium allocations are the same, i.e. $\pi^H(t,S) = \pi^{UB}(t,S) \equiv \pi^*(t,S)$, $\forall t \in T'$ and $\pi^H(t,H) = \pi^{UB}(t,UB) \equiv \pi^*(t,NS)$, $\forall t \in T'$. Moreover,*

$$\pi^*((t,S)) = \min_{\{S_a \subset T': t \in S_a\}} \ \max_{\{S_b \subset T': t \in S_b\}} \ \tilde{V}(W(S_a) \cap B(S_b)), \ \forall t \in T',$$

$$\pi^*((t,NS)) = \min\{\pi^*((t,S)), v(t)\}, \ \forall t \in T'. \tag{12}$$

The equilibrium actions for strategic types in a game with honest senders (or unbiased senders) are the same as one without honest senders but where the receiver has "more favorable" preferences to the sender: his best response to all subsets shift up from V_f to $\tilde{V}$. On the other hand the receiver obtains his bliss point for any honest sender with value less than the equilibrium action of his strategic counterpart.

The reason why π^* is the receiver optimal equilibrium allocation of $\tilde{UB}$ is as follows. First, receiver and strategic sender incentive compatibility are directly transferred from that in $\tilde{H}$. Second, unbiased sender's who obtain $v(t)$ are at their bliss point. And third, unbiased senders who obtain $\pi^*((t,NS)) < v(t)$ must only be able to deviate to lower actions which are further from their bliss point. Otherwise their strategic counterparts would deviate in $\tilde{H}$. The reason why π^* is receiver optimal in $\tilde{UB}$ is that the receiver can always separate unbiased senders who have value less than their equilibrium action by making these sender's truthfully reveal.

6.1.2. Comparative Statics

Since $\tilde{H}$ and $\tilde{UB}$ can be seen as standard disclosure games, Theorem 1 applies. In either game, an increase in p, the probability of strategic types, can be seen as a more evidence shift. This is apparent in Figure 4, as increasing p corresponds to shifting probability "up the disclosure order" from the red honest types to the blue strategic types. By Theorem 1 this change induces more skepticism. Dziuda (2011) establishes this conclusion for honest types in a specific disclosure framework, however the

---

[24] It is without loss to assume that honest senders prefer higher actions. In general their preferences are irrelevant to the equilibrium because they only have one message available.

next corollary shows that the result is general to any disclosure structure. For any p, denote the prior over T as h_p.

Corollary 3. *Let $p \geq p'$, then $h_p \geq_{ME} h_{p'}$ and $h_p \geq_{MS} h_{p'}$ in both $\tilde{UB}$ and $\tilde{H}$.*

6.2. Senders with Unknown Bias

There are other reasons why the sender may not always prefer higher actions. The sender may be biased but in an unknown direction. For example, a police officer may want to exonerate a suspect with whom he has a relationship. When will the sender credibly convey the direction of his bias to the receiver? What beliefs about his preferences does the sender want to induce in the receiver?

The sender prefers higher actions (is type H; note that H types no longer refers to honest senders), $U^S(a) = a$, with probability p, and prefers lower actions (is type L), $U^S(a) = -a$, with probability $1 - p$. In addition to private information about his preferences, the sender obtains disclosable evidence from $(T', \succeq_d')$. The total type space is given by $T' \times \{H, L\}$. The distribution over evidence can depend on the preference of the sender: let $f^H, f^L \in \Delta T'$ be the marginal distributions for H and L preferences respectively. Denote the unconditional distribution, $g \in \Delta T$. The receiver's preferences can also depend on the senders preferences as well as the evidence, i.e. $U^R : T \times A \to \mathbb{R}$.

In addition to disclosing evidence, the sender can make a cheap talk declaration of his preference type. The set of available messages to each type t is $\{s : t \succeq_d' s\} \times \{H, L\}$. Call this communication game $\tilde{C}$, and let $\pi^{\tilde{C}} : T \to \mathbb{R}$ be the corresponding receiver optimal equilibrium allocation of actions to types. Note that $\tilde{C}$ does not fit into the basic framework.

Unlike in the original model, the inclusion of cheap talk messages can alter the set of equilibria. When it is known that the sender prefers high actions, two on path cheap talk messages cannot induce different actions, otherwise all sender types would deviate to the one that induces the higher action. With both H and L senders, two on-path cheap talk messages can lead to different actions: H senders induce the high action and L senders induce the low action. For this reason, I include the possibility of cheap talk about preferences.

6.2.1. An Equivalent Disclosure Game

Consider a related disclosure game, in which the sender has known preferences towards higher actions. The disclosure ordered type space is $(T, \succeq_d)$, where $T \equiv T' \times \{H, L\}$ remains unchanged and $\succeq_d$ is defined as follows:

$$(t, H) \succeq_d (t', H) \iff t \succeq_d' t',$$
$$(t, L) \succeq_d (t', L) \iff t' \succeq_d' t,$$
$$(t, H) \succeq_d (t', L) \iff \exists s : t \succeq_d' s, \text{ and } t' \succeq_d' s.$$

The disclosure order $\succeq_d$ (i) maintains $\succeq_d'$ when comparing two H types, (ii) reverses $\succeq_d'$ when comparing two L types, And (iii) ranks an H type above an L when both types can mimic some

common evidence type $s \in T'$. The distribution over types, $g \in \Delta T$ remains unchanged. Call the associated game $\tilde{D}$ and let $\pi^{\tilde{D}} : T \to \mathbb{R}$ be the receiver optimal equilibrium allocation of actions to types. I illustrate the construction of $\tilde{D}$ in Figure 10. Panel (a) illustrates the evidence space T' and the associated disclosure order $\succeq'_d$ in $\tilde{C}$. Panel (b) illustrates the disclosure order in the associated $\tilde{D}$.

(a) $(T', \succeq'_d)$ in $\tilde{C}$
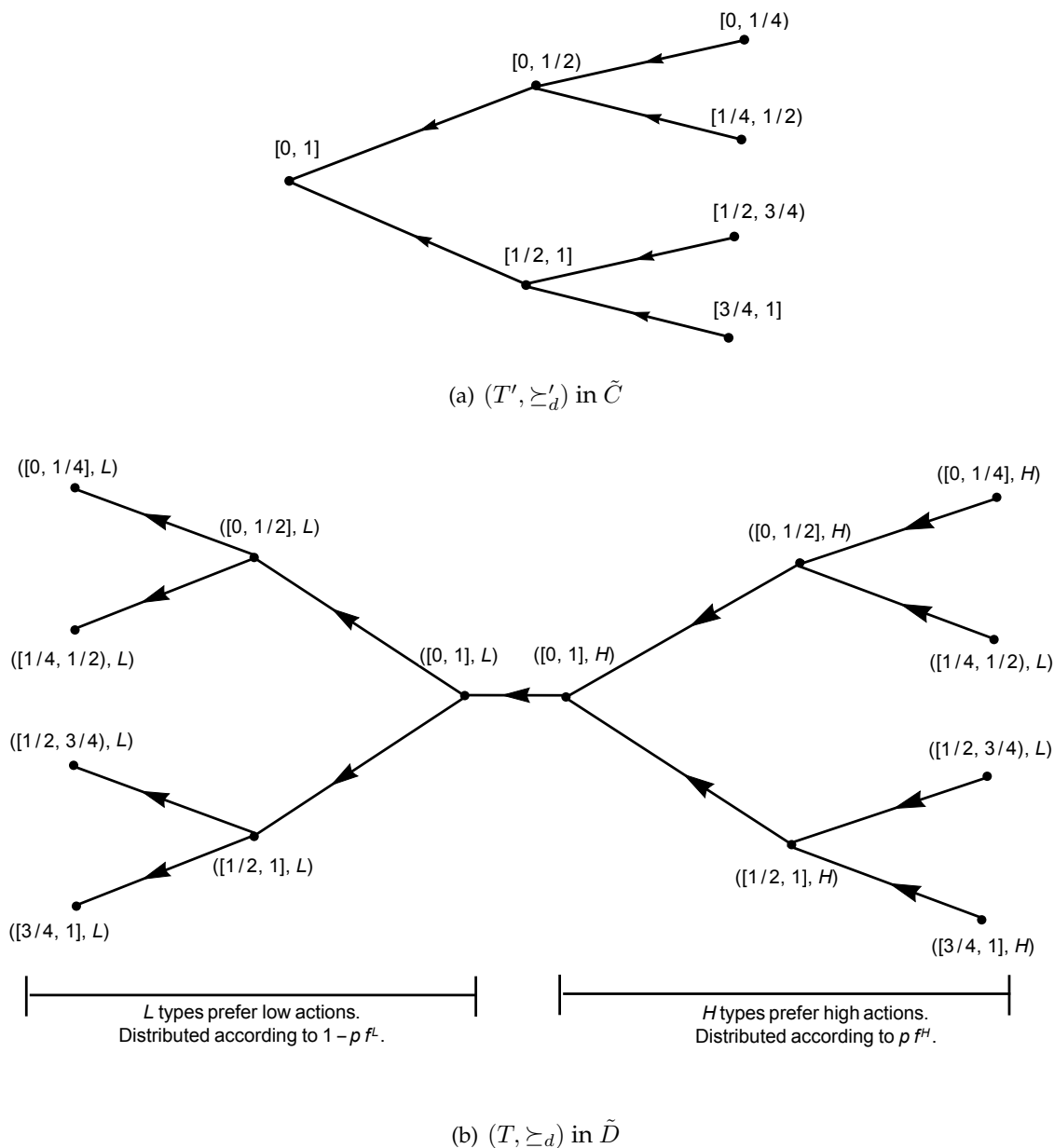

(b) $(T, \succeq_d)$ in $\tilde{D}$

Figure 10: Construction of $\tilde{D}$ from $\tilde{C}$

Proposition 5. $\pi^{\tilde{D}}(t) = \pi^{\tilde{C}}(t), \; \forall t \in T.$

Since $\tilde{D}$ fits into the framework of Section 2, Theorem 1 applies to $\pi^{\tilde{D}}$ and therefore $\pi^{\tilde{C}}(t)$. In $\tilde{D}$,

increasing p constitutes a more evidence change in the distribution over $(T, \succeq_d)$.

Corollary 4. *Let $p > p'$ with $g_p, g_{p'} \in \Delta T$ as the corresponding prior distributions. $g_p \geq_{ME} g_{p'}$ on $(T, \succeq_d)$ so $g_p \geq_{MS} g_{p'}$.*

Note that more skepticism, a decrease in equilibrium actions, no longer corresponds to a decrease in utility. Indeed, the L type senders are better off facing a receiver with a more skeptical prior. Thus L types benefit from the presence of H types and vice versa. If L and H types pool together, the H types that are misreporting will tend to have lower value than the receiver's best response to the pooled set. Thus, increasing the probability of H types decreases the receiver's best response, which benefits the L types. The construction of $\tilde{D}$ also provides some insights into the structure of the receiver optimal equilibrium in $\tilde{C}$.

Proposition 6. *Let $(T', \succeq'_d)$ have a lower bound $\underline{s}$, $f^H = f^L$, and $U^R(a, (t, H)) = U^R(a, (t, L))$, $\forall t \in T'$. Then there exists exactly one on path receiver optimal equilibrium action at which H and L types pool. This pooled set includes $\underline{s}$.*

The result says that there is exactly one message in which the sender does not credibly reveal his preference type. The latter two assumptions impose that the receiver's utility and the distribution over evidence are independent of the sender's preference type. The first assumption is that there exists some type $\underline{s}$ in which all other evidence types can mimic. $\underline{s}$ need not have positive probability, so the assumption only ensures its availability as a message. The presence of $\underline{s}$ can be justified by noting that the sender can always choose to present nothing. Note that unlike in the original model, the presence of zero probability types does affect the receiver optimal equilibrium, as it can change the disclosure order $\succeq_d$ in $\tilde{D}$.[25]

The reason why there is *at most one* equilibrium action in which H and L types pool is that by incentive compatibility all L types must obtain lower actions than $\underline{s}$, and all H types must obtain higher actions than $\underline{s}$. Alternatively, it is immediate from Figure 10 that at most one interval can intersect both the H and L "sides" of the directed graph, and that this interval must contain $\underline{s}$. To see why there is *exactly one* equilibrium action in which H and L types pool, suppose that the $(\underline{s}, H)$ type obtains a different equilibrium action from $(\underline{s}, L)$. Incentive compatibility implies that,

$$\pi^{\tilde{C}}((t, H) \geq \pi^{\tilde{C}}((\underline{s}, H) > \pi^{\tilde{C}}((\underline{s}, L) \geq \pi^{\tilde{C}}((t, L), \ \forall t \in T'.$$

But this implies that the receiver's action for all L types is lower than that for all H types. Since by assumption, the distribution of payoff relevant types T' is the same across H and L senders, this violates Lemma 1. This means that there is always positive probability that the sender does not reveal his preference type.

---

[25] This point relates to Seidmann & Winter (1997) who study a vagueness model in which the sender's direction of bias is also unknown. Even though the sender is "informed" with probability 1,The ability to be vague, i.e. mimic zero probability types destroys the truthful revelation equilibrium.

7. Application 2: Dynamic Disclosure

7.1. Dynamic Arrival and Inspection

The static arrival of evidence can be unrealistic. Consider an entrepreneur strategically disclosing the consumer reviews of his product to an investor. It is likely that the customer reviews arrive gradually. Thus, the investor could also request to see the reviews at some intermediate stage before the actual investment decision. The question is whether the investor could benefit from these additional early inspections, as compared to the situation in which he only communicates with the entrepreneur after all reviews have come in.

This comes down to whether the sender can disclose evidence acquired early in order to induce a more favorable impression from the receiver in a later period. But intuitively, a sender who obtains more evidence early on is more likely to have more evidence tomorrow: the entrepreneur that has one customer review today is more likely to have two customer reviews by tomorrow than an entrepreneur that has zero customer reviews today. Combined with Theorem 1, this suggests that disclosing evidence early is not beneficial for the sender since it will induce more skepticism in the receiver. I introduce a model in which the sender obtains and can disclose information over multiple periods and identify the conditions on the evidence structure so that there is no benefit to early inspections.

7.1.1. Model

The evidence space and messaging are still given by $(T, \succeq_d)$. There are two periods over which the sender can obtain evidence. The period 1 probability distribution of evidence is given by $g_1 \in \Delta T$ which is assumed to have full support. The probability of obtaining t_2 in period 2 given possession of t_1 in period 1 is given by

$$
pr(t_2|t_1) = \begin{cases} \frac{g_2(t_2)}{G_2(B(t_1))} & if \ t_2 \in B(t_1) \\ 0 & otherwise \end{cases},
$$

for some $g_2 \in \Delta T$. This implies that possessing more evidence in period 1 makes one more likely to have more evidence in period 2. Indeed, if $f_t \in \Delta T$ represents the probability distribution over period 2 evidence after acquiring t in period 1, $t' \succeq_d t'' \implies f_{t'} \geq_{ME} f_{t''}$. Note that for $t_1 \succeq_d t_2$, there is a zero probability of obtaining t_2 in period 2 having possessed t_1 in period 1, i.e. the sender does not lose evidence over time. In addition, the distribution of period 2 evidence depends on period 1 evidence only through its upper contour subset. That is, if two types, s, s', both have positive probability in period 2 after two different period 1 realizations, then the relative probability of s to s' will be the same after both period 1 realizations. At the end of the section I discuss the implications of relaxing the assumptions on the evolution of evidence.

In each period the sender can declare any type s such that his current evidence t satisfies $t \succeq_d s$. This means that in period 2, the sender cannot credibly convey his period 1 evidence. The receiver,

having observed the declarations in periods 1 and 2 takes an action $a \in A$. The receiver's utility, $U^R(t_2, a)$, depends on the action and the sender's final type. Thus, period 1 disclosures are only potentially useful as signals for future evidence. The sender still prefers higher actions, i.e. $U^S(a) = a$.

I maintain that the equilibrium in period 2 is receiver optimal given the receiver's beliefs. This can be justified in a setting where the receiver has commitment power within each period but not inter-temporally as in Skreta (2006). Because the receiver only takes an action in the last period, this means that the receiver can commit at time 2 to an action plan, $a : T \to A$, but not before. This justification uses the fact that the receiver optimal equilibrium is also the commitment solution.

For a given equilibrium of this dynamic disclosure game, denote $\tilde{\pi} : T \to \Delta A$ as the distribution of actions given to the sender with period 2 type t.[26] The potential randomness in $\tilde{\pi}$ arises due to different period 1 disclosures leading to the same period 2 disclosure, rather than due to action randomization by the receiver.

I say that the receiver "benefits from early inspections" if his expected utility in some equilibrium of the dynamic disclosure game is higher than that if the receiver were to only communicate with the sender in period 2. Let the ex-ante beliefs over period 2 types be defined by

$$\tilde{g}(t) \equiv g_2(t) \sum_{s \in M(t)} \frac{g_1(s)}{G_2(B(s))}.$$

The receiver benefits from early inspections if there exists an equilibrium allocation, $\tilde{\pi} : T \to \Delta A$ such that

$$\sum_{t \in T} \left(\sum_a U^R(a, t) \tilde{\pi}_t(a) \right) \tilde{g}(t) > \sum_{t \in T} U^R(\pi_{\tilde{g}(t)}(t | U^R), t) \tilde{g}(t).$$

7.1.2. The Unique Evidence Path Property

I next introduce the pivotal feature of a disclosure order that will determine whether the receiver can benefit from early insepctions. Define $t \perp_d t'$ if t and t' are not comparable under $\succeq_d$.

Definition 6. A disclosure ordered type space $(T, \succeq_d)$ has the unique evidence path property (UEPP) if $\forall s', s'' \in T, s' \perp_d s'' \implies B(s') \cap B(s'') = \emptyset$.

The UEPP says there is a unique "path" in the disclosure order to each type. This can be summarized by two properties: (i) the sender type can always report nothing or "no evidence" and (ii) the directed graph representation of $\succeq_d$ has no cycles. Many classic examples fit this property such as the Dye model of Subsection 2.3. A more interesting example from the dynamic disclosure perspective is the following extended version of the vagueness model.

---

[26] As this is the only payoff relevant information for both parties, I exclude any description of how $\tilde{\pi} : T \to \Delta A$ depends on period 1 disclosures.

Example 5. Consider a sequence of partitions of some set X, $R^1 = (R_1^1, ..., R_{m_1}^1), ..., R_n = (R_1^n, ..., R_{m_n}^n)$ with R^i finer than R^{i-1} for every i. The type space, $T \equiv \cup_{i,j} R_i^j$, is the set of all parts of the partitions. The disclosure order is given by $R' \succeq_d R'' \iff R' \subset R''$, with the interpretation that obtaining evidence R_i^j is learning that the "state" is in R_i^j, and one can be vague about his knowledge. $R' \perp_d R'' \implies R' \cap R'' = \emptyset$ since the different partitions are ordered. This means that $\tilde{R} \subset R' \implies \tilde{R} \cap R'' = \emptyset$ which confirms that the UEPP holds. In fact, every disclosure order in which the UEPP holds can be interpreted as above. Figure 11 displays an example of such a disclosure order with three increasingly fine partitions of $[0,1]$: $R_1 = ([0,1])$, $R_2 = ([0,1/2), [1/2,1])$, and $R_3 = ([0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1])$. Dominance in the disclosure order is indicated by a directed path. $\triangle$
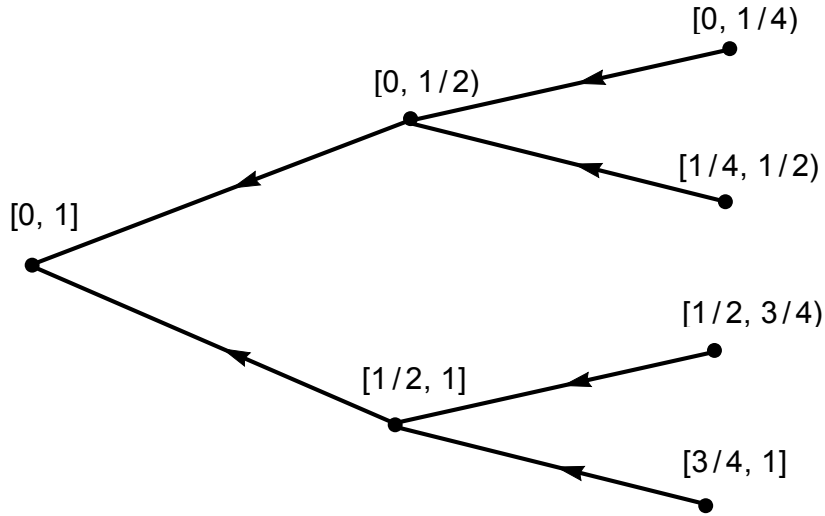

Figure 11: Extended Vagueness Model Satisfies the UEPP

Notice that under the UEPP, any two period 1 types that cannot mimic each other lead to period 2 type realizations that also cannot mimic each other. In this sense, any potential for separation in period 1 is maintained in period 2. To illustrate why this is important for signaling consider a sender who obtains a sample of unknown size from the set $\{0,1\}$ and can disclose any subset. This does not satisfy the UEPP as $01 \succeq_d 0$ and $01 \succeq_d 1$, but $0 \perp_d 1$. If the receiver waits until period 2 to communicate with the sender, he could miss out on the opportunity to separate period 1 realizations of 0 and 1, because these two paths to the period 2 type 01 are indistinguishable in period 2.

Proposition 7. *If $(T, \succeq_d)$ satisfies the UEPP, then the only equilibrium allocation $\tilde{\pi}$ is degenerate on*

$$\min_{\{S_a: t_N \in S_a\}} \max_{\{S_b: t \in S_b\}} V_{\tilde{g}}(W(S_a) \cap B(S_b)),$$

33

and the receiver does not benefit from early inspections. Moreover, if $(T, \succeq_d)$ does not satisfy the UEPP, then there exists g_1, g_2, and U^R such that the receiver benefits from early inspections.

The result is somewhat surprising as different period 1 evidence induces different distributions over period 2 evidence and thereby different preferences over receiver beliefs. The idea is that under the UEPP, for any sender strategies, two different period 1 disclosures will induce beliefs that are ranked by the more evidence relation from the perspective of some type.

To see this, consider that $T = (t_1, ..., t_n)$ is completely ordered as in Subsection 2.3, and an equilibrium in which there are only two on path period 1 disclosures s' and s'' that induce receiver beliefs, $f_{s'}$, $f_{s''}$. Suppose that, in contradiction to Proposition 7, these beliefs admit different period 2 action profiles, $\pi_{f_{s'}}$, $\pi_{f_{s''}}$.

The most dominant type, t_n, according to the disclosure order, is in the highest payoff part under both s' and s''. Because the sender can only gain evidence as time goes on, a sender with type t_n in period 1 is certain that he will pool with this highest payoff part in period 2. Thus, if in period 2, s' induces a lower payoff for t_n than s'', then t_n will not disclose s' in period 1 and neither will any type that pools with t_n under s''. The main observation is that in this case, $f_{s''} \geq_{ME} f_{s'}$ on the pooled set containing t_n. But this means that $f_{s''} \geq_{MS} f_{s'}$ on this pooled set. This contradicts the fact that s'' induced a higher payoff for t_n.

This means that $\pi_{f_{s'}}(t_n) = \pi_{f_{s''}}(t_n)$. But similarly, a sender with type t_{n-1} in period 1 can only realize types t_{n-1} or t_n in period 2. Thus, the sender with t_{n-1}, chooses his period 1 declaration based only on the comparison between $\pi_{f_{s'}}(t_{n-1})$ and $\pi_{f_{s''}}(t_{n-1})$. One can repeat the previous argument to obtain that these two quantities are equal. Continually applying this argument leads any informative dynamic signaling to "unravel" (reminiscent of the original argument in Milgrom (1981)), and so $\pi_{f_{s'}} = \pi_{f_{s''}}$.

To see how the receiver can benefit from early inspections when the UEPP does not hold, consider the following example.

Example 6. The type space is $\{1, 2, 3, 4, 5, 6\}$ with U^R as quadratic loss, and with the value of each type v_i and $\succeq_d$ both illustrated in the left panel of Figure 12. Notice that the UEPP does not hold as types 3 and 2 are not ordered but are both dominated by type 5. The right panel shows the equilibrium partition over types under any distribution in period 2. More specifically, types $1, 3, 6$ declare 1, types $2, 5$ declare 2, and type 4 declares 4.

Consider also using this strategy in the right panel of Figure 12 in period 1. Note that 4 truthfully reveals under all period 1 declarations and is thereby indifferent across them. Thus, the only incentive to check is that types 2 and 5 do not want to deviate to declare type 1 in period 1. Let f_i be the receiver's period 2 belief in ΔT following declaration i in period 1. Types 2 and 5 do not want to deviate if $V_{f_1}(\{2, 5\} \geq V_{f_2}(\{2, 5\})$. This comes down to comparing the likelihood ratio between 2

34

and 5 in period 2 under the two period 1 declarations. Incentive compatibility holds if

$$\frac{\frac{g_1(2)}{G_2(\{2,4,5\})}}{\frac{g_1(2)}{G_2(\{2,4,5\})} + \frac{g_1(5)}{g_2(5)}} \geq \frac{g_1(1)}{g_1(1) + \frac{g_1(3)}{G_2(\{3,5,6\})}}.$$

An example thatr satisfies this inequality is when g_1 and g_2 are the uniform distribution. Since the receiver's behavior is sequentially optimal and the period 2 equilibrium vector is different following the two on path messages in period 1, the receiver benefits from early inspections. $\triangle$

(a) $(T, \succeq_d)$ does not satisfy the UEPP
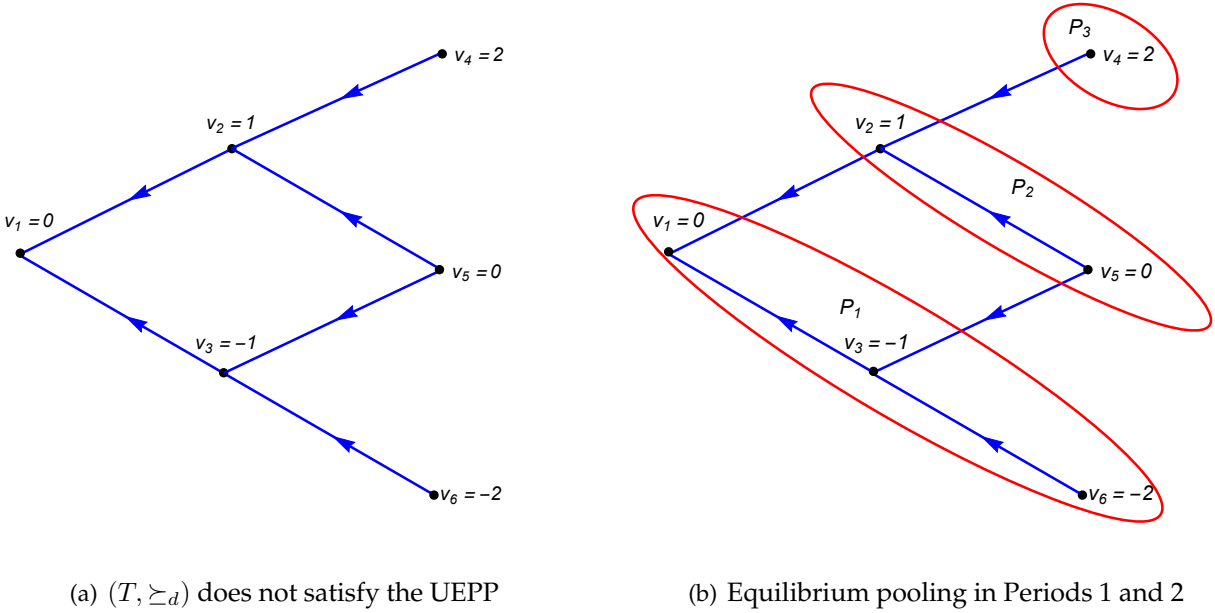
(b) Equilibrium pooling in Periods 1 and 2

Figure 12: Informative Dynamic Signaling Without the UEPP

This section identifies the UEPP as the condition on the disclosure order such that the receiver does not benefit from early inspections. One potential weakness of this result is the assumption on how evidence evolves over time. A more general framework would assume that the probability of period evidence t_2 given period 1 evidence t_1 is given by $g_2(t_2, t_1)$, while maintaining the substantive assumptions that (i) evidence is not lost over time, i.e. $t_1 \succeq_d t_2 \implies g_2(t_2, t_1) = 0$, and (ii) that $t_1' \succeq_d t_1'' \implies g_2(\cdot | t_1') \geq_{ME} g_2(\cdot | t_1'')$. One can show that under the UEPP, if the disclosure order is "large enough" (has a chain of at least 4 types), the receiver does not benefit from early inspections regardless of his preferences or g_1 if and only if g_2 satisfies the assumptions of this section.

7.2. Multiple Receivers

One reason that there is no use for dynamic signaling is that the evidence type in period 1 is payoff irrelevant. Recall the entrepreneur releasing customer reviews to an investor. In addition to signaling value to investors at the angel round, the entrepreneur may make an early disclosure in

order to obtain funding from separate investors at the seed round. The entrepreneur must balance the incentive to get early funding with how his early disclosure will affect funding in later rounds. Does the seed investor benefit or suffer from the entrepreneur's dynamic incentives? Is the amount of seed funding negatively or positively related to the amount of angel funding?

7.2.1. Model

I augment the previous dynamic model with an additional receiver who takes an action $a_1 \in A$, following the first period disclosure. The period 1 and period 2 receiver's preferences are represented by $U^1, U^2 : T \times A \to \mathbb{R}$ respectively. For period 1 and period 2 action choices, a_1, a_2 the sender's utility is given by $\delta a_1 + (1 - \delta)a_2$. The timing is as follows. The sender obtains type t_1 in period 1 and makes a disclosure $d_1 \in M(t_1)$. Receiver 1 observes d_1, takes an action a_1, and obtains utility $U^1(a_1, t_1)$. Then the sender obtains t_2 in period 2 and discloses $d_2 \in M(t_2)$. Receiver 2 observes d_1 and d_2, takes an action a_2, and obtains utility $U^2(a_2, t_2)$.

The sender's strategy is a first period reporting strategy, $\sigma : T \to \Delta T$; and a second period conditional reporting strategy, $\gamma : T^2 \to \Delta T$. Receiver 1's strategy is $a_1 : T \to A$ and Receiver 2's strategy is $a_2 : T^2 \to A$. I assume that in period 2, γ and a_2 are played according to the receiver 2 optimal equilibrium. Thus the equilibrium action in period 2 for type t after a disclosure s in period 1 is given by $\pi_{h_s}(t|U^2)$, where $h_s \in \Delta T$ is the interim distribution over period 2 types given σ and after observing s in period 1. For tractability I focus on interval equilibria.

Definition 7. An interval equilibrium is one in which $\{t : a_1(t) = c\}$ is is an interval for every $c \in \mathbb{R}$.

Lemma 3. If $(T, \succeq_d)$ satisfies the UEPP, then any interval equilibrium (a_1, σ) has, $t \succeq_d t' \implies a_1(t) \geq a_1(t')$. More over if $t \succeq_d t'$ and $a_1(t) > a_1(t')$ then $h_t \geq_{ME} h_{t'}$.

This result says that if the sender makes a disclosure in period 1, then he must be given a higher action than if he did not disclose (or disclosed less). The idea is that the period 1 disclosure induces the receiver to believe that the sender has more evidence. Since this induces more skepticism in the receiver the sender must be compensated in the form of a higher first period action.

Since first period actions are increasing in the disclosure order, increasing δ, the sender's weight on the first period action, further dis-incentivizes the sender from withholding. Thus any equilibrium action profile under δ will also be one under $\delta' > \delta$.

Proposition 8. Let $(T, \succeq_d)$ satisfy the UEPP. Define $\Pi(\delta)$ to be the set of interval equilibrium period 1 action profiles, i.e. all a_1 such that there exists σ with (a_1, σ) constituting an equilibrium.

$$\forall \delta < \delta', \ \Pi(\delta) \subset \Pi(\delta').$$

Let $E^1(\delta)$ be the equilibrium that is optimal from the perspective of the period 1 receiver. A straightforward corollary is that receiver 1's expected utility from $E^1(\delta)$ increases in δ. Thus, period 2 incentives harm the period 1 receiver. The intuition is that because the sender can always wait to

disclose, he will never be induced to disclose more in period 1 by dynamic incentives. Moreover, all else equal, the sender would prefer to wait to disclose because first period disclosures induce more skepticism in receiver 2. Therefore, the sender's dynamic incentives make him to "disclose less".

8. Conclusion

This paper has two main contributions: (i) it characterizes the receiver optimal equilibrium in a large class of verifiable disclosure games and (ii) it shows that distributions which induce greater skepticism in the receiver are characterized by the more evidence relation. While quite general, the disclosure model does not incorporate all related examples from the literature. A prime example is that I do not allow for message dependent disclosure costs such as that of Verrecchia (1983). With disclosure costs, the equilibrium action for any type is not pinned down by the set of types with with which he pools. This impedes the equilibrium partition approach. Another open generalization is allowing the sender to have type dependent preferences.

References

ACHARYA, VIRAL V., DEMARZO, PETER, & KREMER, ILAN. 2011. Endogenous Information Flows and the Clustering of Announcements. *American Economic Review*, **101**(7), 2955–79.

BEN-PORATH, ELCHANAN, DEKEL, EDDIE, & LIPMAN, BART. 2017. Mechanisms with Evidence: Commitment and Robustness. *Working Paper*.

BERTOMEU, JEREMY, & CIANCIARUSO, DAVIDE. 2016. Verifiable Disclosure. *Available at SSRN*.

BHATTACHARYA, SOURAV, & MUKHERJEE, ARIJIT. 2013. Strategic information revelation when experts compete to influence. *The RAND Journal of Economics*, **44**(3), 522–544.

BULL, JESSE, & WATSON, JOEL. 2004. Evidence disclosure and verifiability. *Journal of Economic Theory*, **118**(1), 1 – 31.

DRANOVE, DAVID, & JIN, GINGER ZHE. 2010. Quality Disclosure and Certification: Theory and Practice. *Journal of Economic Literature*, **48**(4), 935–963.

DYE, RONALD A. 1985. Disclosure of Nonproprietary Information. *Journal of Accounting Research*, **23**(1), 123–145.

DZIUDA, WIOLETTA. 2011. Strategic argumentation. *Journal of Economic Theory*, **146**(4), 1362 – 1397.

EINHORN, ETI. 2007. Voluntary disclosure under uncertainty about the reporting objective. *Journal of Accounting and Economics*, **43**(2), 245 – 274.

EINHORN, ETI, & ZIV, AMIR. 2008. Intertemporal Dynamics of Corporate Voluntary Disclosures. *Journal of Accounting Research*, **46**(3), 568–589.

GLAZER, JACOB, & RUBINSTEIN, ARIEL. 2004. On Optimal Rules of Persuasion. *Econometrica*, **72**(6), 1715–1736.

GREEN, JERRY R., & LAFFONT, JEAN-JACQUES. 1986. Partially Verifiable Information and Mechanism Design. *Review of Economic Studies*, **53**(3), 447–456.

GROSSMAN, S. J., & HART, O. D. 1980. Disclosure Laws and Takeover Bids. *The Journal of Finance*, **35**(2), 323–334.

GROSSMAN, SANFORD. 1981. The Informational Role of Warranties and Private Disclosure about Product Quality. *Journal of Law and Economics*, **24**(3), 461–83.

GRUBB, MICHAEL D. 2011. Developing a Reputation for Reticence. *Journal of Economics and Management Strategy*, **20**(1), 225–268.

GUTTMAN, ILAN, KREMER, ILAN, & SKRZYPACZ, ANDRZEJ. 2014. Not Only What but Also When: A Theory of Dynamic Voluntary Disclosure. *American Economic Review*, **104**(8), 2400–2420.

HAGENBACH, JEANNE, KOESSLER, FRÉDÉRIC, & PEREZ-RICHET, EDUARDO. 2014. Certifiable Pre-Play Communication: Full Disclosure. *Econometrica*, **82**(3), 1093–1131.

HART, SERGIEU, KREMER, ILAN, & PERRY, MOTTY. 2017. Evidence Games: Truth and Commitment. *American Economic Review, Forthcoming*.

JUNG, WOON-OH, & KWON, YOUNG K. 1988. Disclosure When the Market Is Unsure of Information Endowment of Managers. *Journal of Accounting Research*, **26**(1), 146–153.

KIM, KYUNGMIN, & POGACH, JONATHAN. 2014. Honesty vs. advocacy. *Journal of Economic Behavior Organization*, **105**(Supplement C), 51 – 74.

MATHIS, JÉRÔME. 2008. Full revelation of information in Sender–Receiver games of persuasion. *Journal of Economic Theory*, **143**(1), 571 – 584.

MILGROM, PAUL. 2008. What the Seller Won't Tell You: Persuasion and Disclosure in Markets. *Journal of Economic Perspectives*, **22**(2), 115–131.

MILGROM, PAUL R. 1981. Good News and Bad News: Representation Theorems and Applications. *Bell Journal of Economics*, **12**(2), 380–391.

RUBINSTEIN, ARIEL, & GLAZER, JACOB. 2006. A study in the pragmatics of persuasion: a game theoretical approach. *Theoretical Economics*, **1**(4), 395–410.

SEIDMANN, DANIEL J., & WINTER, EYAL. 1997. Strategic Information Transmission with Verifiable Messages. *Econometrica*, **65**(1), 163–169.

SHELTON, DONALD, KIM, YOUNG, & BARAK, GREGG. 2009. Examining the "CSI-effect" in the cases of circumstantial evidence and eyewitness testimony: Multivariate and path analyses. *Journal of Criminal Justice*, **37**(5), 452–460.

SHER, ITAI. 2011. Credibility and determinism in a game of persuasion. *Games and Economic Behavior*, **71**(2), 409 – 419.

SHER, ITAI. 2014. Persuasion and dynamic communication. *Theoretical Economics*, **9**(1), 99–136.

SHIN, HYUN SONG. 2003. Disclosures and Asset Returns. *Econometrica*, **71**(1), 105–133.

SKRETA, VASILIKI. 2006. Sequentially Optimal Mechanisms. *Review of Economic Studies*, **73**, 1085–1111.

VERRECCHIA, ROBERT E. 1983. Discretionary disclosure. *Journal of Accounting and Economics*, **5**, 179 – 194.

A. General implications of Algorithm 2

This section establishes a general comparative statics result that concerns the output of Algorithm 2. For a function $r : X \to \mathbb{R}$ and distribution $h \in \Delta X$, denote the conditional expectation function as $E_h^r(S) \equiv \mathbb{E}[r(x)|x \in S, x \sim h]$.

Proposition 9. *Let X be a finite set, $f \in \Delta X$, and $r : X \to \mathbb{R}$. For any $g \in \Delta X$, there exists an interval partition $P = (P_1, ..., P_m)$ of $(X, \geq_{f/g})$ with,*

$$E_f^r(P_1) < ... < E_f^r(P_m), \tag{13}$$

$$and \ E_f^r(P_i) \leq E_g^r(P_i) \ \forall i, \tag{14}$$

where the partition P does not depend on g given $\geq_{f/g}$.

To interpret Proposition 9, consider its implications in two extreme cases. First, if r is strictly increasing on $(X, \geq_{f/g})$, then the only interval partition satisfying (14) is the complete partition $P = (\{x_1\}, ..., \{x_n\})$. Conversely, if r is decreasing on $(X, \geq_{f/g})$, then the only interval partition satisfying (13) is the trivial partition $P = (X)$.

Broadly, every function has decreasing and increasing portions on $(X, \geq_{f/g})$. On the decreasing regions the expectation is lower under f than under g. The argument for Proposition 9 repeatedly applies this result, collapsing decreasing sequences to their average, until the interval partition has "no more decreasing regions". The consequences are (13) and (14). The former says that the conditional expectation is strictly increasing in the part's index. The latter says that for each part the conditional expectation is lower under f than under g.[27]

---

[27] The last part of the result says that P only depends on g through $\geq_{f/g}$; i.e. if P satisfies (13) and (14) for $g' \in \Delta X$, then P will also satisfy these conditions for any g'' such that $\geq_{f/g'} = \geq_{f/g''}$.

A.1. Proof of Proposition 9

Proof. I prove the existence of a partition satisfying (13) and (14) by construction: the output of Algorithm 2 satisfies (13) and (14).

Input $(X, \geq_{f/g})$ for some $f, g \in \Delta X$, and the conditional expectation function E_f^r associated with $r : X \to \mathbb{R}$. Because X is a finite set, and the algorithm repeatedly returns coarser and coarser partitions, the process must terminate at some stage T. At this point $P^T = P^{T+1}$ which means that $E_f^r(P_1^T) < ... < E_f^r(P_m^T)$. Thus P^T satisfies (13). I will show that P^T also satisfies (14), thereby proving the result.

Consider the partition $P^i = (P_1^i, ..., P_m^i)$ generated at stage $i > E^r 1$. Each part P_j^i is the union of an "interval" of parts from the previous partition P^{i-1}. More specifically, for each j there exists $\underline{k}(j) \leq \overline{k}(j)$ such that $P_j^i = \cup_{l=\underline{k}(j)}^{\overline{k}(j)} P_l^{i-1}$. Because P^{i-1} is an interval partition of $(X, \geq_{f/g})$, and $E_f^r(P_l^{i-1})$ is decreasing for $\underline{k}(j) \leq l \leq \overline{k}(j)$ one can use Remark 4 on the set P_j^i, to obtain,

$$\frac{1}{G(P_j^i)} \sum_{l=\underline{k}(j)}^{\overline{k}(j)} E_f^r(P_l^{i-1}) G(P_l^{i-1}) \geq \frac{1}{F(P_j^i)} \sum_{l=\underline{k}(j)}^{\overline{k}(j)} E_f^r(P_l^{i-1}) F(P_l^{i-1}).$$

Consider a given part P_k^T of the final partition P^T. One can iteratively use the above inequality for every part of the interval partition at every stage of the algorithm to obtain the following string of inequalities,

$$E_g^r(P_k^T) = \frac{1}{G(P_k^T)} \sum_{t_i \in P_k^T} r(t_i) g(t_i) = \frac{1}{G(P_k^T)} \sum_{P_j^2 \subset P_k^T} \left(\frac{1}{G(P_j^2)} \sum_{t_i \in P_j^2} r(t_i) g(t_i) \right) G(P_j^2)$$

$$\geq \frac{1}{G(P_k^T)} \sum_j \left(\frac{1}{F(P_j^2)} \sum_{t_k \in P_j^2} r(t_i) f(t_i) \right) G(P_j^2) = \frac{1}{G(P_k^T)} \sum_j E_f^r(P_j^2) G(P_j^2)$$

$$...$$

$$\geq \frac{1}{G(P_k^T)} \left(\frac{1}{G(P_k^T)} \sum_{P_l^{T-1} \subset P_k^T} E_f^r(P_l^{T-1}) G(P_l^{T-1}) \right) G(P_k^T)$$

$$\geq \frac{1}{G(P_k^T)} \left(\frac{1}{F(P_k^T)} \sum_{P_l^{T-1} \subset P_k^T} E_f^r(P_l^{T-1}) F(P_l^{T-1}) \right) G(P_k^T)$$

$$= E_f^r(P_k^T).$$

Combining these inequalities gives $E_g^r(P_k^T) \geq E_f^r(P_k^t)$, $\forall k$ establishing (14) and thereby Proposition 9.

Q.E.D.

B. Preliminaries

B.1. Proof of Lemma 1

Proof of (1)

Proof. Let $a_1 \equiv \min_i a^*(q_i)$ and $a_2 \equiv \max_i a^*(q_i)$. Note that because U^R is strictly concave, $a^*(q) \geq a \iff \sum_t U_a^R(a,t)q_i(t) > 0$. This means that $\sum_t U_a^R(a_2,t)q_1(t) < 0$ and $\sum_t U_a^R(a_1,t)q_2(t) > 0$. This implies that

$$\sum_t U_a^R(a_2,t)(\lambda q_1(t) + (1-\lambda)q_2(t))$$
$$= \lambda \sum_t U_a^R(a_2,t)q_1(t) + (1-\lambda)\sum_t U_a^R(a_2,t)q_2(t)$$
$$= \lambda \sum_t U_a^R(a_2,t)q_1(t) < 0.$$

This implies that $a^*(\lambda q_1 + (1-\lambda)q_2) < a_2$. The argument is symmetric for $a_1 < a^*(\lambda q_1 + (1-\lambda)q_2)$.

Q.E.D.

B.2. Truth leaning Equilibrium Refinement

Hart *et al.* (2017) show that the truth leaning refinement selects the receiver optimal equilibrium. For any sender strategy σ define $Supp(\sigma) \equiv \cup_t Supp(\sigma_t)$. An equilibrium (σ, a) is truth leaning if,

$$t \notin Supp(\sigma) \implies a(t) = v(t), \text{ and} \tag{15}$$
$$t \in \arg\max_{s \in M(t)} a(s) \implies \sigma_t(t) = 1. \tag{16}$$

I recall one result from HKP concerning truth leaning equilibria.

Lemma 4. *If (σ, a) constitute a truth leaning equilibrium, then for every $t \in T$ exactly one of the following holds,*

$$\sigma_t(t) = 1, \text{ and } \pi_h(t|U^R) = a(t) \leq v(t), \text{ or} \tag{17}$$
$$\sigma_s(t) = 0, \ \forall s, \text{ and } \pi_h(t|U^R) > v(t) = a(t). \tag{18}$$

Proof. See Hart *et al.* (2017).

Q.E.D.

C. Proofs from Section 4

C.1. Proof of Proposition 1

Proof. Suppose P is an equilibrium partition but not an interval partition. This means that $t \succeq_d s$ where $t \in P_i$, $s \in P_j$, and $j > i$. But then t can deviate to the strategy of s and obtain a strictly higher action.

Now suppose that P is an interval partition, and that there exists σ, such that $t \in P_i \implies Supp(\sigma_t) \subset P_i$ and $s \in P_i \cap Supp(\sigma) \implies \arg\max_a \sum U^R(a,t)\sigma_t(s)h(t) = V_h(P_i)$. That is, σ is self contained in each part, and induces the receiver to take the same action for each declaration within each part. If such a σ exists then P is an interval partition. To complete the argument I prove the following claim.

Claim 1. *For a subset $S \subset T$. There exist truth leaning mutual best responses $\sigma : S \to \Delta S$ and $a : S \to A$, such that $s \in Supp(\sigma) \implies a(s) = V_h(S)$ if and only if V_h is downward biased on $(S, \succeq_d)$.*

Proof. " $\implies$ " Now say that, V_h is not downward biased on $(S, \succeq_d)$, i.e. there exists $\tilde{S} \subset S$ such that $V_h(W(\tilde{S}) \cap S) < V_h(S)$. Let $\tilde{W} \equiv W(\tilde{S}) \cap S$. Take the set of declared types in $\tilde{W}$ to be $\tilde{W}_d \equiv \cup_t Supp(\sigma_t) \cap \tilde{W}$. Because $\tilde{W}$ is a lower contour subset of S, $\cup_{t \in \tilde{W}} Supp(\sigma_t) \subset \tilde{W}_d$. By receiver incentive compatibility,

$$\sum_s U_a^R(V_h(S), s)\sigma_s(t)h(s) = 0, \ \forall t \in \tilde{W}_d.$$

Summing over $t \in \tilde{W}_d$,

$$0 = \sum_{t \in \tilde{W}_d} \sum_s U_a^R(V_h(S), s)\sigma_s(t)h(s)$$
$$= \sum_{t \in \tilde{W}_d} \sum_{s \notin \tilde{W}} U_a^R(V_h(S), s)\sigma_s(t)h(s) + \sum_{s \in \tilde{W}} U_a^R(V_h(S), s)h(s) \sum_{t \in \tilde{W}_d} \sigma_s(t)$$
$$= \sum_{t \in \tilde{W}_d} \sum_{s \notin \tilde{W}} U_a^R(V_h(S), s)\sigma_s(t)h(s) + \sum_{s \in \tilde{W}} U_a^R(V_h(S), s)h(s). \tag{19}$$

The last equality follows from the fact that $\cup_{t \in \tilde{W}} Supp(\sigma_t) \subset \tilde{W}_d$ and so $\sum_{t \in \tilde{W}_d} \sigma_s(t) = 1, \ \forall s \in \tilde{W}$. By assumption $V_h(\tilde{W}) < V_h(S)$ and so the second term in (19) is negative i.e.,

$$\sum_{s \in \tilde{W}} U_a^R(V_h(S), s)h(s) < 0.$$

This means that the first term must be positive. Lemma 4 states that every type s that declares some other type t must have lower value than the action $a(t)$ which type s obtains. This means that for $s \notin \tilde{W}$ such that $\sigma_s(t) > 0$, $v(s) < a(t) = V_h(S)$. Thus, every summand in the first term in (19), $U_a^R(V_h(S), s) < 0$. But this is a contradiction.

" $\impliedby$ "

Now say that V_h is downward biased on $(S, \succeq_d)$. Bipartition S into $U \equiv \{t : \ v(t) \geq V_h(S)\}$, $D \equiv U^c$. Now consider any feasible strategy $\eta : D \to \Delta U$, and define $\sigma : S \to \Delta S$ as $\sigma_t(s) = \eta_t(s), \ \forall t \in D$ and $\sigma_t(t) = 1, \ \forall t \in U$. Denote Z the set of all such σ.

First, note that Z is non-empty. Z is empty only if $\exists s, \ s.t. \ M(s) \cap U = \emptyset$, i.e. $M(s) \cap S \subset D$. By Lemma 1, $V_h(M(s) \cap S) < V_h(S)$, which contradicts the fact that V_h is downward biased on $(S, \succeq_d)$.

Thus the restricted set of strategies Z is nonempty. Define the receiver's unique truth-leaning best response to $\sigma \in Z$ as $a^\sigma : S \to \mathbb{R}$. Now consider the problem,

$$\min_{\sigma \in Z} \; f(\sigma) \equiv \sum_{t \in U} (a^\sigma(t) - V_h(S))^2. \tag{20}$$

The objective in (20) is continuous in a and σ. The receiver can be taken to maximize over a compact set ($[\min_t v(t), \max_t v(t)]$) so the theorem of the maximum holds. Moreover the maximum is unique, so for each $t \in U$, $a^\sigma(t)$ is continuous in σ. Since Z is a compact set, the problem has a solution σ^* by Weierstrauss theorem. Let the corresponding receiver best response be a^{σ^*}.

Now say $f(\sigma^*) > 0$, i.e. $\exists t \in U : a^{\sigma^*}(t) \neq V_h(S)$. Notice that $\cup_s \text{Supp}(\sigma_s) = U$, so by Lemma 1, $U^B \equiv \{s \in U : a^{\sigma^*}(s) \geq V_h(S)\}$ and $U^W \equiv \{s \in U : a^{\sigma^*}(s) < V_h(S)\}$ are both nonempty. Let,

$$X \equiv \{s : \exists t \in \text{Supp}(\sigma_s^*), \; V_h(S) > a^{\sigma^*}(t) > v(s)\}, \; and$$
$$Y \equiv \{s : U^B \cap M(s) \neq \emptyset\}.$$

X is the set of types that obtain an action with positive probability that is less than $V_h(S)$, but greater than its own value. Y is the set of types that have value less than $V_h(S)$, but have the ability to obtain an action greater than $V_h(S)$. I will show that $X \cap Y \neq \emptyset$.

First note that $X \neq \emptyset$. This is because U^W is non-empty, meaning there exists a type $t \in U$ with $a(t) < V_h(S)$. But by Lemma 1, the best response action must be in between the values of the types that declare it. Since $t \in U$, $v(t) \geq V_h(S)$, and $\sigma_t^*(t) = 1$. This in turn means there must exist an s that declares t such that $v(s) < a^{\sigma^*}(t)$, or in other words X is non-empty. Let $\tilde{W} \equiv W(X) \cap S$.

Now suppose that $X \cap Y = \emptyset$. Consider the sender strategy of types in $\tilde{W}$, and recompute receiver best responses as $a_{\tilde{W}}^{\sigma^*} : \tilde{W} \to \mathbb{R}$. That is, let

$$a_{\tilde{W}}^{\sigma^*}(s) \equiv \arg\max_a \sum_{t \in \tilde{W}} U^R(a, t) \sigma_t^*(s) h(t).$$

I claim that $a_{\tilde{W}}^{\sigma^*}(s) < V_h(S)$. $\forall s \in \tilde{W}$. To see this, first note that $a^{\sigma^*}(s) < V_h(S)$, $\forall s \in \tilde{W}$, because by assumption $X \cap Y = \emptyset$, i.e. types in X cannot obtain a higher action than $V_h(S)$. Now inspect the types that declare some $s \in \tilde{W}$ under σ^*, but are not in $\tilde{W}$. Call this set

$$R \equiv \{t \notin \tilde{W} : \exists s \in \tilde{W} \cap \text{Supp}(\sigma_t^*)\}.$$

Consider $t \in R$. By definition $t \notin X$ because $t \notin \tilde{W}$ and $X \subset \tilde{W}$. Now consider $s \in \tilde{W} \cap \text{Supp}(\sigma_t^*)$. Because, $s \in \tilde{W}$, $a^{\sigma^*}(s) < V_h(S)$. Then, because $t \notin X$ and $s \in \text{Supp}(\sigma_t^*)$, $v(t) \geq a^{\sigma^*}(s)$. Thus, excluding types in R from σ^* decreases $\sigma_t(s)$ which decreases the best response of the receiver establishing that $a_{\tilde{W}}^{\sigma^*}(s) < V_h(S) \; \forall s \in \tilde{W}$.

Also by Lemma 1, the best response action to the entire set $\tilde{W}$ must be in between the best

response actions to each type for any strategy, i.e.

$$\min_{s \in \tilde{W}} a_{\tilde{W}}^{\sigma^*}(s) \leq V_h(\tilde{W}) \leq \max_{s \in \tilde{W}} a_{\tilde{W}}^{\sigma^*}(s).$$

This implies that $V_h(\tilde{W}) < V_h(S)$. But this is a contradiction, because V_h is downward biased on $(S, \succeq_d)$.

Thus $X \cap Y \neq \emptyset$. Let $s \in X \cap Y$. This means $\exists t', t'' \in S$ such that $t' \in \mathrm{Supp}(\sigma_s^*)$ with $v(s) < a^{\sigma^*}(t') < V_h(S)$ and $t'' \in \tilde{W}$ with $a^{\sigma^*}(t'') \geq V_h(S)$. Now construct a new strategy $\tilde{\sigma} \in Z$ as follows.

$$\tilde{\sigma}_{\tilde{s}}(\tilde{t}) = \begin{cases} \sigma_{\tilde{s}}^*(\tilde{t}) & if \ \tilde{s} \neq s, \tilde{t} \neq t', t'' \\ \sigma_{\tilde{s}}^*(\tilde{t}) - \varepsilon & if \ \tilde{s} = s, \tilde{t} = t' \\ \sigma_{\tilde{s}}^*(\tilde{t}) + \varepsilon & if \ \tilde{s} = s, \tilde{t} = t'' \end{cases},$$

with $\varepsilon > 0$ small enough such that this is a feasible strategy. Because $V_h(S) > a^{\sigma^*}(t') > v(s)$, and this new strategy decreases the probability that s declares t', this change increases the best response $a^{\sigma^*}(t)$. This decreases the objective in (20). Similarly because $a(t'') \geq V_h(S) > v(s)$ and this new strategy increases the probability that s declares t'', this change decreases the best response $a^{\sigma^*}(t'')$. This also decreases the objective in (20).[28] All other best responses remain unchanged. This change thereby decreases the objective in (20) so that $f(\tilde{\sigma}) < f(\sigma^*)$, contradicting the minimality of σ^*. So at the minimum, $f(\sigma^*) = 0 \implies a^{\sigma^*}(t) = V_h(S) \ \forall t \in U$. *Q.E.D.*

Q.E.D.

C.2. Proof of Lemma 2

Proof. Let $S^* \in \arg\min_{\tilde{S} \subset S} V_h(W(\tilde{S}) \cap S)$ with value $\overline{V}$, let $\overline{W} \equiv W(S^*) \cap S$. I prove that,

$$V_h(W(\tilde{S}) \cap \overline{W}) \geq V_h(\overline{W}), \ \forall \tilde{S} \subset \overline{W}.$$

Suppose not, and take $\overline{W}' \equiv W(\tilde{S}) \cap \overline{W}$ such that $V_h(\overline{W}') < V_h(\overline{W})$. Note that $W(\overline{W}') \cap S = \overline{W}'$, which contradicts the minimality of $\overline{W}$ in the above problem. Thus each minimizer of the above problem is downward biased.

Now take $J \subset \arg\min_{\tilde{S} \subset S} V_h(W(\tilde{S}) \cap S)$ with $J \equiv (S_1, ..., S_c)$ and $\overline{W}_i \equiv W(S_i) \cap S$ and $\overline{W} \equiv \cup_{k=1}^{c} \overline{W}_k$. Note that because each $\overline{W}_i$ is downward biased, for each i, $V_h(W_i \setminus \cup_{k=1}^{i-1} \overline{W}_i) \leq V_h(W_i) = \overline{V}$. Since $\overline{W}$ is the disjoint union of these sets, i.e. $\overline{W} = \cup_{i=1}^{c}(W_i \setminus \cup_{k=1}^{i-1} \overline{W}_i)$, Lemma 1 implies that $V_h(\overline{W}) \geq \overline{V}$. Thus $\overline{W} \in \arg\max_{\tilde{S} \subset S} V_h(W(\tilde{S}) \cap S)$, and so by the previous argument $\overline{W}$ is downward biased. *Q.E.D.*

---

[28] If $V_h(S) = a^{\sigma^*}(t'')$ then this change actually increases $(a^{\sigma^*}(t'') - V_h(S))^2$. However, for small enough ε this increase will be second order.

C.3. Proof of Proposition 2

Proof. Algorithm 1 produces a partition of T into disjoint sets $(P_1, P_2, ..., P_m)$. I argue that this partition satisfies the requirements of Proposition 1, and thereby constitutes an equilibrium partition. Lemma 2 implies that each P_i is a downward biased set. One must only check that $i > j \implies V_h(P_i) > V_h(P_j)$ and that $i > j \implies W(P_j) \cap P_i = \emptyset$.

First, suppose that $V_h(P_i) \geq V_h(P_{i+1})$. Note that $W(P_i \cup P_{i+1}) \cap S_i = P_i \cup P_{i+1}$. This is because, $W(P_{i+1}) \cap S_{i+1} = P_{i+1}$ by definition, and $W(P_{i+1}) \cap S_i \subset P_i \cup P_{i+1}$ by the construction in Algorithm 1. Thus, $V_h((P_i \cup P_{i+1}) \cap S_i) \leq V_h(P_i)$ by Lemma 1. But this means that P_i could not have been the output of the algorithm at stage i as $(P_i \cup P_{i+1}) \cap S_i$ is a larger (in the sense of set inclusion) minimizing set than P_i.

Second, notice that for $i < j$, $P_j \subset S_i$. But $W(P_i) \cap S_i = P_i$. Since $P_i \cap P_j = \emptyset$, this means that $W(P_i) \cap P_j = \emptyset$.
$\hfill$ Q.E.D.

C.4. Proof of Theorem 3

Proof. Take the equilibrium partition $(P_1, ..., P_m)$. For $t \in P_i$, $V_h(P_i) = \pi_h(t|U^R)$. Thus, I prove that the solution to the problem on the right hand side of (9) is $V_h(P_k)$.

Consider setting $S_a^* \equiv \cup_{k=1}^i P_k$ and $S_b^* \equiv \cup_{k=i}^m P_k$. S_a^* is the union of parts of the equilibrium partition that have value less than or equal to P_i, and S_b^* is the union of parts of the equilibrium partition that have value greater than or equal to P_i. Note that $W(S_a^*) = S_a^*$ and $B(S_b^*) = S_b^*$. Using these choices in equation (9) gives,

$$V_h(W(S_a^*) \cap B(S_b^*)) = V_h(S_a^* \cap S_b^*) = V_h(P_i).$$

First, I show that given S_a^*, S_b^* solves the partial problem $\max_{\{S_b : t \in S_b\}} V_h(W(S_a^*) \cap B(S_b))$. This shows that $V_h(P_i)$ is achievable in the problem in (9). Second, I show that for any feasible S_a, choosing $S_b = S_b^*$ gives $V_h(S_a \cap S_b^*) \geq V_h(P_i)$. This means that for any feasible S_a, the value to the partial problem $\max_{\{S_b : t \in S_b\}} V_h(W(S_a) \cap B(S_b)) \geq V_h(P_i)$. Thus, choosing $S_a = S_a^*$ achieves the minimum value, establishing the result.

Step 1: Take any feasible S_b. $B(S_b) \cap W(S_a^*) = \cup_{k=1}^i (B(S_b) \cap P_k)$. By the fact that V_h is downward biased on each part P_k, for $k \leq i$ (whenever non-empty) $V_h((B(S_b) \cap P_k) \leq V_h(P_k) \leq V_h(P_i)$. Thus, by Lemma 1, $V_h(\cup_{k=1}^i (B(S_b) \cap P_k)) \leq V_h(P_i)$. Since S_b^* achieves $V_h(P_i)$, S_b^* solves the partial problem $\max_{\{S_b : t \in S_b\}} V_h(W(S_a^*) \cap B(S_b))$.

Step 2: Take any feasible S_a. Notice that $B(S_b^*) \cap W(S_a) = \cup_{k=i}^m (W(S_a) \cap P_k)$. Because V_h is downward biased on each part P_k, for $k \geq i$ (whenever non-empty) $V_h((W(S_a) \cap P_k) \geq V_h(P_k) \geq V_h(P_i)$. Thus, by Lemma 1 that $V_h(\cup_{k=i}^m (W(S_a) \cap P_k)) \geq V_h(P_i)$. Since S_b^* is feasible, the solution to the partial problem $\max_{\{S_b : t \in S_b\}} V_h(W(S_a) \cap B(S_b)) \geq V_h(P_i)$. Since choosing $S_a = S_a^*$ achieves $V_h(P_i)$, this choice achieves the minimal value.

D. Proofs from Section 5

D.1. Proof of Proposition 3

Proof. $" \Longleftarrow "$

Let $r : S \to \mathbb{R}$ be defined as $r(s) \equiv U_a^R(V_f(S), s)$. Notice that because U^R is strictly concave,

$$V_h(\tilde{S}) > V_f(S) \iff E_g^r(\tilde{S}) > E_f^r(S) = 0, \ \forall \tilde{S} \subset S, h \in \Delta S. \tag{21}$$

Thus, because V_f is downward biased on $(S, \succeq_d)$, E_f^r is downward biased on $(S, \succeq_d)$. Because f has more evidence than g, $(S, \geq_{f/g})$ is a refinement of $(S, \succeq_d)$, and so E_f^r is also downward biased on $(S, \geq_{f/g})$.

Using E_f^r in Proposition 9 produces an interval partition $P = (P_1, ..., P_m)$ of $(S, \geq_{f/g})$ such that (13) and (14) hold. If P is not the trivial partition, then because of (13) $E_f^r(P_1) < E_f^r(S)$. But this is a contradiction because P_1 is a lower contour subset of $(S, \geq_{f/g})$ and E_f^r is downward biased on $(S, \geq_{f/g})$. Since P must be the trivial partition, (14) gives $E_f^r(S) \leq E_g^r(S)$.

$" \Longrightarrow "$

suppose V_f is not downward biased on $(S, \succeq_d)$. This means there exists a lower contour subset $L = W(L) \subset S$, such that $V_f(L) < V_f(S) \implies V_f(L) < V_f(S \setminus L)$. Define $g(s) = \frac{f(s)}{F(L)}$ if $s \in L$ and $g(s) = 0$ otherwise. $f \geq_{ME} g$ but $V_f(S) > V_g(S)$. $\hfill Q.E.D.$

E. Proofs from Section 6

E.1. Proof of Proposition 4

Proof. That the expression in (12) corresponds to the receiver optimal equilibrium in $\tilde{H}$, flows directly from Theorem 3, and taking $S_b \equiv R' \times \{NS\}$ in the definition of $\tilde{V}$.

The fact that π^* is an equilibrium allocation of $\tilde{UB}$ is as follows. Receiver and strategic sender incentive compatibility are directly transferred from that in $\tilde{H}$. Unbiased sender's who obtain $v(t)$ are at their bliss point. unbiased senders who obtain $\pi^*(t, NS) < v(t)$ can only potentially deviate to lower actions which are further from their bliss point, otherwise their strategic counterparts would deviate in $\tilde{H}$.

Now I show that π^* is receiver optimal in $\tilde{UB}$. Consider some other equilibrium allocation π' in $\tilde{UB}$. Define the following commitment allocation, $\hat{\pi}$, in $\tilde{UB}$ defined by:

$$\hat{\pi}((t, S)) = \pi'((t, S)),$$
$$\hat{\pi}((t, UB)) = \min\{\pi'((t, UB)), v(t)\}.$$

Take any $k, k' \in T' \times \{S, UB\}$ and let $k \succeq_d^{UB} k'$. If k is strategic then $\pi'(k) \geq \pi'(k') \geq \hat{\pi}(k')$ so $\hat{\pi}$ is incentive compatible for strategic senders. If k is unbiased, then either $\hat{\pi}(k) = v(k)$, or $v(k) > \hat{\pi}(k) = \pi'(k) \geq \pi'(k') \geq \hat{\pi}(k')$ so $\hat{\pi}$ is also incentive compatible for unbiased senders. The receiver is better off under $\hat{\pi}$ than π'. Also $\hat{\pi}$ is an incentive compatible commitment allocation in $\tilde{H}$. Since the optimal commitment allocation is the receiver optimal equilibrium allocation in $\tilde{H}$, the receiver prefers π^* to π'. *Q.E.D.*

E.2. Proof of Proposition 5

Proof. *Step 1*: Every equilibrium of $\tilde{C}$ is an incentive compatible commitment allocation of $\tilde{D}$ Consider any equilibrium allocation of $\tilde{C}$ given by $\pi : T \to \mathbb{R}$. Consider altering $\tilde{D}$ to a game in which the receiver can commit ex-ante to an action $a : T \to \mathbb{R}$. Say that π is not incentive compatible in $\tilde{D}$ with commitment. There exists $t, t' \in T'$ such that either

$$\pi((t, H)) < \pi((t', H)), \ t \succeq_d' t', \tag{22}$$

$$\pi((t, L)) > \pi((t', L)), \ t \succeq_d' t', \tag{23}$$

$$\pi((t, H)) < \pi((t', L)), \ \exists s \in M_{\succeq_d'}(t) \cap M_{\succeq_d'}(t'). \tag{24}$$

The statements in (22) and (23) directly violate incentive compatibility in $\tilde{C}$. incentive compatibility in $\tilde{C}$ requires that $\pi((t, H)) > \pi((s, H))$ and $\pi((t', L)) < \pi((s, L))$, which combined with (24) gives $\pi((s, H)) < \pi((s, L))$ violating incentive compatibility in $\tilde{C}$.

Step 2: $\pi^{\tilde{D}}$ is an equilibrium in $\tilde{C}$, The receiver optimal commitment allocation in $\tilde{D}$ is equivalent to that in $\tilde{D}$ by Sher (2011). By Proposition 1 this corresponds to a interval partition of $(T, \succeq_d)$, $(P_1, ..., P_m)$ such that V_g is downward biased on each $(P_i, \succeq_d)$. I will show that condition (B) of Proposition 12 is satisfied so that there exists a pooling strategy. Let $P_i \cap \{T' \times \{H\} \equiv R_i$ and $P_i \cap \{T' \times \{L\} \equiv Q_i$. Take the set of non-dominant elements of $(P_i, \succeq_d')$ to be W_i. Consider $W \subset W_i$. Define

$$Q_i \equiv \{(t, H) : M_{\succeq_d'}((t, H)) \subset W \cap T' \times \{H\}\} \cup B_{\succeq_d'}(W \cap T' \times \{L\}).$$

Notice that Q_i is a lower contour set of $(P_i, \succeq_d)$, and so by Proposition 1 $V_g(Q_i) \geq V_g(P_i)$. This verifies condition (B) and therefore there exists a pooling strategy for each P_i under $\succeq_c$. Thus

step 3 $\pi^{\tilde{D}} = \pi^{\tilde{C}}$. Notice that the receiver's expected utility for any allocation in $\tilde{C}$ is the same as that in $\tilde{D}$. Step 1 shows that the set of equilibrium allocations in $\tilde{C}$ is smaller than the set of equilibrium allocations of the commitment version of $\tilde{D}$. Second, the set of equilibrium allocations of the commitment version of $\tilde{D}$ is larger than the set of equilibrium allocations of the commitment version of $\tilde{D}$. Third, step 2 shows that the receiver optimal commitment allocation in $\tilde{D}$ is incentive compatible in $\tilde{C}$ and in $\tilde{D}$. Thus, this allocation is both $\pi^{\tilde{D}}$ and $\pi^{\tilde{C}}$. *Q.E.D.*

F. Proofs from Section 7

F.1. Proof of Proposition 7

Proof. Consider any equilibrium allocation $\tilde{\pi} : T \to \Delta A$ with sender strategies σ and γ. Let the perceived distribution over T for period 2 following declaration t in period 1 be $f_t \in \Delta T$ with corresponding equilibrium partition $(Z^t, P_1^t, ..., P_m^t)$. Note that $Supp(f_t) = B(S)$ for some $S \in T$, i.e. $Z^t \equiv \{s : f_t(s) = 0\}$ is a lower contour set. Also note that,

$$f_t(s) = \frac{\sum_{r \in M(s)} \frac{g_2(s)}{G_2(B(r))} \gamma_r(t) g_1(r)}{\sum_s \sum_{r \in M(s)} \frac{g_2(s)}{G_2(B(r))} \gamma_r(t) g_1(r)}. \tag{25}$$

Note that if $\tilde{\pi}$ is degenerate then the result holds by Lemma 1 and Theorem 3. So in search of a contradiction suppose that $\tilde{\pi}$ is not degenerate. There exists $s, t', t'' \in T$ with $s \in P_{k'}^{t'} \cap P_{k''}^{t''}$ such that $\pi_{f_t'}(s|U^R) > \pi_{f_t''}(s|U^R)$ and $\pi_{f_t'}(s'|U^R) = \pi_{f_t''}(s'|U^R) \ \forall s' \in B(P_{k'}^{t'}) \setminus P_{k'}^{t'}$. In words, there is some most dominant part of the equilibrium partition in period 2 over which the payoffs to different period 1 declarations differ. This means that in period 1 type s expects a payoff difference between declarations t' and t'' at $\pi_{f_t'}(s|U^R) - \pi_{f_t''}(s|U^R)$ in period 2.

This means that $\forall s \in P^{t'}$, declaring t'' in the first period is strictly dominated by declaring t', i.e. $\gamma_s(t') = 0 \ \forall s \in P_{k'}^{t'}$. I will show that $f_{t'} \geq_{ME} f_{t''}$ on $P_{k'}^{t'}$. Then by using Proposition 11, this will imply that $V_{f_{t'}}(P_{k'}^{t'}) \leq V_{f_{t''}}(P_{k'}^{t'})$. This means that by Proposition 2 $V_{f_{t'}}(P_{k'}^{t'}) \geq V_{f_{t''}}(P_{k'}^{t'})$, contradicting the fact that $\pi_{f_t'}(s|U^R) > \pi_{f_t''}(s|U^R)$.

The argument for why $f_{t'} \geq_{ME} f_{t''}$ on $P_{k'}^{t'}$ is as follows. Take $s' \succeq_d s''$. First suppose that $s'' \notin Supp(f_{t''})$. If $s' \in Supp(f_{t''})$ then $\exists \tilde{s} \in M(s')$ such that $\gamma_{\tilde{s}}(t'') > 0$ and $\tilde{s} \notin M(s')$. This means that $s' \succeq_d s''$ and $s' \succeq_d \tilde{s}$, so because of the UEPP, $\tilde{s} \not\perp_d s''$. Since $\tilde{s} \notin M(s')$ it must be that $\tilde{s} \succeq_d s'$. Given that $s', s'' \in P_{k'}^{t'}$ and $P_{k'}^{t'}$ is an interval, $\tilde{s} \in P_{k'}^{t'}$. But this means that $\gamma_{\tilde{s}}(t'') = 0$, a contradiction. Thus $s' \notin Supp(f_{t''})$ and $f_{t'}(s') f_{t''}(s'') \geq f_{t'}(s'') f_{t''}(s')$ holds.

Now consider that $s'' \in Supp(f_{t''})$.

$$\frac{f_{t''}(s')}{f_{t''}(s'')} = \frac{\sum_{r \in M(s')} \frac{g_2(s')}{G_2(B(r))} \gamma_r(t'') h_1(r)}{\sum_{r \in M(s'')} \frac{g_2(s')}{G_2(B(r))} \gamma_r(t'') h_1(r)}$$

$$= \frac{g_2(s')}{g_2(s'')} \frac{\sum_{r \in M(s')} \frac{\gamma_r(t'') h_1(r)}{G_2(B(r))}}{\sum_{r \in M(s'')} \frac{\gamma_r(t'') h_1(r)}{G_2(B(r))}}.$$

This motivates the definition of the following term for any s', s'', t,

$$R_{s',s''}(t) \equiv \frac{\sum_{r \in M(s')} \frac{\sigma_r(t) h_{N-1}(r)}{G_N(B(r))}}{\sum_{r \in M(s'')} \frac{\sigma_r(t) h_{N-1}(r)}{G_N(B(r))}}.$$

Since $s' \succeq_d s''$, $R_{s',s''}(t'') \geq 1$. Now if $R_{s',s''}(t'') > 1$, this means there exists $r \in M(s') \setminus M(s'')$ such

48

that $\gamma_r(t'') > 0$ which implies that $t'' \in M(r)$. Also note that because $s'' \in Supp(f_{t''})$, $\exists r' \in M(s'')$ with $\gamma_{r'}(t'') > 0$. This means that $t'' \in M(s'')$. But since the UEPP holds, $s'' \not\perp_d r$, i.e. $r \succeq_d s''$. But since $s', s'' \in P_{k'}^{t'}$ and $P_{k'}^{t'}$ is an interval, we have that $r \in P_{k'}^{t'}$. But this means that $\gamma_r(t'') = 0$, a contradiction. This means that $R_{s',s''}(t'') = 1$. But then $R_{s',s''}(t') \geq R_{s',s''}(t'')$ which implies that $f_{t'}(s')f_{t''}(s'') \geq f_{t'}(s'')f_{t''}(s')$ holds. Thus $f_{t'} \geq_{ME} f_{t''}$ on $P_{k'}^{t'}$, completing the argument.

<div align="right">Q.E.D.</div>

F.2. Proof of Lemma 3

Proof. Suppose not, i.e. $\exists t, t' : t \succeq_d t', a_1(t) < a_1(t')$. Take the interval of types who induce $a_1(t)$ and $a_1(t')$ to be I and I' respectively. Let f and f' be the distributions induced after a declaration x inducing $a_1(t)$ and x' inducing $a_1(t')$ respectively. Because $t \succeq_d t'$ and I and I' are intervals, $W(I') \cap I = \emptyset$.

Now take $s \succeq_d s'$. We will show that $f \geq_{ME} f'$. If $f'(s') = 0$, $f(s') > 0$, and $f'(s) > 0$ then $\exists k \in I : s' \succeq_d k$ and $k' \in I' : s \succeq_d k'$, $s' \not\succeq_d k'$. By the UEPP $k' \not\perp k$ so $k' \succeq_d k$. But this means that $k \in W(I') \cap I_t$, a contradiction. Also if $f(s') > 0$, then $f(s) > 0$. Thus one need only consider cases in which $f(s), f(s'), f'(s), f'(s')$ are all strictly positive. I will show that,

$$\frac{f(s)}{f(s')} = \frac{g_2(s')}{g_2(s)} \frac{\sum_{r \in M(s')} \frac{\sigma_r(x)h_1(r)}{G_2(B(r))}}{\sum_{r \in M(s)} \frac{\sigma_r(x)h_1(r)}{G_2(B(r))}} \leq \frac{g_2(s')}{g_2(s)} \frac{\sum_{r \in M(s')} \frac{\sigma_r(x')h_1(r)}{G_2(B(r))}}{\sum_{r \in M(s)} \frac{\sigma_r(x')h_1(r)}{G_2(B(r))}} = \frac{f'(s)}{f'(s')}.$$

Notice that if $r \in M(s) \setminus M(s') \cap I'$, then $f(s') = 0$. By the UEPP $r \succeq_d s'$. So if $f(s') > 0$. $\exists r' \in M(s') \cap I$. But then $r \succeq_d r'$ contradicting that $W(I') \cap I = \emptyset$. Thus if $f(s), f(s'), f'(s), f'(s')$ are all strictly positive, then $R_{s,s'}(x) = 1$, its maximum value.

Since $f \geq_{ME} f'$, by Theorem 1, $f \geq_{MS} f'$. Thus $\mathbb{E}[\pi_f(s|U^2)|t] \leq \mathbb{E}[\pi_{f'}(s|U^2)|t]$. But since t induces $a_1(t)$ instead of $a_1(t)$,

$$a_1(t) + \mathbb{E}[\pi_f(s|U^2)|t] \geq a_1(t') + \mathbb{E}[\pi_{f'}(s|U^2)|t].$$

This gives the desired result that $a_1(t) \geq a_1(t')$. Q.E.D.

G. Generalization to Non-Full Support Distributions

G.1. Results from Section 4

It turns out that all the construction results go through without further refinement of the equilibrium concept. Adding zero probability types is like order embedding $(\text{Supp}(h), \succeq_d)$ into some larger ordered type set $(T, \succeq_d')$. This change enlarges the message sets of positive probability types. One might think that one may have to deter positive probability types from taking advantage of these new messages by appropriately refining the off path best response to these zero probability types. Also the set of equilibrium payoffs changes by adding these zero probability types.

However the receiver optimal equilibrium does not change and does not involve declaration of zero measure types with positive probability. This is because the truth leaning refinement which identifies the commitment solution ensures that zero probability types will be off path. Consider some $t \in T \setminus \text{Supp}(h)$. Suppose $t \in \cup_{t \in \text{Supp}(h)} \text{Supp}(\sigma_t)$, with best response $a(t)$, for truth leaning a and σ. By Lemma 1 there exists some type $s \in \text{Supp}(h)$ such that $\sigma_s(t) > 0$ and $v(s) \geq a(t)$. But since $s \neq t$, and $\pi_h(s|U^R) = a(t)$, this is a contradiction to Lemma 4 which says that $\pi(s|U^R) \leq v(t) \implies \sigma_s(s) = 1$. This allows one to ignore zero probability types in the construction of equilibrium. However, the truth leaning refinement must be adjusted to allow for more flexibility for off path declarations of zero probability types,

$$t \in \arg\max_{s' \in M(t)} a(s') \implies \sigma_t(t) = 1,$$

$$s \notin \cup_t \text{Supp}(\sigma_t) \implies \begin{cases} a(s) = v(s) & \forall s \in \text{Supp}(h) \\ a(s) = \min_{s' \in B(s)} v(s) & \forall s \notin \text{Supp}(h) \end{cases}.$$

Without this modification, the equilibrium may not exist. Notice that the modification still selects a PBNE. The above refinement derives the same equilibrium payoff vector for types in $\text{Supp}(h)$ as when T is restricted to $\text{Supp}(h)$. Thus it is without loss to focus on $\tilde{T} \equiv \text{Supp}(h)$ and apply the results of Section 4.

G.2. Comparative Statics Results from Section 5 and Appendix A

In this section I compare equilibrium utilities for distributions without full support. Because the supports may differ across the distributions one cannot simply discard the zero probability types as in the previous subsection. The main problem that arises is that the best response function V_h will not be defined on all sets if h does not have full support. However, I show that the main theorem, Theorem 1, goes through with little modification.

First I make some notions robust to general distributions. Consider some finite set X, and $f, g \in \Delta X$ such that without loss $X = \text{Supp}(f) \cup \text{Supp}(g)$.[29] Define the $f - g$ likelihood ratio order $\geq_{f/g}$, as

$$x \geq x' \implies f(x)g(x') \geq_{f/g} f(x')g(x).$$

This definition reduces to the one in the main text for full support distributions. It is straightforward to verify that $(X, \geq_{f/g})$ is a completely preordered set.

I next state the revised versions of the results from Section 5 and Appendix A.

Proposition 10. *Let X be a finite set, $r : X \to \mathbb{R}$, and $f \in \Delta X$ with full support. For any distribution*

---

[29] One can disregard elements not in $\text{Supp}(f) \cup \text{Supp}(g)$ by the same logic presented above.

$g \in \Delta X$, there exists an interval partition $P = (P_1, ..., P_m)$ of $(X, \geq_{f/g})$ with,

$$E_f^\varepsilon(P_1) < ... < E_f^r(P_m), \text{ and} \tag{26}$$

$$E_f^r(P_i) \leq E_g^r(P_i) \ \forall i : G(P_i) > 0. \tag{27}$$

Moreover, P is independent of g given $\geq_{f/g}$.

Proposition 11. *Let $f \in \Delta T$. If $S \subset \mathrm{Supp}(f)$ and $S \cap \mathrm{Supp}(g) \neq \emptyset$, then $V_f(S) \leq V_g(S), \forall g : f \geq_{ME} g, S \cap \mathrm{Supp}(g) \neq \emptyset \iff V_f$ is downward biased on $(S, \succeq_d)$.*

Theorem 4. *Let $f, g \in \Delta T$, where $T = \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$. $f \geq_{ME} g \implies$*

$$\pi_f(t|U^R) \leq \pi_g(t|U^R) \ \forall t \in \mathrm{Supp}(f) \cap \mathrm{Supp}(g), \ \forall U^R \in \Upsilon.$$

Moreover, if $\frac{f(t)}{g(t)} < \frac{f(t')}{g(t')}$ for some $t \succeq_d t'$ then $\exists U^R \in \Upsilon$ such that $\pi_f(t'|U^R) > \pi_g(t'|U^R)$.

The only real amendment in the above results is in [Theorem 4]. The reason is that one cannot compare the utility of types that are not in $\mathrm{Supp}(f) \cap \mathrm{Supp}(g)$ simply because $\pi_f(t|U^R)$ does not exist if $f(t) = 0$.

The proofs of the first three results are identical to those of their original versions. The proof of [Theorem 4] is slightly modified.

H. Proofs of Equivalence Between $\geq_{MS}$ and $\geq_{ME}$

H.1. Proof of [Theorem 1] and [Theorem 4]

Proof. $'' \implies ''$

The key observation in proving the theorem is that if $f \geq_{ME} g$ then $Z^g \equiv \{t : g(t) = 0\}$ is an upper contour set of $(T, \succeq_d)$ and $Z^f \equiv \{t : f(t) = 0\}$ is a lower contour set of $(T, \succeq_d)$, i.e. $B(Z_g) = Z_g$ and $W(Z_f) = Z_f$.

Let $P^f = (Z^f, P_1^f, ..., P_m^f)$ and $P^g = (P_1^g, ..., P_l^g, Z^g)$ be the equilibrium partitions under f and g respectively.[30] $\forall t \in \mathrm{Supp}(f) \cap \mathrm{Supp}(g)$, $t \in P_i^f \cap P_j^g$ for some i, j. I will show that $\pi_f(t|U^R) = V_f(P_i^f) \leq V_g(P_j^g) = \pi_g(t|U^R)$ proving the result. Now let $D^g \equiv \cup_{k=1}^j P_k^g$ and let $U^f \equiv \cup_{k=i}^m P_k^f$.

Now consider the set $R \equiv U^f \cap D^g$. This set is the union of disjoint subsets, $R = \cup_{k=i}^m (P_k^f \cap D^g)$. Also whenever non-empty $W(P_k^f \cap D^g) \cap P_k^f = P_k^f \cap D^g$, because $W(D^g) = D^g$ as P^g is an equilibrium partition. Now since P_k^f is poolable under f, [Proposition 1] implies that V_f is downward biased on $(P_k^f, \succeq_d)$. This means that, whenever non-empty, $V_f(P_k^f \cap D^g) \geq V_f(P_k^f)$. Also because P^f is an equilibrium partition, [Proposition 1] implies that $V_f(P_k^f) \geq V_f(P_i^f) \ \forall k \geq i$. Putting these

---

[30] Since I do not use any refinement beyond truth leaning, one can set the off path actions for declarations in Z^g and Z_f arbitrarily to guarantee that this is an equilibrium.

together gives $V_f(P_k^f \cap D^g) \geq V_f(P_i^f) \; \forall k \geq i$. Since R is the union over these disjoint sets, Lemma 1 implies, $V_f(R) \geq V_f(P_i^f)$. Now consider the problem,

$$\max_{\tilde{S} \subset D^g \backslash Z^f} V_f(B(\tilde{S}) \cap (D^g \setminus Z^f)), \tag{28}$$

with corresponding solution $\overline{S}$ with $\overline{R} \equiv B(\overline{S}) \cap (D^g \setminus Z^f)$. Because $B(R) \cap (D^g \setminus Z^f) = R$, $V_f(\overline{R}) \geq V_f(R)$. Moreover $\overline{R} \subset \text{Supp}(f) \cap \text{Supp}(g)$ and so because of Lemma 2, $\overline{R}$ is poolable under f. Using Proposition 11, this means that $V_g(\overline{R}) \geq V_f(\overline{R})$. Now notice that by Proposition 2,

$$V_g(P_j^g) = \max_{\tilde{S} \subset D^g} V_g(B(\tilde{S}) \cap D^g).$$

Thus since $\overline{R}$ is feasible in this problem, by optimality $V_g(P_j^g) \geq V_g(\overline{R})$. Putting this string of inequalities together,

$$V_g(P_j^g) \geq V_g(\overline{R}) \geq V_f(\overline{R}) \geq V_f(R) \geq V_f(P_i^f),$$

proving the result.

$'' \Longleftarrow ''$

Let $A = [\underline{a}, \overline{a}]$. Define $S \equiv M(t) \cap B(t')$, and $\tilde{S} \equiv S \setminus \{t, t'\}$. By assumption, $\{t, t'\} \subset \text{Supp}(f) \cap \text{Supp}(g)$. I prove the case in which $F(\tilde{S}) \geq G(\tilde{S})$; the opposite case is symmetric. Let U^R be quadratic loss, with $v(s) = \overline{a} \; \forall s \notin M(t)$, $v(s) = \underline{a} \; \forall s \in M(t) \setminus S$, $v(s) = \overline{a} \forall s \in \tilde{S} \cup \{t'\}$, and $v(t) = \underline{A}$. The equilibrium partition is clearly $(M(t) \setminus S, S, M(t)^c)$ and prior independent. For any $h \in \Delta T$ with $H(S) > 0$, $V_f(S) = (H(\tilde{S} \cup \{t'\})\overline{a} + h(t)\underline{a})/H(S)$. Thus $V_f(S) > V_g(S)$ if $\frac{f(t)}{F(S \cup \{t'\})} < \frac{g(t)}{G(S \cup \{t'\})}$ which holds by assumption. Thus $\pi_f(t'|U^R) > \pi_g(t'|U^R)$. $\qquad$ *Q.E.D.*

H.2. Proof of Theorem 2

Proof. I only prove the $\Longleftarrow$ direction, as the proof of $\Longrightarrow$ is equivalent to that for Theorem 1.

$'' \Longleftarrow ''$

Take the receiver optimal equilibrium partition under $\succeq_d$, $P = (P_1, ..., P_m)$. I show that P is also the receiver optimal equilibrium partition under $\succeq_{d,v}$. The result then follows from Theorem 1.

Since $\succeq_{d,v}$ is coarser than $\succeq_d$, P remains an interval partition. Thus by Proposition 1 all that remains is to check that V_h is downward biased on $(P_i, \succeq_{d,v})$, $\forall i$.

Suppose not. Take $S \equiv \arg\min_{S' \in P_i} V_h(W_{\succeq_{d,v}}(S'))$, with $V_h(S) < V_h(P_i)$ by assumption. Note that by Lemma 2, V_h is downward biased $(S, \succeq_{d,v})$. Let $R \equiv \{t \in P_i : t \in W(S) \setminus S\}$. $\forall t' \in R$, $t \in B(t') \cap S \implies v(t) \geq v(t')$. This means that $V_h(B(t') \cap S) > v(t')$ and since $B(t') \cap S$ is an upper contour subset of S, $V_h(S) \geq V_h(B(t') \cap S) \geq v(t')$. This establishes that $V_h(R) \leq V_h(S)$ which implies $V_h(W(S)) < V_h(P_i)$ violating that P is the receiver optimal partition under $\succeq_d$. $\qquad$ *Q.E.D.*

I. Strict Comparative Statics Results

Theorem 1 says that receiver optimal equilibrium actions weakly decrease when the sender has more evidence. More evidence is also defined in terms of a weak inequality. In this section I characterize changes in the prior distribution that characterize when receiver optimal equilibrium actions will strictly decrease. I next introduce the *strictly more evidence* relation over prior distributions. For simplicity, assume that all distributions have full support over the type space.

Definition 8. Let $f, g \in \Delta T$. Distribution f has *strictly more evidence* than g ($f \geq_{SME} g$) if

$$t \succeq_d t' \implies \frac{f(t)}{g(t)} > \frac{f(t')}{g(t')}.$$

This definition is not sufficient for receiver optimal equilibrium actions to be strictly lower under f than g. The reason is that some types may completely separate in equilibrium in which case the receiver's best response will be constant across f and g. Thus, the "strict" version of more skepticism can only obtain a strict decrease in actions for types that pool together. Moreover, I incorporate strictness into the definition of pooling.

Definition 9. For the receiver equilibrium partition $P = (P_1, ..., P_m)$, $P' = (P'_1, ..., P'_n)$ is the *strict equilibrium partition* defined by

> 1) P is an interval partition of $(T, \succeq_d)$.
> 2) $V_h(P_i)$ is weakly increasing in i.
> 3) $\forall S \subset P_i : P_i \not\subset W(S), \ V_h(S) > V_h(P_i), \ \forall i$.

P and P' differ in their pooled sets but not in their equilibrium outcomes, and so the strict equilibrium partition is merely a different representation of the same equilibrium. Higher parts in the strict equilibrium partition can have the same action as those in lower parts, while lower contour subsets of strict equilibrium parts merit strictly higher actions. Finding P' is simple given P, as P' is finer than P, and for all $P'_i \subset P_j$, $V_h(P'_i) = V_h(P_j)$.

This notion of pooling may seem less observable than that in the main text because types in different parts can obtain the same action. However, one can distinguish strict equilibrium parts through messaging behavior, as there exists a sender strategy σ behind the strict equilibrium partition has that $t \in P'_i \implies Supp(\sigma_t) \subset P'_i$. That is, the receiver can distinguish the strict equilibrium part from the message that is sent. P' is the finest partition that "represents" the receiver optimal equilibrium. I define the *strictly more skepticism* relation over prior distributions. For any partition P and $t \in T$, let $P(t)$ be the part that contains t.

Definition 10. Let $f, g \in \Delta T$ with receiver optimal strict equilibrium partitions P^f, P^g under U^R. Distribution f induces *strictly more skepticism* than g ($f \geq_{SMS} g$) if $f \geq_{MS} g$ and $P^f(t) \neq \{t\}$ or $P^g(t) \neq \{t\} \implies \pi_f(t|U^R) < \pi_g(t|U^R), \ \forall U^R \in \Upsilon$.

SMS strengthens MS so equilibrium actions are weakly lower for all types under a strictly more skeptical distribution. The addition is that types that pool together in the sense of the the strict equilibrium partition obtain strictly lower actions under the more skeptical distribution.

Theorem 5. *The strictly more skeptical and strictly more evidence relations are equivalent, i.e.*

$$f \geq_{SME} g \iff f \geq_{SMS} g.$$

Proof. " $\implies$ "

Let $f \geq_{SMS} g$. Take $h_\alpha \in \Delta T$ defined by $h_\alpha(t) \equiv \alpha f(t) + (1 - \alpha) g(t)$. The first point of the theorem is guaranteed by Theorem 1. I will prove the case when $P^f(t)$ is not a singleton, and omit the analogous case when $P^g(t)$ is not a singleton.

Claim 2. *For any $\alpha \in (0, 1]$ $V_{h_\alpha}(S) > V_f(S)$ whenever S is not a singleton and satisfies (3) in Definition 9.*

If one can find an α small enough such that the equilibrium strict partition does not change, this completes the proof because $h_\alpha \geq_{MS} g$ by Theorem 1. Since the receiver's best response to subsets is continuous in α, the only occasion when the strict partition does change for any small α is when pooled sets $P_k, P_{k+1}, ..., P_j$ all of the same value join together. Say that for any small α $P_k, P_k + 1, ..., P_j$ pool together. Since P_j is an upper contour subset of this new pooled set and $V_{h_\alpha}(S) > V_f(P_j)$, $V_{h_\alpha}(P_k \cup, ..., \cup P_j) > V_f(P_j)$. Thus proving the claim completes the argument.

Proof of Claim 2: First let $V_h(S)$ be the conditional expectation of $v(t)$ over S and under distribution h. Define the complete order $\geq'_{f/g}$ on S as $t \geq'_{f/g} t'$ if $\frac{f(t)}{g(t)} > \frac{f(t')}{g(t')}$ or $\frac{f(t)}{g(t)} = \frac{f(t')}{g(t')}$ and $v(t) > v(t')$.[31] Perform a slightly modified version of Algorithm 2 where maximal *strictly* decreasing sequences are collected, on $(S, \geq'_{f/g})$.

Let the partition formed at the first stage be $Q_1, ..., Q_m$. For each Q_i that is not a singleton $V_f(Q_i) < V_{h_\alpha}(Q_i)$. This uses the strict version of Remark 4, i.e. the expectation of a strictly decreasing function is strictly lower under a strict monotone likelihood ratio increase in the distribution. The one subtlety is that f strictly likelihood ratio dominates h_α which comes from the construction of $\geq'_{f/g}$. If at lease one Q_i is not a singleton,

$$\begin{aligned} V_{h_\alpha}(S) &= \sum_t v(t) h_\alpha(t) \\ &= \sum_i V_{h_\alpha}(Q_i) h_\alpha(Q_i) \\ &> \sum_i V_f(Q_i) h_\alpha(Q_i). \end{aligned}$$

The algorithm concludes when each maximal strictly decreasing sequence is a singleton, i.e. V_f is weakly increasing on each part. Because $f \geq_{SME} h_\alpha$, $t \geq'_{f/g} t'$ is a completion of $\succeq_d$. Thus there

---

[31] Break other ties arbitrarily.

does not exist a non-trivial interval partition of S such that V_f is weakly increasing on each part by (3) in Definition 9. Moreover since S is not a singleton Q_i must not be a singleton for some i. Thus by the logic of the proof of Proposition 9

$$\sum_i V_f(Q_i) h_\alpha(Q_i) \geq V_f(S).$$

This completes the proof for when the best response is an expectation. For the more general case I apply the logic of the proof of Proposition 3.

$'' \Longleftarrow ''$

Say $f \not\succeq_{SME} g$, i.e. $t' \succeq_d t''$ but $\frac{f(t')}{g(t')} \leq \frac{f(t'')}{g(t'')}$. It is without loss to take t', t'' such that $\nexists s : t' \succeq_d s \succeq_d t''$. Take $U^R(a,t) \equiv -(a - v(t))^2$ with $v(t') = 0, v(t'') = 1, v(t) = 1, \forall t \in B(t'') \setminus \{t\}$, and $v(t) = 0$ otherwise. The strict equilibrium partition is $(\{s_1\}, \{s_2\}, ..., \{t', t''\}, \{s_k\}, ..., \{s_n\})$ under both f and g. However $V_f(\{t', t''\}) \geq V_g(\{t', t''\})$, so $f \not\succeq_{SMS} g$. $\hfill$ Q.E.D.

J. Other Equilibria

This appendix characterizes the conditions for an interval partition $(P_1, ..., P_m)$ to constitute an equilibrium partition without imposing receiver optimality.

J.1. Generally Poolable Sets

Let $(P_1, ..., P_m)$ constitutes an equilibrium partition. For each P_i, there exists a sender strategy that induces the receiver to take the same action following each on-path declaration within P_i. I formalize this definition below.

Definition 11. a subset S is *generally poolable* if $\exists$ mutual best responses $\sigma : S \to \Delta S, \ a^\sigma : S \to A$ such that $a^\sigma(t) = V_h(S), \ \forall t \in \cup_{t \in S} Supp(\sigma_t)$.

I next characterize generally poolable sets in terms of the primitives of the model. Let $W^S \equiv \{t \in S : M(t) \cap S = \{t\}\} S$, be the set of non-dominant types in S. Define the following conditions:

(A) $\forall s \subset S, Min_{t \in B(s)} \ v(t) \leq V_h(S)$.

(B) $\forall W \subset W^S, \ \exists Q \subset B(W) : \ S \setminus Q \subset B(W^S \setminus W) \ and \ V_h(Q) \geq V_h(S)$.

Proposition 12. *A set S is generally poolable $\Longleftrightarrow$ (A) and (B) hold.*

In considering existence of sender strategies that induce pooling, it is without loss to focus on those such that a declarations is on path if and only if it is contained in W^S.

Proof. $'' \Longrightarrow ''$

First I will show that (A) must hold. Say, to the contrary that there exists $t \in S$ such that $V_h(S) < Min_{t \in B(s)} v(t)$. It is without loss to take t such that $B(t) \cap S = \{t\}$. If $t \in \cup_{t \in S} Supp(\sigma_t)$, then

$a^\sigma(t) = v(t) = V_h(S)$ by the fact that σ is pooling and a^σ is a best response. This contradicts the hypothesis. However, if $t \notin \cup_{t \in S} Supp(\sigma_t)$, then $a^\sigma(t)$ must be a best response to some belief over types in $B(t)$, which by Lemma 1 implies that $a^\sigma(t) > V_h(S)$. This means that $\sigma_t(t) = 1$ which contradicts that t is off path.

Now I will show that (B) must hold. Take an arbitrary $W \subset W^S$. Let

$$Q \equiv \{s \in S: \ v(s) \geq V_h(S) \,\&\, Supp(\sigma_s) \cap W \neq \emptyset\} \cup \{s: \ M(s) \subset W\},$$

i.e. the types that declare an element of W and have value greater that $V_h(S)$ combined with the types that can only declare elements of W. It is straightforward that $Q \subset B(W)$. The difference between $V_h(Q)$ and $V_h(S)$ can be seen through the following shifts. The first component of Q weakly increases the probability of types with value greater than $V_h(S)$ relative to the induced distribution under σ. Because σ induces a best response $a^\sigma(t) = V_h(S)$, $\forall s \in W$, this change increases the value. Secondly, the construction of Q removes all types that had value less than $V_h(S)$ and could declare something not in W, which also increases the value. Thus, $V_h(Q) \geq V_h(S)$.

$''\Longleftarrow''$

For any arbitrary feasible $\sigma^i: S \to \Delta W^S$, let the receiver best response be $a^\sigma: W^S \to A$. . Begin with arbitrary σ and do the following iterated process.

Initialize $i = 0$.

Step (1): set $W_i \equiv \arg\min_{s \in S} a^\sigma(s)$. If $W_i = W^S$ then σ is a pooling strategy.

Step (2): Take $s \in B(W_i): \ (v(s) \geq V_h(S) \ and \ \exists t \in W_i: \sigma_s(t) < 1)$. If this is not feasible, then take $s \in B(W_i): \ (v(s) \leq V_h(S) \ and \ \exists t \in W_i: \sigma_s(t) > 0)$ One of these must exist otherwise assumption (B) is violated.

Step (3): In the first (second) case increase (decrease) $\sigma_s(t)$ until either, (i) $\exists t' \neq t: a^\sigma(t) = a^\sigma(t')$, or (ii) $\sigma_s(t) \in \{1, 0\}$ occurs. In both cases, set $i = i + 1$ and return to Step (1).

Let the lowest action at stage i be a_i. By construction a_i is weakly increasing at each stage with strict increases guaranteed at a future stage if $W_i \neq W^S$. Also a_i is bounded above by $V_h(S)$ because of Lemma 1. Thus a_i must converge to $V_h(S)$. The associated σ is a pooling strategy.

Q.E.D.

For completeness I present the characterization of equilibrium with no additional refinement.

Proposition 13. *Let $\pi_h: T \to \mathbb{R}$ with equivalence classes $\cup_s\{\pi_h(s|U^R)\} = \{\pi_1 < ... < \pi_m\}$ and let $P_i \equiv \{s: \pi_h(s|U^R) = \pi_i\}$. π_h is an equilibrium sender payoff vector $\Longleftrightarrow$*

$$P_i \text{ satisfies (A) and (B) } \forall i, \tag{29}$$

$$(P_1, ..., P_m) \text{ is an interval partition of } (T, \succeq_d), \tag{30}$$

$$\text{and } \pi_i = V_h(P_i) \ \forall i. \tag{31}$$