Durable Goods Monopoly with Stochastic Costs

Juan Ortner*
Princeton University

December 20, 2011

JOB MARKET PAPER

Abstract

I study the problem of a durable good monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. When costs do not change over time, the Coase conjecture holds: the monopolist sets an opening price equal to marginal cost and the market outcome is competitive. Time-varying costs modify the results on the Coase conjecture. When the distribution of consumer valuations is discrete, the monopolist is able to exercise market power and the outcome is inefficient. In contrast, with a continuous distribution the monopolist is unable to extract additional surplus from buyers with higher valuations. Moreover, the outcome is efficient in this setting: the monopolist serves consumers sequentially as costs decrease, precisely at the point in time that maximizes total surplus. The model is set up in continuous time and the monopolist's marginal cost evolves as a diffusion process. Continuous time methods lead to a tractable characterization of the equilibrium.

*I am indebted to Faruk Gul, Wolfgang Pesendorfer and Sylvain Chassang for guidance, encouragement and support. I also thank Dilip Abreu, Jan De Loecker, Myrto Kalouptsidi, Stephen Morris, Satoru Takahashi, Ben Brooks, Edoardo Grillo and Leandro Gorno for helpful comments. All remaining errors are my own. Address: Department of Economics, Fisher Hall, Princeton University, Princeton NJ 08544-1021. E-mail: jortner@princeton.edu.

1 Introduction

Consider the problem of a monopolist who produces a durable good and who cannot commit to a path of prices. For settings in which marginal costs do not change over time, Coase (1972) argued that such a producer would not be able to sell at the static monopoly price. After selling the initial quantity, the monopolist has the temptation to reduce prices to reach consumers with lower valuation. This temptation leads the monopolist to continue cutting prices after each sale. Forward looking consumers expect prices to fall, so they are unwilling to pay a high price. Coase conjectured that these forces would lead the monopolist to post an opening price arbitrarily close to marginal cost. The monopolist would then serve the entire market "in the twinkling of an eye", and the outcome would be fully efficient. The classic papers on durable goods monopoly (Stokey, 1981 and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist's opening price converges to the lowest consumer valuation. In the limit, all consumers trade immediately and the monopolist earns the same profits she would get if all consumers had the lowest valuation (zero "excess profits").

The purpose of this paper is to study the problem of a durable good monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. The assumption that marginal costs are subject to stochastic shocks is natural in many markets. For instance, costs may vary over time as a consequence of changes in input prices. Time-varying costs may also arise as a result of changes in productivity. An example of this is high-tech consumer goods, whose costs of production typically fall rapidly over time (e.g., Conlon, 2010). Finally, fluctuations in exchange rates will also lead to time-varying costs if the monopolist sells an imported good, or if she uses imported inputs.

Time-varying costs introduce an option value of delaying trade. The efficient outcome in this setting is that the monopolist serves consumers with valuation $v > 0$ the first time costs fall below a threshold z_v. This threshold is decreasing in the valuation, so under the optimal outcome the monopolist serves consumers sequentially as costs decrease. Selling to all consumers immediately is therefore inconsistent with efficiency in this setting, so at least one of these features of Coase's original conjecture cannot hold.

In this paper, I show that the Coase conjecture fails to hold in its entirety when costs are time-varying *and* the distribution of consumer valuations is discrete. With discrete valuations, the monopolist can truthfully commit to delay trade with low type consumers until costs decrease. This allows the monopolist to extract additional surplus from consumers with higher valuations, enabling her to obtain excess profits. Moreover, the outcome in this setting involves inefficiencies in the form of delayed trade. In contrast, a generalization of Coase's theorem does hold when the distribution of valuations is continuous. In this case, the monopolist has an incentive to serve the next buyer arbitrarily soon after her last sale. This forces the monopolist's profits down to what she would earn if all consumers had the lowest valuation (i.e., zero excess profits), and the outcome is fully efficient. Consumers with higher valuations trade earlier, and end up paying higher prices.

Coase's original arguments illustrate the forces that prevent a monopolist producer of a durable good from exercising market power. The results in this paper show that these forces are more general than what Coase described. In particular, these forces do not rely on serving the entire market immediately, nor on serving every consumer at the same price. To attain efficiency and zero excess profits, it is enough that the monopolist cannot credibly commit to delay trade from one sale to the next.

The model is set up in continuous time and the monopolist's marginal cost x_t evolves as a diffusion process. Continuous time methods are especially suitable to perform the option value calculations that arise with time-varying costs, allowing me to obtain a tractable characterization of the equilibrium. I show that the monopolist's profits solve an ordinary differential equation with appropriate boundary conditions. The model delivers simple expressions for the prices at which buyers are willing to trade as a function of costs, allowing the computation of profit margins as a function of costs and the level of market penetration.

To see how time-varying costs modify the results on the Coase conjecture, consider first a setting with two types of consumers: high types, who value the good at v_H, and low types, who value the good at $v_L < v_H$. Low type consumers buy when the price is weakly lower than v_L. After high types leave the market, the monopolist's problem is to choose when to sell to the remaining low valuation consumers. When costs do not change over time, it is optimal for the monopolist to sell to low types immediately after selling to high types. This is the force behind the Coase conjecture: high valuation consumers are not willing to pay a high price, since they expect prices to fall rapidly after they buy. Time-varying costs give the monopolist the option value of delaying trade with low type buyers until costs decrease. In this case, the monopolist will only sell to low types when costs fall below a threshold $z_L < v_L$. High valuation consumers know that it will take a non-negligible amount of time for prices to drop to v_L when costs are above z_L, so they are willing to pay a higher price. In a sense, time-varying costs provide commitment power to the monopolist.

The equilibrium dynamics with two types of buyers are as follows. If costs are initially larger than a threshold $\overline{x}_0 > z_L$, the monopolist first sells to all high valuation consumers, and then sells to all low type buyers when costs fall below z_L. When costs are initially below a threshold $\underline{x}_0 < z_L$, the monopolist sells immediately to high and low valuation consumers and the market closes. When costs initially lie between $\underline{x}_0$ and $\overline{x}_0$, the monopolist gradually sells to high type buyers and continues to do so until costs either fall below $\underline{x}_t < z_L$ or rise above $\overline{x}_t > z_L$. The cutoffs $\underline{x}_t$ and $\overline{x}_t$ change over time as the level of market penetration increases. When costs fall below $\underline{x}_t$, the monopolist sells to all remaining consumers (high and low types) and the market closes. When costs rise above $\overline{x}_t$, the monopolist sells to all remaining high type buyers, and then sells to low types when costs fall below z_L.

The intuition for the delayed trade when $x_0 \in (\underline{x}_0, \overline{x}_0)$ is as follows. High type consumers expect prices to fall rapidly to v_L after they have all left the market when x_0 lies in this region. Thus, the monopolist would not be able to charge a price significantly larger than v_L if she were to sell to all high type consumers. However, the monopolist has the option value of waiting and obtaining a larger profit margin in the future. The cutoffs $\underline{x}_0$ and $\overline{x}_0$ are such that the monopolist gets a larger payoff by waiting than by selling to all high type consumers immediately when $x_0 \in (\underline{x}_0, \overline{x}_0)$. When x_0 lies within this region, the monopolist gradually sells to high type consumers at a price that leaves her indifferent between selling now or waiting and obtaining a larger margin in the future. High valuation consumers are willing to postpone their purchase since they expect prices to fall at a rate that compensates their cost of delay. The cutoffs $\underline{x}_t$ and $\overline{x}_t$ change over time, since the gains from delaying trade change as the number of high type consumers in the market decreases.

The equilibrium outcome is inefficient when costs initially lie between $\underline{x}_0$ and $\overline{x}_0$. The monopolist serves high type consumers sequentially in this case, but the first-best outcome is that all high valuation consumers trade immediately. In addition, the level of market penetration at each moment in time $s > 0$ depends upon the entire history of costs when x_0 lies in this region. Since prices are a function of costs and market penetration, the prices that the monopolist charges also display history dependence. Finally, the monopolist is able to obtain excess profits, since time-varying costs allow her to extract additional surplus from high valuation consumers. These results generalize to settings in which the distribution of valuations takes any finite number of values.

I study markets in which the distribution of consumer valuations is continuous by analyzing a sequence of models with discrete valuations that approximate the desired continuous distribution. I show that the equilibrium outcome converges to the efficient outcome as the distribution becomes continuous. In the limit, the monopolist serves consumers sequentially as costs decrease, precisely at the point in time that maximizes total surplus. Moreover, the monopolist's profits converge to what she would earn if all consumers had the lowest valuation (i.e., zero excess profits).

To see the intuition behind these results, suppose first that the distribution of valuations is discrete, taking values $v_1 < ... < v_n$. After consumers with valuation v_k leave the market, the monopolist can truthfully commit to delaying trade with consumers with valuation v_{k-1} until costs decrease. This allows the monopolist to charge v_k-consumers a price significantly larger than the price v_{k-1}-consumers are willing to pay. This commitment power disappears as the gap between valuations becomes vanishingly small, since now the monopolist will serve v_{k-1}-consumers arbitrarily soon after serving consumers with valuation v_k. The monopolist is therefore unable to obtain excess profits as the distribution becomes continuous, and the limiting equilibrium outcome is fully efficient.

1.1 Related literature

The literature on durable goods monopoly has identified different ways in which a monopolist can exercise market power. For instance, a durable good monopolist can ameliorate her lack of commitment by renting her good rather than selling it (Bulow, 1982), or by introducing best-price provisions (Butz, 1990). The Coase conjecture also fails when the monopolist faces capacity constraints (Kahn, 1986, and McAfee and Wiseman, 2008), or when consumers use non-stationary strategies (Ausubel and Deneckere, 1989 and Sobel, 1991). The current paper identifies a new setting in which a dynamic monopolist can exercise market power. When marginal costs vary over time and the distribution of valuations is discrete, a monopolist producer of a durable good can commit to delaying trade with low valuation consumers until costs decrease. This allows the monopolist to extract more surplus from consumers with higher valuation, enabling her to obtain excess profits.[1]

This paper also shares some features with models of bargaining with one-sided incomplete information and one-sided offers (Fudenberg, Levine and Tirole, 1985). Deneckere and Liang (2006) study a bargaining game in which the valuation of the buyer is correlated with the cost of the seller (see also Evans, 1989 and Vincent, 1989). They show that trade occurs via atoms in this setting, with short periods of high probability of agreement followed by long periods of inaction. In the current paper's model, trade also occurs via atoms when the distribution of types is discrete. For instance, with two types of buyers the monopolist will first sell to all high types whenever costs are initially large, and will then sell to all low types when costs fall below z_L.

[1] Other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment pricing path of a durable good monopolist when costs evolve deterministically over time. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Biehl (2001) studies a setting in which the buyers' valuations are subject to shocks.

Fuchs and Skrzypacz (2010) study a one-sided incomplete information bargaining model in which a new trader may arrive according to a Poisson process. When a new trader arrives, the seller runs a second price auction between the two potential buyers. Fuchs and Skrzypacz (2010) show that a generalization of the Coase conjecture holds in this setting: the seller's inability to commit to a path of offers drives her profits down to her outside option of waiting for the arrival of a new buyer. Moreover, the possibility of arrivals leads to inefficient delays, with the seller slowly screening out high type buyers. In the current paper, the monopolist is also unable to obtain excess profits when the distribution of valuations is continuous. However, the equilibrium outcome is fully efficient, with the seller serving the different consumers exactly at the point in time that maximizes total surplus.

Finally, this paper adds to the growing literature that uses continuous time methods to analyze strategic interactions.[2] The analysis of games in continuous time presents technical difficulties. First, there are measurability problems related to the fact that players can condition their actions on "instantaneous" events (e.g., Simon and Stinchcombe, 1989). Second, subgame perfection has less bite when the monopolist can change her price in continuous time, leading to a multiplicity of equilibria. The reason for this is that consumers do not face a cost of delay after rejecting a price when the game is in continuous time, since they can always accept a new price within the next instant. Following the recent literature on continuous time games (e.g., Sannikov 2007, 2008), I deal with the first issue by imposing measurability conditions on strategies that guarantee that outcomes and payoffs are well defined. I deal with the second issue by imposing intuitive conditions on strategies that resemble the conditions that would necessarily arise in a subgame perfect equilibrium of a discrete time durable good monopoly game.[3]

2 Model

A monopolist faces a unit measure of non-atomic consumers indexed by $i \in [0,1]$. Consumers are in the market to buy one unit of the monopolist's good. Time is continuous, and consumers can make their purchase at any time $t \in [0,\infty)$. The valuations of the consumers are defined by the non-increasing and left-continuous function $f : [0,1] \to [\underline{v}, \overline{v}]$ with $\overline{v} > \underline{v} > 0$; consumer i has valuation $f(i)$. Consumers and the monopolist are risk-neutral

[2]For instance, continuous time methods have been used to study the provision of incentives in dynamic settings (Sannikov 2007, 2008), political campaigns (Gul and Pesendorfer, 2011) and dynamic markets for lemons (Daley and Green, 2011).

[3]I follow a similar approach in Ortner (2011), where I study a continuous time bilateral bargaining model in which the players' relative bargaining power varies stochastically over time.

expected utility maximizers and discount future payoffs at rate $r > 0$. I assume that f is a step function taking n values $v_1, ..., v_n$, with $0 < v_1 < v_2 ... < v_n$. For $k = 1, .., n$, let $\alpha_k = \max\{i \in [0, 1] : f(i) = v_k\}$. That is, α_k is the highest indexed consumer with valuation v_k. Section 6 considers the case in which f approximates a continuous function h.

Let $B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.[4] The Brownian motion B drives the monopolist's marginal cost x_t,

$$dx_t = \mu x_t dt + \sigma x_t B_t, \tag{1}$$

with $x_0 = x > 0$, $\sigma > 0$ and $|\mu| < r$. At time t the monopolist can produce any desired quantity at marginal cost x_t. The assumption that $|\mu| < r$ guarantees that the monopolist will always produce on demand: under this condition it is never optimal for the monopolist to produce when costs are low to sell in the future when costs are high.[5] The constants μ and σ measure the expected rate of change of x_t and the volatility of x_t, respectively. The process x_t is publicly observable and its underlying structure is common knowledge: monopolist and consumers commonly know that x_t evolves as (1). The assumption that x_t evolves as (1) is for convenience. The main results in this paper continue to hold if x_t follows a more general diffusion process (see Section 7).

A (stationary) strategy for consumer $i \in [0, 1]$ is a function $P : \mathbb{R}_+ \to \mathbb{R}_+$ that describes the maximum price that i is willing to pay for the good given any level of marginal costs. Suppose consumer i is still in the market at time t. Then, under strategy $P(\cdot)$ consumer i purchases the good at time t if and only if the price that the monopolist charges is weakly lower than $P(x_t)$.

Let $\mathbf{P} = P(x, i)$ be a strategy profile for the consumers, with $P(\cdot, i)$ denoting the strategy of consumer $i \in [0, 1]$. In equilibrium, the strategy profile of the consumers must satisfy the *skimming property*: for all $i < j$, $P(x, i) \geq P(x, j)$ for all x. That is, consumers with higher valuation are willing to pay higher prices. The reason for this is that it is more costly for consumers with higher valuation to delay their purchase: if consumers with valuation v_k find it weakly optimal to purchase at some time t given a future path of prices, then consumers with valuation $v_{k'} > v_k$ will find it strictly optimal to purchase at time t. I will restrict attention to strategy profiles such that $P(x, i)$ is left-continuous in i and continuous in x.

The skimming property implies that at any time t there exists $a_t \in [0, 1]$ such that

[4] The filtration $\{\mathcal{F}_t^B : 0 \leq t < \infty\}$ is assumed to include all sets of measure zero, and is therefore right-continuous: for every $t \geq 0$, $\mathcal{F}_t^B = \cap_{s>0} \mathcal{F}_{t+s}^B$.

[5] This assumption also guarantees that the stopping problems in equation (6) have a finite solution.

consumers $i \leq a_t$ have already left the market, while consumers $i > a_t$ are still in the market. The cutoff a_t describes the level of market penetration at time t. At each time t, the level of market penetration a_t and the monopolist's marginal cost x_t describe the payoff relevant state of the game.

Given a strategy profile $\mathbf{P}$, the problem of the monopolist is to choose a path of prices to maximize her profits. Since $\mathbf{P}$ satisfies the skimming property, by setting a price p the monopolist effectively chooses the level of market penetration: if the monopolist sets price p at time t, there will be an $a \in [0,1]$ such that $P(x_t, i) \geq p$ if and only if $i \leq a$. Moreover, the monopolist will charge $P(x_t, a)$ if consumer a is the marginal buyer at time t. Thus, I can alternatively specify the monopolist's problem as choosing a non-decreasing process $\{a_t\}$ with $a_0 = 0$ and $a_t \leq 1$ for all t, describing the level of market penetration at any time t. With this specification, under strategy $\{a_t\}$ the monopolist charges price $P(x_t, a_t)$ at every time t, and at this price all consumers $i \leq a_t$ who are still in the market buy.

Remark 1 With this specification, the process $\{a_t\}$ must satisfy the following condition: suppose $P(x,i) = p$ for all $i \in [l,h] \subseteq [0,1]$ and the monopolist chooses a strategy $\{a_t\}$ such that $da_t > 0$ when $a_{t^-} \in (l,h)$ and $x_t = x$ (i.e., the monopolist makes some sales at state (a,x) with $a \in (l,h)$). Then, in this case it must be that $da_t \geq h - a_{t^-}$: in order to sell at time t with $a_{t^-} \in (l,h)$ and $x_t = x$ the monopolist has to set a price of at most $P(x, a_{t^-})$; and at this price all consumers $i \in [a_{t^-}, h]$ will buy the good. Thus, in this case the level of market penetration $\{a_t\}$ jumps at time t.

Monopolist's problem: Given a strategy profile $\mathbf{P}$ of the consumers, a strategy for the seller is an $\mathcal{F}_t$-progressively measurable process $\{a_t\}$ satisfying the conditions in Remark 1 such that $a_0 = 0$, a_t is non-decreasing with $a_t \leq 1$ for all t, and $\{a_t\}$ is right-continuous with left-hand limits.[6] Let $\mathcal{A}^{\mathbf{P}}$ denote the set of all such processes. Given a strategy profile $\mathbf{P}$ of the consumers and a strategy $\{a_t\} \in \mathcal{A}^{\mathbf{P}}$, the monopolist's discounted profits are[7]

$$\Pi = E\left[\int_{[0,\infty]} e^{-rt} \left(P(x_t, a_t) - x_t\right) da_t\right]. \tag{2}$$

Let $\Pi(x,a)$ denote the monopolist's future discounted profits conditional on the current state being (x,a), and let $\mathcal{A}^{\mathbf{P}}_{a,t}$ denote the set of processes in $\mathcal{A}^{\mathbf{P}}$ such that $a_t = a$. Then, the

[6] These requirements on $\{a_t\}$ together with the continuity requirements on $P(x,i)$ guarantee that the integrals in (2) and (3) are well-defined.

[7] Given the discontinuities in $\{a_t\}$, I use set notation in the integrals to avoid ambiguities: $\int_{(s,T]} f(a_t)\, da_t$ denotes the integral between time s and T, whereas $\int_{[s,T]} f(a_t)\, da_t$ denotes the integral between s^- and T.

monopolist's payoffs conditional on state (x_t, a_t) are

$$\Pi(x_t, a_t) = \sup_{\{a_t\} \in \mathcal{A}^{\mathbf{P}}_{a_t,t}} E\left[\int_{(t,\infty)} e^{-r(s-t)} \left(P(x_s, a_s) - x_s\right) da_s \middle| x_t, a_t\right]. \tag{3}$$

Condition (3) is the requirement that the monopolist's strategy $\{a_t\}$ is subgame perfect (i.e., time-consistent), since $\{a_t\}$ must be optimal at every state (x_t, a_t).

Consumer's problem: Given a strategy of the monopolist $\{a_t\}$ and a strategy profile $\mathbf{P}$ of the consumers, the path of prices is $\{P(x_t, a_t)\}$. The strategy $P(x, i)$ of each consumer i must be optimal given the path of prices $\{P(x_t, a_t)\}$: the payoff that consumer i gets from buying at the time strategy $P(x, i)$ tells her to buy must be weakly larger than what she would get from purchasing at any other point in time.

I impose two additional conditions on the consumers' strategies. First,

$$\forall i \text{ such that } f(i) = v_1,\ P(x; i) = v_1 \text{ for all } x. \tag{4}$$

In words, all consumers with the lowest valuation are willing to pay a price equal to their valuation. The second condition I impose is as follows. Fix a strategy profile $(\mathbf{P}, \{a_t\})$. Recall that for $k = 1, ..., n$, α_k is the highest indexed consumer with valuation v_k. For $k = 1, ..., n$, let $\tau(k)$ denote the (possibly random) time at which the monopolist starts selling to consumers with valuation v_k, i.e., $\tau(k) = \inf\{t : a_t > \alpha_{k-1}\}$. Then, for $k = 2, ..., n$,

$$v_k - P(x_t, \alpha_k) = E\left[e^{-r(\tau(k-1)-t)} \left(v_k - P\left(x_{\tau(k-1)}, a_{\tau(k-1)}\right)\right) \middle| x_t, \alpha_k\right]. \tag{5}$$

Equation (5) is an incentive compatibility condition stating that the price consumer α_k is willing to pay must leave her indifferent between buying at that price or waiting and buying at the price at which consumers with valuation v_{k-1} start buying.

Definition 1 *A strategy profile $(\mathbf{P}, \{a_t\})$ is an equilibrium if:*

(i) $\{a_t\}$ is optimal for all (x_t, a_t) given $\mathbf{P}$,

(ii) For each i, $P(x, i)$ is optimal given $\{a_t\}$ and $\mathbf{P}$, and

(iii) $\mathbf{P}$ satisfies conditions (4) and (5), given $\{a_t\}$.

On games played in continuous time: The analysis of games in continuous time presents technical difficulties. First, there are measurability problems related to the fact that players can condition their actions on "instantaneous" events (e.g., Simon and Stinchcombe, 1989). In this paper, I deal with these issues by restricting consumers to use stationary strategies and by restricting the strategy $\{a_t\}$ of the monopolist to be $\mathcal{F}_t$-progressively measurable. These restrictions guarantee that payoffs and outcomes are well-defined.

Second, in continuous time durable good monopoly games the notion of subgame perfection has less bite, leading to a multiplicity of equilibria. The reason for this is that consumers do not face a cost of delay after they reject a price when the monopolist can change prices in continuous time, since they can always accept a new price within the next instant. To see this, suppose that the game I described so far was in discrete time. In that case, one can easily show that the following two conditions would hold in any SPE: (a) the monopolist would never charge a price below the lowest consumer valuation $v_1 > 0$ (so v_1-consumers would always accept a price of v_1), and (b) the price that the last buyer with valuation v_k is willing to pay leaves her indifferent between trading at that price or delaying trade until the purchase of the next buyer.

In this paper, I directly impose these conditions in the definition of equilibrium; see condition (iii) in Definition 1. When players can take actions in continuous time, there are equilibria that don't satisfy these conditions. For example, in continuous time the strategy profile in which the monopolist always charges a price equal to marginal cost and in which consumers choose optimally the time at which to buy (given that prices will always be equal to x_t) satisfies conditions (i) and (ii) in Definition 1. Under this strategy profile, consumers always reject prices higher than x_t because they expect the monopolist to charge a lower price within the next instant. Against this strategy of the consumers, the monopolist can do no better than to charge a price equal to x_t at all times. Conditions (4) and (5) should be seen as a refinement, which rule out non-intuitive equilibria (like the one in which the monopolist always sets price equal to marginal cost) that violate them.

3 First-best outcome

This section computes the first-best outcome. Recall that the function $f : [0,1] \to [\underline{v}, \overline{v}]$ describing the valuation of the consumers is a step function taking n values $v_1 < v_2 < ... < v_n$. To compute the efficient outcome, consider first the problem of choosing the surplus maximizing time at which to serve a homogeneous group of consumers with valuation v_k,

$$V_k(x) = \sup_{\tau \in T} E\left[e^{-r\tau} (v_k - x_\tau) \right| x_0 = x \Big], \tag{6}$$

where T is the set of stopping times. Let λ be the negative root of $\frac{1}{2}\sigma^2 y(y-1) + \mu y = r$, and for $k = 1, ..., n$ let $z_k := \frac{-\lambda}{1-\lambda} v_k$.

Lemma 1 *The stopping time $\tau_k = \inf\{t : x_t \leq z_k\}$ solves (6). Moreover,*

$$V_k(x) = \begin{cases} (v_k - z_k)\left(\frac{x}{z_k}\right)^{\lambda} & x > z_k, \\ v_k - x & x \leq z_k. \end{cases}$$

Proof. See Appendix A1. ■

Lemma 1 captures the option value that arises when the monopolist's costs vary over time. The total surplus from serving consumers with valuation v_k is maximized by waiting until costs fall below z_k. One can show that $\partial z_k / \partial \mu > 0$ and $\partial z_k / \partial \sigma < 0$, so that it is optimal to wait longer when costs fall faster or when they are more volatile. By Lemma 1, the first-best outcome is that the monopolist serves consumers with valuation v_k at time τ_k. When $x_0 > z_n$, under the optimal outcome the monopolist serves consumers with valuation v_n the first time $x_t = z_n$. After that, the monopolists serves to consumers with valuation v_{n-1} the first time $x_t = z_{n-1}$, and so on. On the other hand, when $x_0 < z_n$ the optimal outcome is that the monopolist sells immediately to all consumers whose valuation v_k is such that $x_0 \leq z_k$. After this initial sale, the monopolist sells to the remaining groups of consumers sequentially as costs decrease.

4 Markets with two-types of consumers

4.1 Equilibrium

In this section, I characterize the equilibrium dynamics for markets with two types of consumers. That is, I consider the case in which

$$f(i) = \begin{cases} v_2 & i \in [0, \alpha], \\ v_1 & i \in (\alpha, 1], \end{cases}$$

with $v_2 > v_1 > 0$ and $\alpha \in (0, 1)$.

By equation (4), consumers with valuation v_1 will only buy when the price equals v_1. That is, $\forall i \in (\alpha, 1]$, $P(x, i) = v_1$ for all x. Let $\Pi(x, \alpha)$ denote the monopolist's profits when the only consumers left in the market are those with valuation v_1 (i.e., when the level of market penetration is α). Since all consumers with valuation v_1 buy at the same instant, at state (x, α) the problem of the monopolist is to optimally choose the time at which to sell to all consumers remaining in the market: $\Pi(x, \alpha) = (1-\alpha) \sup_\tau E[e^{-r\tau}(v_1 - x_\tau) | x_0 = x]$. By Lemma 1, the solution to this problem is $\tau_1 = \inf\{t : x_t \leq z_1\}$, and

$$\Pi(x, \alpha) = \begin{cases} (1-\alpha)(v_1 - z_1)\left(\frac{x}{z_1}\right)^\lambda & x > z_1, \\ (1-\alpha)(v_1 - x) & x \leq z_1. \end{cases} \tag{7}$$

For future reference, note that $\Pi(x, \alpha) \in C^1$ in x.

Consider next the case in which the level of market penetration is $a \in [0, \alpha)$, so there are $\alpha - a$ high valuation consumers remaining in the market. To study equilibrium behavior at these states, I proceed in two steps. First, I establish a lower bound $L(x, a)$ on the monopolist's payoffs for states (x, a) with $a \in [0, \alpha)$. Second, I show that in equilibrium the monopolist's profits are exactly equal to this lower bound $L(x, a)$.

Consider the strategy $P(x, \alpha)$ of consumer α, the highest indexed consumer with valuation v_2. After consumer α buys and leaves the market, the monopolist faces only consumers with valuation v_1. After consumer α makes her purchase, the monopolist will sell to the remaining low valuation consumers when costs fall below the threshold z_1. Therefore, by equation (5), $P(x, \alpha)$ must satisfy

$$P(x, \alpha) = v_2 - E\left[e^{-r\tau_1}(v_2 - v_1)\middle| x_0 = x\right]. \tag{8}$$

That is, for all $x > 0$ consumer α must be indifferent between buying at price $P(x, \alpha)$ or waiting until costs fall below z_1 and obtaining the good at price v_1. Equation (8) highlights the commitment power that time-varying costs provide to the monopolist. When $x_t > z_1$, consumer α knows that prices will not fall to v_1 until costs fall below z_1, so she is willing to pay a price strictly larger than v_1 (see Figure 1 for a plot of $P(x, \alpha)$).

Lemma 2 *$P(x, \alpha) - x > V_1(x)$ for all $x \in (z_1, z_2]$. Moreover,*

$$P(x, \alpha) = \begin{cases} v_2 - (v_2 - v_1)\left(\frac{x}{z_1}\right)^\lambda & x > z_1, \\ v_1 & x \leq z_1. \end{cases} \tag{9}$$

Proof. See Appendix A1. ■

Since the strategy profile of consumers satisfies the skimming property, for all $i < \alpha$,

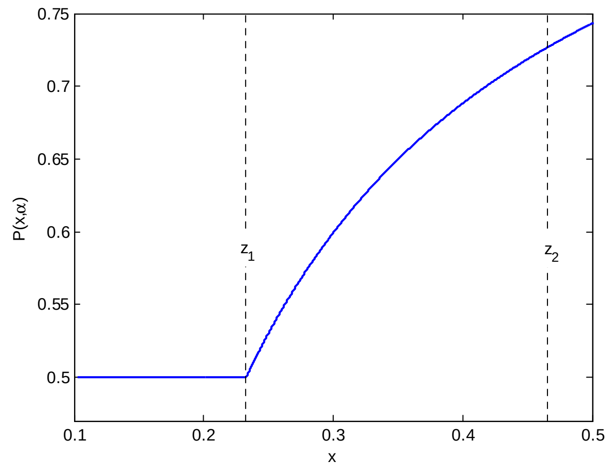

Figure 1: Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

$P(x, i) \geq P(x, \alpha)$ for all x. This implies that, at any time t, the monopolist can sell to all remaining high type buyers at price $P(x_t, \alpha)$. Therefore, for all states (x, a) with $a \in [0, \alpha)$ the monopolist's profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} E\left[e^{-r\tau}\left[(\alpha - a)\left(P(x_\tau, \alpha) - x_\tau\right) + \Pi(x_\tau, \alpha)\right]\middle|\, x_0 = x\right], \tag{10}$$

where $P(x, \alpha)$ and $\Pi(x, \alpha)$ are given by (7) and (9), respectively. That is, at states (x, a) with $a < \alpha$ the monopolist can choose optimally the time τ at which to sell to the remaining high valuation consumers at price $P(x_\tau, \alpha)$, obtaining profits of $(\alpha - a)(P(x_\tau, \alpha) - x_\tau)$ from these sales plus a continuation payoff of $\Pi(x_\tau, \alpha)$.

Lemma 3 *For every $a \in [0, \alpha)$, there exists $\underline{x}(a) \in (0, z_1)$ and $\overline{x}(a) \in (z_1, z_2)$ such that $\tau(a) = \inf\{t : x_t \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]\}$ is a solution to (10). Moreover, $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{a \to \alpha} \underline{x}(a) = \lim_{a \to \alpha} \overline{x}(a) = z_1$.*

Proof. See Appendix A2. ■

To gain intuition behind the solution to (10), let $g(x, a) := (\alpha - a)(P(x, \alpha) - x) + \Pi(x, \alpha)$. This implies that $L(x, a) = \sup_{\tau \in T} E[e^{-r\tau} g(x_\tau, a) |\, x_0 = x]$. Since $P(x, \alpha)$ has a convex kink at z_1 (see Figure 1) and $\Pi(x, \alpha) \in C^1$, $g(x, a)$ also has a convex kink at z_1. Therefore, when $x \in (\underline{x}(a), \overline{x}(a))$ the monopolist can obtain larger profits by delaying trade with high type consumers than by serving all of them at price $P(x, \alpha)$ (see Figure 2). The solution to (10) also involves delaying when costs are above z_2: serving high types is too expensive when

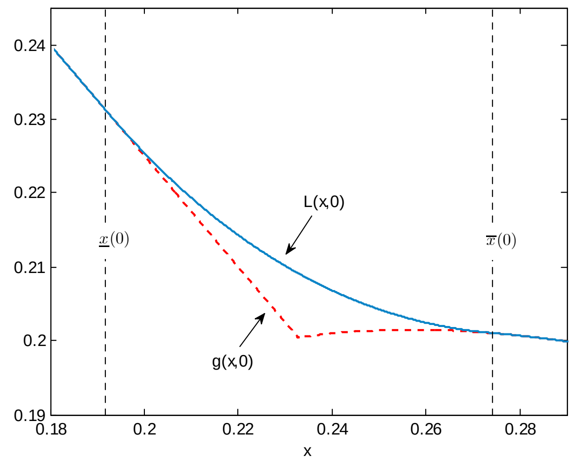

Figure 2: Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

$x > z_2$, so in this case it is optimal to wait for costs to fall.

For all $x \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]$, $L(x,a) = g(x,a)$. The proof of Lemma 3 shows that

$$rL(x,a) = \mu x L_x(x,a) + \frac{1}{2}\sigma^2 L_{xx}(x,a) \text{ for all } x \in (\underline{x}(a), \overline{x}(a)). \tag{11}$$

The general solution to (11) is $L(x,a) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2}\sigma^2 y(y-1) + \mu y = r$, and A and B are constants. There are four unknowns: A and B and the thresholds $\underline{x}(a)$ and $\overline{x}(a)$. The four equations that determine these unknowns are

$$L(\underline{x}(a), a) = g(\underline{x}(a), a), L(\overline{x}(a), a) = g(\overline{x}(a), a), \tag{VM}$$

$$L_x(\underline{x}(a), a) = g_x(\underline{x}(a), a), L_x(\overline{x}(a), a) = g_x(\overline{x}(a), a). \tag{SP}$$

The proof of Lemma 3 shows that there exists a unique solution to this system of equations, with $\underline{x}(a) < z_1 < \overline{x}(a) < z_2$.

The optimal stopping problem (10) is defined for all $a \in [0, \alpha)$. That is, for each $a \in [0, \alpha)$ there are cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ such that the solution to (10) involves stopping the first time $x_t \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]$. Lemma 3 shows that $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{a\to\alpha} \underline{x}(a) = \lim_{a\to\alpha} \overline{x}(a) = z_1$. In words, the delay region $(\underline{x}(a), \overline{x}(a))$ shrinks as a increases, and in the limit as a converges to α it becomes optimal to stop when $x_t \leq z_2$. The reason for this is that the kink that $g(x,a)$ has at z_1 gradually disappears as a increases; and therefore delaying when x is around z_1 becomes less profitable. Intuitively, the gains from delaying trade decrease when there are fewer high type consumers remaining in the market.

Let $(\mathbf{P}, \{a_t\})$ be an equilibrium and let $\Pi(x,a)$ denote the monopolist's profit function. An equilibrium $(\mathbf{P}, \{a_t\})$ is *regular* if $\Pi(x,a)$ is piecewise $C^{2,1}$.

Theorem 1 *There exists a unique regular equilibrium. In this equilibrium, at every state (x,a) with $a \in [0,\alpha)$ the monopolist's profits are $L(x,a)$. Moreover, for all $t \geq 0$*

(i) if $x_t > z_2$, the monopolist doesn't sell (so $da_t = 0$),

(ii) if $x_t \in [\overline{x}(a_t), z_2]$, the monopolist sells to all remaining high type consumers at price $P(x_t, \alpha)$ (so $da_t = \alpha - a_{t-}$),

(iii) if $x_t \leq \underline{x}(a_t)$, the monopolist sells to all remaining consumers (high and low type) at price v_1 (so $da_t = 1 - a_{t-}$),

(iv) while $x_t \in (\underline{x}(a_t), \underline{x}(a_t))$, the monopolist gradually sells to high type consumers at price $P(x_t, a_t) = x_t - L_a(x_t, a_t)$ (so a_t is continuously increasing).

Proof. See Appendix A3. ■

Theorem 1 shows that the monopolist's profits are equal to the lower bound $L(x,a)$ for every state (x,a) with $a \in [0,\alpha)$. When $x_t \in [\overline{x}(a_t), z_2]$, the monopolist sells to all remaining high type buyers at price $P(x_t,\alpha)$, and then sells to low types when costs drop below z_1. When $x_t \leq \underline{x}(a_t)$, the monopolist sells to both low and high type consumers at price v_1 and the market closes. When $x_t > z_2$, the monopolist waits for costs to decrease.

When $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$, it is never optimal for the monopolist to sell to all remaining high type buyers immediately: by doing this the monopolist earns $g(x_t, a_t) < L(x_t, a_t)$. On the other hand, it cannot be an equilibrium for the monopolist to wait until $\tau(a_t) := \inf\{s > t : x_s \notin (\underline{x}(a_t), \overline{x}(a_t))\}$ and sell to all high types at that time. By doing this, the monopolist would earn $E[e^{-r(\tau(a_t)-t)}\left(P\left(x_{\tau(a_t)}, \alpha\right) - x_{\tau(a_t)}\right)\big| x_t]$ on each high type consumer. In this case, for all $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$ the marginal buyer a_t^+ would be willing to buy at a price $P\left(x_t, a_t^+\right) = v_2 - E[e^{-r(\tau(a_t)-t)}\left(v_2 - P\left(x_{\tau(a_t)}, \alpha\right)\right)\big| x_t]$. That is, if the monopolist were to delay sales until $\tau(a_t)$, the marginal buyer a_t^+ would be willing to pay a price $P\left(x_t, a_t^+\right)$ that leaves her indifferent between buying at that price or waiting until time $\tau(a_t)$ and buying at price $P\left(x_{\tau(a_t)}, \alpha\right)$. Note then that, for $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$,

$$\begin{aligned} & P\left(x_t, a_t^+\right) - x_t - E\left[e^{-r(\tau(a_t)-t)}\left(P\left(x_{\tau(a_t)}, \alpha\right) - x_{\tau(a_t)}\right)\big| x_t\right] \\ = \; & v_2 - x_t - E\left[e^{-r(\tau(a_t)-t)}\left(v_2 - x_{\tau(a_t)}\right)\big| x_t\right] > 0, \end{aligned}$$

where the inequality follows from the fact that $v_2 - x = \sup_\tau E[e^{-r\tau}(v_2 - x_\tau)|\, x_0 = x]$ for all $x_t \leq z_2$ (Lemma 1). This implies that it cannot be optimal for the monopolist to delay sales until time $\tau(a_t)$ when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. Therefore, the monopolist must sell gradually to high types when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$.

I now show how to determine the price that the monopolist charges and the rate at which she sells when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. Suppose $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$ and let $\tau = \inf\{s > t : x_s \notin (\underline{x}(a_s), \overline{x}(a_s))\}$. By Theorem 1, $\{a_s\}$ is continuously increasing in s for $s \in [t, \tau)$, so $da_s = \dot{a}_s ds$. At t the monopolist's discounted profits (which by Theorem 1 are $L(x_t, a_t)$) are

$$L(x_t, a_t) = E\left[\int_{(t,\tau]} e^{-r(s-t)}(P(x_s, a_s) - x_s)\dot{a}_s ds + e^{-r(\tau-t)} L(x_\tau, a_\tau)\bigg|\, x_t, a_t\right].$$

By the Law of Iterated Expectations, the process

$$\begin{aligned} Y_t &= \int_{[0,t]} e^{-rs}(P(x_s, a_s) - x_s)\, da_s + e^{-rt} L(x_t, a_t) \\ &= E\left[\int_{[0,\tau]} e^{-rs}(P(x_s, a_s) - x_s)\, da_s + e^{-r\tau} L(x_\tau, a_\tau)\bigg|\, \mathcal{F}_t\right], \end{aligned} \tag{12}$$

is a continuous martingale for all $t < \tau$. By the Martingale Representation Theorem (Karatzas and Shreve, page 182), there exists a progressively measurable process $\beta \in \mathcal{L}^*$ such that $dY_t = e^{-rt}\beta_t dB_t$.[8] Differentiating the left-hand side of (12) with respect to t and using the fact that $dY_t = e^{-rt}\beta_t dB_t$ gives

$$\begin{aligned} dY_t &= e^{-rt}(P(x_t, a_t) - x_t)\dot{a}_t dt - re^{-rt} L(x_t, a_t)\, dt + e^{-rt} dL(x_t, a_t) \Rightarrow \\ dL(x_t, a_t) &= (rL(x_t, a_t) - (P(x_t, a_t) - x_t)\dot{a}_t)\, dt + \beta_t dB_t. \end{aligned}$$

Since $L(x, a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$ (Lemma A5), by Ito's Lemma

$$dL(x_t, a_t) = \left(\mu x_t L_x(x_t, a_t) + \frac{1}{2}\sigma^2 x_t^2 L_{xx}(x_t, a_t)\right) dt + L_a(x_t, a_t)\dot{a}_t dt + \sigma x L_x(x, a)\, dB_t.$$

Combining these two equations,

$$rL(x_t, a_t) = (P(x_t, a_t) - x_t)\dot{a}_t + L_a(x_t, a_t)\dot{a}_t + \mu x_t L_x(x_t, a_t) + \frac{1}{2}\sigma^2 x_t^2 L_{xx}(x_t, a_t). \tag{13}$$

The left-hand side of (13) is the monopolist's expected flow payoff at state (x_t, a_t), while the

[8] A process β belongs to $\mathcal{L}^*$ if $E[\int_0^t \beta_s^2 ds] < \infty$ for all $t \in [0, \infty)$.

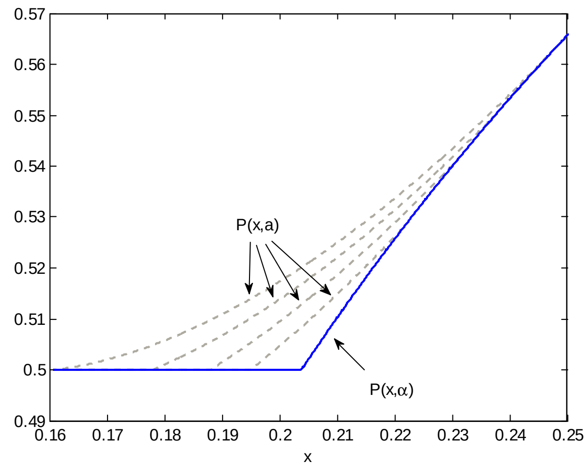

Figure 3: Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

right-hand side shows the sources of this flow payoff. The term $(P(x_t, a_t) - x_t)\dot{a}_t$ represents the flow payoff that the monopolist gets from her sales, while the term $L_a(x_t, a_t)\,\dot{a}_t$ represents the drop in the monopolist's continuation payoff due to the fact that consumers are leaving the market at rate $\dot{a}_t$. Finally, the term $\mu x_t L_x + \frac{1}{2}\sigma^2 x_t^2 L_{xx}$ gives the change in the monopolist's continuation payoff due to changes in marginal cost.

Comparing equations (13) and (11), it follows that

$$P(x_t, a_t) = x_t - L_a(x_t, a_t), \tag{14}$$

for all (x_t, a_t) such that $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. That is, the profit margin $P(x_t, a_t) - x_t$ that the monopolist earns on each sale must be equal to the cost $-L_a(x_t, a_t)$ that she incurs in terms of a lower continuation payoff. Equation (14) has the following interpretation. The monopolist sells at rate $\dot{a}_t > 0$ when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. If $P(x_t, a_t) - x_t > -L_a(x_t, a_t)$, the monopolist could increase her profits by selling at a faster rate. Similarly, if $P(x_t, a_t) - x_t < -L_a(x_t, a_t)$ the monopolist would be better off not selling at all. Therefore, for $\dot{a}_t > 0$ to be optimal, equation (14) must hold for all t such that $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. Since $L(x, a) = E[e^{-r\tau(a)} g(x_{\tau(a)}, a) | \, x_0 = x]$, it follows that $-L_a(x, a) = E[e^{-r\tau(a)} \left(P(x_{\tau(a)}, \alpha) - x_{\tau(a)}\right) | \, x_0 = x]$. Figure 3 plots the price $P(x, a) = x - L_a(x, a)$ that the monopolist charges when $x \in (\underline{x}(a), \overline{x}(a))$, for different values of a.

To close the equilibrium, I need to pin down the rate $\dot{a}_t$ at which the monopolist sells to high valuation consumers when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. In equilibrium, all high valuation consumers must get the same payoff; otherwise, it would be profitable for a consumer who gets a lower payoff to mimic the strategy of one who is getting a larger payoff. Since the monopolist serves high type buyers sequentially while $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$, prices must evolve in such a way that high type consumers are indifferent between purchasing at any time $s \in [t, \tau]$ (where $\tau = \inf\{s > t : x_s \notin (\underline{x}(a_s), \overline{x}(a_s))\}$). That is, for any $s, u \in [t, \tau]$, $s < u$,

$$\begin{aligned} v_2 - P(x_s, a_s) &= E\left[e^{-r(u-s)}(v_2 - P(x_u, a_u)) \middle| x_s, a_s\right] \Rightarrow \\ e^{-rs}(v_2 - P(x_s, a_s)) &= E\left[e^{-ru}(v_2 - P(x_u, a_u)) \middle| x_s, a_s\right]. \end{aligned} \tag{15}$$

By the Law of Iterated Expectations, $M_s := E[e^{-ru}(v_2 - P(x_u, a_u))| x_s, a_s]$ is a continuous martingale. By the Martingale Representation Theorem, there exists a progressively measurable process $\gamma \in \mathcal{L}^*$ such that $dM_s = e^{-rs}\gamma_s dB_s$. Differentiating (15) with respect to s and using $dM_s = e^{-rs}\gamma_s dB_s$, gives

$$\begin{aligned} d\left(e^{-rs}(v_2 - P(x_s, a_s))\right) &= -re^{-rs}(v_2 - P(x_s, a_s))\,ds - e^{-rs}dP(x_s, a_s) = e^{-rs}\gamma_s dB_s \Rightarrow \\ dP(x_s, a_s) &= -r(v_2 - P(x_s, a_s))\,ds - \gamma_s dB_s. \end{aligned} \tag{16}$$

Equation (16) shows that (in expectation) prices must fall at rate $-r(v_2 - P(x_s, a_s))$ in order to maintain high valuation buyers indifferent. By equation (14), $P(x_s, a_s) = x_s - L_a(x_s, a_s)$ for all $s \in [t, \tau)$. The proof of Lemma 3 shows that $L(x, a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$, so $P(x, a) \in C^{2,1}$ for all $x \in (\underline{x}(a), \overline{x}(a))$. Ito's Lemma then implies that for all $s \in [t, \tau]$,

$$dP(x_s, a_s) = \left(\mu x P_x(x_s, a_s) + \frac{1}{2}\sigma^2 x^2 P_{xx}(x_s, a_s) + P_a(x_s, a_s)\dot{a}_s\right) ds + P_x(x_s, a_s)\sigma x dB_s.$$

Combining these two expressions and rearranging gives

$$\dot{a}_s = \frac{-r(v_2 - P(x_s, a_s)) - \mu x P_x(x_s, a_s) - \frac{1}{2}\sigma^2 x^2 P_{xx}(x_s, a_s)}{P_a(x_s, a_s)}.$$

Finally, the proof of Lemma 3 also shows that $L_a(x, a)$ solves

$$rL_a(x, a) = \mu x L_{ax}(x, a) + \frac{1}{2}\sigma^2 x^2 L_{axx}(x, a) \text{ for all } x \in (\underline{x}(a), \overline{x}(a)).$$

Using this together with equation (14) gives

$$\dot{a}_s = -\frac{r(v_2 - x_s) + \mu x_s}{P_a(x_s, a_s)} = \frac{r(v_2 - x_s) + \mu x_s}{L_{aa}(x_s, a_s)} > 0, \tag{17}$$

where the inequality follows from the fact that $L(x,a)$ is strictly convex in a for all $x \in (\underline{x}(a), \overline{x}(a))$ (Lemma A6) and from the fact that $r(v_2 - x) + \mu x > 0$ for all $x < z_2$.[9] Equation (17) gives the rate at which the monopolist sells while $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$.

4.2 Features of the equilibrium

Failure of the Coase conjecture: In his classic paper, Coase (1972) conjectured that a durable good monopolist would be post an initial price arbitrarily close to marginal cost. The monopolist would then serve the entire market "in the twinkling of an eye", and the market outcome would be competitive. The classic papers on durable goods monopoly (Stokey, 1981 and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist's opening price converges to the lowest consumer valuation. In the limit, all consumers trade immediately and the monopolist earns the same profits she would get if all consumers had the lowest valuation.

Time-varying costs introduce an option value of delaying trade. By Lemma 1, the efficient outcome in this setting is that the monopolist serves consumers with valuation v_k the first time costs fall below z_k. This threshold is decreasing in the valuation, so under the optimal outcome the monopolist serves consumers sequentially as costs decrease. Selling to all consumers immediately is therefore inconsistent with efficiency in this setting, so at least one of these features of Coase's original conjecture will not hold.

With time-varying costs, the profits a monopolist would earn if all consumers had the lowest valuation v_1 are $V_1(x) = \sup_\tau E[e^{-r\tau}(v_1 - x_\tau)|\, x_0 = x]$. Say that a monopolist producer of a durable good earns *zero excess profits* if her payoffs are exactly equal to $V_1(x)$. A natural generalization of the Coase conjecture to this paper's setting is that the monopolist earns zero excess profits, and the equilibrium outcome is fully efficient.

This *generalized Coase conjecture* fails to hold when there are two types of consumers in the market. First, the equilibrium is inefficient when $x_0 \in (\underline{x}(0), \overline{x}(0))$. The monopolist sells to high type consumers at a rate given by (17) when costs initially lie within this range, but the efficient outcome is to serve them immediately. When $x_0 \in (\underline{x}(0), \overline{x}(0))$, it is never optimal for the monopolist to sell to all remaining high types. Instead, the monopolist serves high type buyers gradually; and these buyers are willing to postpone their purchases since they expect prices to fall at a rate that compensates their cost of delay.

Second, time-varying costs allow the monopolist to obtain excess profits. By Lemma 2, $P(x,\alpha) - x > V_1(x)$ for all $x \in (z_1, z_2]$, so $L(x,0) \geq g(x,0) > V_1(x)$ for all $x \in (z_1, z_2]$. The

[9] One can check that $rv_2 > z_2(r-\mu)$ whenever $|\mu| < r$. Thus, $rv_2 - x(r-\mu) > 0$ for all $x < z_2$.

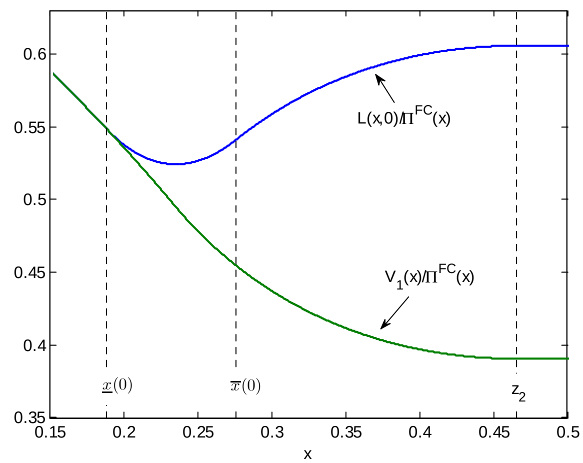

Figure 4: Parameters: $\alpha = 0.7$, $v_1 = 0.5$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

intuition for why the monopolist is able to obtain excess profits is as follows. When marginal costs are fixed, a monopolist lacking commitment power will sell to low type consumers immediately after selling to those with high valuation. This limits the price high valuation buyers are willing to pay, since they expect prices to fall rapidly after they buy. With time-varying costs, the monopolist can truthfully commit to wait and serve low valuation consumers when costs fall below z_1. If x_t is large, high types know that it will take a non-negligible amount of time for prices to drop to v_1, so the monopolist is able to extract more surplus from them.

The monopolist's profits under full commitment are $\Pi^{FC}(x) = \sup_\tau E[e^{-r\tau}\alpha(v_2 - x_\tau)|\, x_0 = x]$ when $\alpha v_2 > v_1$. That is, a monopolist who can commit to a path of prices would find it optimal to sell only to high types (at a price of v_2) when the share of high types is large. High type buyers would be willing to pay a price equal to v_2 in this case, since the monopolist can commit to keep prices above v_2 after they purchase.[10] Figure 4 shows that the monopolist obtains a large fraction of the full commitment profits when costs vary over time.

History dependence: Suppose $x_0 \in (\underline{x}(0), \overline{x}(0))$ and let $\tau = \inf\{t : x_t \notin (\underline{x}(a_t), \overline{x}(a_t))\}$. For all $s \in [0, \tau)$, the rate $\dot{a}_s$ at which the monopolist sells to high valuation consumers at time s depends on the current marginal cost x_s and on the current level of market penetration a_s (see equation 17). Therefore, for all $t \in [0, \tau]$ the level of market penetration $a_t = \int_0^t \dot{a}_s ds$ depends upon the entire path of costs from time zero to t. This implies that the price

[10] Clearly, this strategy of the monopolistis not time-consistent: after selling to high type buyers, it is in the monopolist's best interest to sell to low types when costs fall below z_1

$P(x_t, a_t)$ that the monopolist charges at time t depends upon the path of x_s up to time t. In other words, the price that the monopolist charges at each instant in time $t \in [0, \tau]$ is not Markovian on x_t, but depends upon the entire history of costs.

Upward sloping demand: Suppose $x_0 \in (\underline{x}(0), \overline{x}(0))$, and again let $\tau = \inf\{t : x_t \notin (\underline{x}(a_t), \overline{x}(a_t))\}$. Consider histories in which $x_\tau = \overline{x}(a_\tau)$. At such histories, at time τ all high type buyers remaining in the market buy at a price $P(\overline{x}(a_\tau), a_\tau) = P(\overline{x}(a_\tau), \alpha)$. Since $P(x, a)$ is increasing in x, under such histories a mass of consumers buys at a moment in which prices are actually increasing. If we plotted prices and quantities sold after such histories, we would observe that demand is (locally) upward sloping.

Rate of price changes and costs: The model in this section predicts that prices fall at a faster rate when costs are lower. For $x_t \in [\overline{x}(a_t), z_2]$, the monopolist charges a price $P(x_t, \alpha)$. Applying Ito's Lemma on equation (9) gives

$$dP(x_t, \alpha) = -r(v_2 - P(x_t, \alpha))\, dt + \sigma x P_x(x_t, \alpha)\, dB_t, \tag{18}$$

for all $x \in [\overline{x}(a), z_2]$. Similarly, by equation (16) the drift of $P(x_s, a_s)$ is also $-r(v_2 - P(x_s, a_s))$ for all $x_s \in (\underline{x}(a_s), \overline{x}(a_s))$. Since $P(x, a)$ is strictly increasing in x for all $x \in (\underline{x}(a), \overline{x}(a))$ and since $P(x, \alpha)$ is strictly increasing in x for $x \in [z_1, z_2]$, it follows that prices fall (on average) at a faster rate when marginal costs are lower. The intuition behind this is as follows. Prices must evolve in such a way that high type consumers are indifferent between purchasing at any time. High type consumers get a larger payoff from buying when prices are low (i.e., when costs are low). Therefore, when costs are low, prices need to fall faster to compensate high type consumers for their cost of delay.

Gap vs. no gap: The literature on the Coase conjecture distinguishes two cases: (i) the case in which there is a positive gap between the lowest consumer valuation and the monopolist's marginal cost, and (ii) the case in which this gap is zero. With fixed costs and a positive gap, there is a unique equilibrium, which is stationary and satisfies the Coase conjecture (Gul, Sonnenschein and Wilson, 1986). In the no-gap case, there are also non-stationary equilibria in which the monopolist obtains excess profits (Ausubel and Deneckere, 1989).

With fixed costs and a positive gap, the price that the monopolist charges to high type consumers is increasing in the gap. In this paper's setting, we can think of v_1 as measuring the "gap". Interestingly, with two types of consumers and time-varying costs the price that the monopolist can charge to high type buyers may be increasing or decreasing in the gap.

By equation (9),

$$\frac{\partial P(x,\alpha)}{\partial v_1} = \frac{1}{v_1}\left(v_1(1-\lambda)+\lambda v_2\right)\left(\frac{x}{z_1}\right)^{\lambda} \text{ for } x > z_1.$$

Thus, $\partial P(x,\alpha)/\partial v_1 < 0$ if and only if $v_1 < \frac{-\lambda}{1-\lambda}v_2 = z_2$. That is, the price that high type buyers are willing to pay is decreasing in v_1 for low values of v_1, and its increasing in v_1 for high values of v_1. The price $P(x,\alpha)$ depends on two quantities: the time τ_1 at which the monopolist starts selling to low type consumers, and the price v_1 that the monopolist charges at τ_1. An increase in v_1 affects both of these quantities: it decreases the stopping time τ_1 and it increases the price v_1 the monopolist charges at τ_1. The second effect dominates when v_1 is large, so an increase in v_1 leads to an increase in $P(x,\alpha)$. In contrast, the first effect dominates when v_1 is low, so an increase in v_1 reduces $P(x,\alpha)$.

It follows from equation (9) that $\lim_{v_1\to 0} P(x,\alpha) = v_2$ for all $x > 0$. The monopolist will wait an arbitrarily long time to sell to low type buyers when v_1 is arbitrarily small. In the limit as $v_1 \to 0$, it is as if the market was comprised only of high valuation consumers, so the monopolist can charge them a price of v_2. This implies that, as $v_1 \to 0$, the monopolist's profits converge to the full commitment profits $\Pi^{FC}(x) = \sup_\tau E\left[e^{-r\tau}\alpha(v_2 - x_\tau)\middle| x_0 = x\right]$.

Consumer heterogeneity and prices: The model in this section gives predictions about how the degree of heterogeneity among consumers affects the evolution of prices. Let $m = \alpha v_2 + (1-\alpha)v_1$ be the average valuation, so $v_1 = (m - \alpha v_2)/(1-\alpha)$. By equation (9),

$$P(x,\alpha) = v_2 - \left(v_2 - \frac{m-\alpha v_2}{1-\alpha}\right)\left(\frac{x}{\frac{-\lambda}{1-\lambda}\frac{m-\alpha v_2}{1-\alpha}}\right)^{\lambda} \text{ for } x > z_1. \tag{19}$$

By varying v_2 in equation (19) we can trace how a mean-preserving change in the distribution of valuations affects prices. Suppose $x_0 \geq z_2$, so in equilibrium prices are $P(x_t,\alpha)$ for all $x_t \leq z_2$. By equation (18), the drift of $P(x_t,\alpha)$ is $-r(v_2 - P(x_t,\alpha))$. Equation (19) then implies that the drift of $P(x_t,\alpha)$ is decreasing in v_2 if and only if $\frac{v_2}{v_1} < \frac{1-\alpha\lambda}{-\alpha\lambda}$. Thus, prices will decrease at a faster rate in markets in which the degree of heterogeneity among consumers is larger, provided v_2/v_1 is small.

Comparative statics with respect to σ: Equation (9) implies that $\partial P(x,\alpha)/\partial\sigma > 0$ if and only if $x > z_1 \exp(1/(1-\lambda))$. In words, at high levels of costs the monopolist is able to charge a higher price when the volatility of x_t is larger. A change in σ has two opposing effects on $P(x,\alpha)$. First, a higher σ lowers the threshold z_1 at which the monopolist starts selling to low valuation consumers. Second, a higher σ means that costs will (on average) reach z_1 faster. The second effect dominates when $x \in (z_1, z_1 \exp(1/(1-\lambda)))$, while the first effect dominates when $x > z_1 \exp(1/(1-\lambda))$.

Comparative statics with respect to μ: By equation (9), $\partial P(x,\alpha)/\partial\mu > 0$ if and only if $x > z_1 \exp(1/(1-\lambda))$: at high levels of costs a monopolist charges higher prices in settings in which costs fall at a slower rate. Again, a change in μ has two opposing effects on $P(x,\alpha)$. First, a higher μ raises the threshold z_1 at which the monopolist starts selling to low type consumers. Second, a higher μ means that costs will (on average) take longer to fall to z_1. The first effect dominates when $x \in (z_1, z_1 \exp(1/(1-\lambda)))$, while the second effect dominates when $x > z_1 \exp(1/(1-\lambda))$.

5 Markets with n types of consumers

In this section, I show how the results in Section 4 generalize to settings in which the function $f : [0,1] \to [\underline{v}, \overline{v}]$ describing the valuations of the consumers is a left-continuous, non-increasing step function taking a finite number of values $v_1 < v_2 < ... < v_n$. For $k = 1,..,n$, let $\alpha_k = \max\{i \in [0,1] : f(i) = v_k\}$ denote the highest indexed consumer with valuation v_k, so $f(i) = v_k$ for all $i \in (\alpha_{k+1}, \alpha_k]$. Let $\alpha_{n+1} = 0$.

As a first step towards analyzing this more general setting, note that at any state (x,a) with $a \geq \alpha_3$ there are either one or two types of consumers remaining in the market: consumers with valuation v_1 and consumers with valuation v_2. Thus, for any state (x,a) with $a \geq \alpha_3$ the equilibrium outcome is the one described in Section 4. At states (x,a) with $a \geq \alpha_2$, there are only consumers with valuation v_1 in the market, so the monopolist's profits are $(1-\alpha_2) V_1(x)$. On the other hand, at states (x,a) with $a \in [\alpha_3, \alpha_2)$ there are $\alpha_2 - a$ consumers with valuation v_2 in the market. In this case, there are cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ such that the monopolist sells to all remaining consumers when $x \leq \underline{x}(a)$, and sells to all remaining consumers with valuation v_2 when $x \in [\overline{x}(a_t), z_2]$. When $x \in (\underline{x}(a), \overline{x}(a))$, the monopolist sells gradually to consumers with valuation v_2 at a rate given by equation (17), and when $x_t > z_2$ the monopolist doesn't sell. For states (x,a) with $a \geq \alpha_3$, let $L(x,a)$ denote the (unique) equilibrium profits of the monopolist (derived in Section 4).

Consider next states (x,a) with $a \in [\alpha_4, \alpha_3)$, so that there are $\alpha_3 - a$ consumers with valuation v_3 still in the market. Let $P_2(x) = \sup_{i \in (\alpha_3, \alpha_2]} P(x,i)$ be the highest price that a consumer with valuation v_2 is willing to pay. The analysis in Section 4 implies that $P_2(x) = P(x, \alpha_2)$ for all $x \in [0, \underline{x}(\alpha_3)] \cup [\overline{x}(\alpha_3), \infty)$ (where $P(x,\alpha_2) = v_2 - E[e^{-r\tau_1}(v_2 - v_1) | x_0 = x])$, and $P_2(x) = x - L_a(x, \alpha_3)$ for all $x \in (\underline{x}(\alpha_3), \overline{x}(\alpha_3))$. By equation (5), the strategy $P(x, \alpha_3)$ of consumer α_3 (the highest indexed consumer with valuation v_3) satisfies

$$P(x, \alpha_3) = v_3 - E\left[e^{-r\tau_2}(v_3 - P_2(x_{\tau_2}))\middle| x_0 = x\right],$$

where $\tau_2 = \inf\{t : x_t \leq z_2\}$ is the time at which the monopolist starts selling to consumers with valuation v_2 when the level of market penetration is α_3 (i.e., when all consumers with valuation v_3 have left the market).

By the skimming property, the monopolist can sell to all remaining v_3-consumers at price $P(x, \alpha_3)$. Therefore, at states (x, a) with $a \in [\alpha_4, \alpha_3)$ her profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_3 - a)(P(x_\tau, \alpha_3) - x_\tau) + e^{-r\tau} L(x_\tau, \alpha_3)\right)\middle| x_0 = x\right]. \tag{20}$$

By arguments similar to those in Lemma 3, there exists thresholds $\underline{x}_1(a)$, $\overline{x}_1(a)$, $\underline{x}_2(a)$, $\overline{x}_2(a)$ with $\underline{x}_1(a) < z_1 < \overline{x}_1(a)$ and $\underline{x}_2(a) < z_2 < \overline{x}_2(a) < z_3$ such that

$$\tau(a) = \inf\{t : x_t \in [0, \underline{x}_1(a)] \cup [\overline{x}_1(a), \underline{x}_2(a)] \cup [\overline{x}_2(a), z_3]\},$$

is a solution to the optimal stopping problem (20). That is, the solution to (20) is such that the monopolist sells immediately to all remaining consumers with valuation v_3 at price $P(x, \alpha_3)$ whenever $x_t \in [0, \underline{x}_1(a_t)] \cup [\overline{x}_1(a), \underline{x}_2(a_t)] \cup [\overline{x}_2(a_t), z_3]$. On the other hand, when $x_t \in (\underline{x}_1(a_t), \overline{x}_1(a_t)) \cup (\underline{x}_2(a_t), \overline{x}_2(a_t))$ it is optimal to delay. By arguments similar to those in Section 4, at states (x_t, a_t) with $a_t \in [\alpha_4, \alpha_3)$ and $x_t \in (\underline{x}_1(a_t), \overline{x}_1(a_t)) \cup (\underline{x}_2(a_t), \overline{x}_2(a_t))$ the monopolist sells gradually to v_3-consumers at a price equal to $x_t - L_a(x_t, a_t)$. The rate at which the monopolist sells when costs are in this region can be derived following the steps leading to equation (17). Finally, for $x > z_3$ the monopolist doesn't sell.

Next, consider states (x, a) with $a \in (\alpha_5, \alpha_4]$. At such states there are $\alpha_4 - a$ consumers with valuation v_4 still in the market. Let $P_3(x) = \sup_{i \in (\alpha_4, \alpha_3]} P(x, i)$ be the highest price that a consumer with valuation v_3 is willing to pay. From the arguments above, $P_3(x) = P(x, \alpha_3)$ for $x \in [0, \underline{x}_1(a_t)] \cup [\overline{x}_1(a), \underline{x}_2(a_t)] \cup [\overline{x}_2(a_t), \infty)$, and $P_3(x) = x_t - L_a(x_t, \alpha_4)$ for all $x \in (\underline{x}_1(a_t), \overline{x}_1(a)) \cup (\underline{x}_2(a_t), \overline{x}_2(a_t))$. By equation (5), the strategy $P(x, \alpha_4)$ of consumer α_4 (the highest indexed consumer with valuation v_4) satisfies

$$P(x, \alpha_4) = v_4 - E\left[e^{-r\tau_3}(v_4 - P_3(x_{\tau_3}))\middle| x_0 = x\right],$$

where $\tau_3 = \inf\{t : x_t \leq z_3\}$ is the time at which the monopolist starts selling to consumers with valuation v_3 after all consumers with valuation v_4 have left the market. The skimming property again implies that at states (x, a) with $a \in (\alpha_5, \alpha_4]$ the monopolist can sell to all remaining consumers with valuation v_4 at price $P(x, \alpha_4)$. Therefore, at all such states the monopolist's profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_4 - a)(P(x_\tau, \alpha_4) - x_\tau) + e^{-r\tau} L(x_\tau, \alpha_4)\right)\Big| x_0 = x\right].$$

Continuing in this way, I can extend the function $L(x, a)$ to all $a \in [0, 1]$ in such a way that, for $k = 1, ..., n$ and all $a \in [\alpha_{k+1}, \alpha_k)$,

$$L(x, a) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_k - a)(P(x_\tau, \alpha_k) - x_\tau) + e^{-r\tau} L(x_\tau, \alpha_k)\right)\Big| x_0 = x\right]. \tag{21}$$

Theorem 2 *In any regular equilibrium, the monopolist's profits are $L(x, a)$ at every state (x, a).*

Proof. See Appendix A4. ■

Theorem 2 shows that the results in Theorem 1 extend to the case in which $f : [0, 1] \to [\underline{v}, \overline{v}]$ takes any finite number of values: in this more general setting, the monopolist's equilibrium profits are also equal to the lower bound $L(x, a)$.

At states (x, a) with $a \in [\alpha_{k+1}, \alpha_k)$, consumers with valuation v_{k+1} and higher have already left the market. For these states, the solution to the optimal stopping problem (21) involves delaying when x is around z_1, z_2, ..., or z_{k-1}, and when $x > z_k$. If $x < z_k$ is in the delay region of the optimal stopping problem (21), the monopolist sells gradually to those consumers with valuation v_k (the highest valuation remaining in the market). By arguments similar to those in Section 4, the rate at which the monopolist sells in this case is such that consumers with valuation v_k are indifferent between buying at time t or delaying their purchase; and the price that the monopolist charges at each instant is $P(x, a) = x - L_a(x, a)$.

If $x > z_k$, the monopolist does not sell until costs fall to z_k (and at this point she sells to all v_k-consumers at price $P(z_k, \alpha_k)$). Finally, if x lies in the stopping region of (21), the monopolist sells to all remaining consumers with valuation v_k at price $P(x, \alpha_k)$, and the state moves to (x, α_k). At state (x, α_k), the solution to the the optimal stopping problem (21) involves delaying when x is around z_1, z_2, ..., or z_{k-2}, and when $x > z_{k-1}$. Again, the monopolist sells gradually to consumers with valuation v_{k-1} (the highest remaining buyers in the market) if $x < z_{k-1}$ lies inside the delay region of the optimal stopping problem (21). If $x > z_{k-1}$, the monopolist waits for costs to fall, while if x lies in the stopping region of

(21) the monopolist sells to all remaining consumers with valuation v_{k-1} at price $P(x, \alpha_k)$, and the state moves to (x, α_{k-1}).

The equilibrium of this more general model shares many of the same features of the two type case analyzed in Section 4. The generalized Coase conjecture also fails to hold in this setting. First, the monopolist is able to obtain excess profits. To see this, note that the monopolist can always sell to all consumers with valuation v_2 and higher at price $P(x, \alpha_2)$, obtaining a margin of $P(x, \alpha_2) - x$. By Lemma 2, $P(x, \alpha_2) - x > V_1(x)$ for all $x \in (z_1, z_2]$. Therefore, the monopolist's profits $L(x, a)$ are strictly larger than $(1-a)V_1(x)$ for all $x \in (z_1, z_2]$. More generally, arguments similar to those in Lemma 2 imply that for $k \geq 2$, $P(x, \alpha_k) - x > V_1(x)$ for all $x \in [z_{k-1}, z_k]$. Since the monopolist can sell to all consumers with valuation v_k and higher at a price of $P(x, \alpha_k)$, this implies that $L(x, a) > (1-a)V_1(x)$ for all $x \in [z_{k-1}, z_k]$. Second, the equilibrium outcome also involves inefficiencies in the form of delayed trade. Suppose $x_0 < z_n$ lies inside the delay region of (21). In this case, the efficient outcome is to serve all consumers with valuation v_n immediately, but the monopolist sells to them gradually. In contrast, the outcome is fully efficient when $x_0 \geq z_n$: in this case, for $k = 1, ..., n$ the monopolist serves v_k-consumers at the surplus maximizing time τ_k.

The equilibrium outcome also displays history dependence when costs initially lie inside the delay region of the optimal stopping problem (21), since the rate at which the monopolist sells at each instant depends on the current level of marginal costs. Finally, in this more general model there will also be histories under which a positive mass of consumers buys after an increase in prices. Suppose x_t lies within the delay region of the optimal stopping problem (21). In this case, there will be a threshold $\overline{x}(a_t)$ such that all consumers with the highest valuation in the market buy if x_t increases above $\overline{x}(a_t)$. Since the price that the monopolist charges is increasing in x, if costs raise rapidly above $\overline{x}(a_t)$ all high valuation consumers will buy at a moment in which prices are actually going up.

6 Continuous distributions and the generalized Coase conjecture

In this section, I study markets in which the valuations of the consumers are described by a continuous and strictly decreasing function $h : [0,1] \to [\underline{v}, \overline{v}]$, with $\overline{v} > \underline{v} > 0$. I study this setting by considering a sequence of models with step functions $\{f^n\}$ such that $\sup_{i \in [0,1]} |f^n(i) - h(i)| \to 0$ as $n \to \infty$. For simplicity, I consider approximations $\{f^n\}$ that satisfy the following property: for $n = 2, 3, ...$, f^n is a left-continuous step function taking n

values $v_1^n, ..., v_n^n$, with $v_1^n = \underline{v}$ and for $k = 2, ..., n$, $v_k^n = v_{k-1}^n + (\overline{v} - \underline{v})/(n-1)$.[11]

For $n = 2, 3, ...$, let $L^n(x, a)$ denote the monopolist's profits at state (x, a) in an environment in which the valuations of the consumers are described by f^n. Recall that $V_1(x) = \sup_\tau E[e^{-r\tau}(\underline{v} - x_\tau) | x_0 = x]$ are the profits that the monopolist would earn if all consumers in the market had the lowest valuation $\underline{v} > 0$.

Theorem 3 *For all states $(x, a) \in \mathbb{R}_+ \times [0, 1]$, $\lim_{n\to\infty} L^n(x, a) = (1 - a) V_1(x)$.*

Proof. See Appendix A5. ■

Theorem 3 shows that the monopolist's profits at state (x, a) converge to $(1 - a)V_1(x)$ in the limit as the distribution of consumer valuations becomes continuous. That is, a monopolist producer of a durable good earns zero excess profits when she faces a continuous distribution of consumer valuations.

To see the intuition behind Theorem 3, consider first a setting with two types of consumers: high types, with valuation $\overline{v}$, and low types, with valuation $\underline{v}$. After high types have left the market, the monopolist will sell to low types when costs fall below $\underline{z} = \frac{-\lambda}{1-\lambda}\underline{v}$. In this case, the monopolist can truthfully commit to maintain high prices until costs fall below $\underline{z}$. High type buyers know that prices will fall to $\underline{v}$ only when $x_t \leq \underline{z}$. Thus, when costs are above $\underline{z}$ they are willing to pay higher prices.

Consider next the case in which there are three types of consumers, with valuations $\overline{v}$, $(\overline{v} + \underline{v})/2$ and $\underline{v}$. After consumers with valuation $\overline{v}$ have left the market, the monopolist can only commit to keep prices high until costs fall below $\frac{-\lambda}{1-\lambda}\frac{(\overline{v}+\underline{v})}{2}$. At this point, it becomes optimal for the monopolist to sell to consumers with intermediate valuation $(\overline{v} + \underline{v})/2$. This puts a limit to the price consumers with valuation $\overline{v}$ are willing to pay when $x_t > \frac{-\lambda}{1-\lambda}\frac{(\overline{v}+\underline{v})}{2}$, since now they can wait until costs fall to $\frac{-\lambda}{1-\lambda}\frac{(\overline{v}+\underline{v})}{2}$ and obtain a lower price.

More generally, the proof of Theorem 3 shows that the price consumers are willing to pay monotonically decreases as $n \to \infty$. In the limit as the gap between valuations becomes vanishingly small, the monopolist's profits fall to what she would earn if all consumers had the lowest valuation $\underline{v}$. In other words, the monopolist losses all commitment power when she faces a continuous distribution of valuations, since in this case she always has an incentive to serve the next buyer arbitrarily soon after her last sale (Figure 5 plots the prices consumers are willing to pay for $n = 2$, 3, 4 and 5).

[11] For instance, we can construct such sequence $\{f^n\}$ as follows. For $k = 1, ..., n$, let $\alpha_k^n = \max\{i \in [0, 1] : f^n(i) = v_k^n\}$ be the highest indexed consumer with valuation v_k^n. Thus, $f^n(i) = v_k^n$ for all $i \in (\alpha_{k+1}^n, \alpha_k^n]$. Let f^n be such that $\alpha_1^n = 1$ and, for $k = 2, ..., n$, $\alpha_k^n = (h^{-1}(v_n^{k-1}) + h^{-1}(v_n^k))/2$. One can check that $\sup_i |f^n(i) - h(i)| \to 0$ as $n \to \infty$.

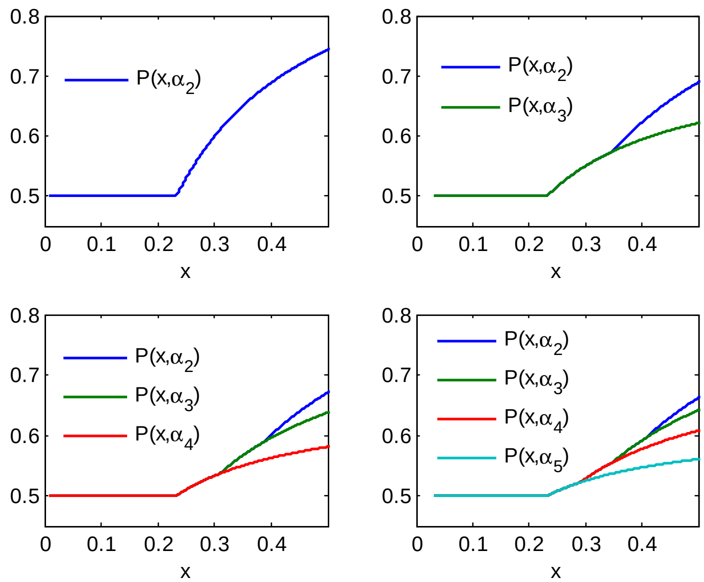

Figure 5: Parameters: $\underline{v} = \frac{1}{2}$, $\overline{v} = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

Corollary 1 *In the limit as $n \to \infty$, the monopolist sells at price $P_t = x_t + V_1(x_t)$ and the equilibrium outcome is fully efficient.*

Corollary 1 and Theorem 1 together imply that the generalized Coase conjecture holds when the distribution of valuations is continuous: the outcome is fully efficient in this case, and the monopolist is unable to obtain excess profits. To see why Corollary 1 must hold, note first that the monopolist can always guarantee herself a profit of $V_1(x_t)$ on every consumer by treating all of them as low types. This implies that the monopolist will never sell at a price below $x_t + V_1(x_t)$: selling at such a price would give her a profit lower than $V_1(x_t)$. On the other hand, the monopolist's profits would be strictly larger than $V_1(x_t)$ if she could sell at prices strictly higher than $x_t + V_1(x_t)$, contradicting Theorem 3.

With a continuous distribution the path of prices is then given by $\{x_t + V_1(x_t)\}$ and the monopolist's profit margin on each sale is $V_1(x_t)$. Given this path of prices, a consumer with valuation $v \in [\underline{v}, \overline{v}]$ chooses optimally when to buy, solving $\sup_\tau E[e^{-r\tau}(v - x_t - V_1(x_t)) | x_0 = x]$. The solution to this stopping problem is $\tau_v = \inf\{t : x_t \leq \frac{-\lambda}{1-\lambda} v\}$: with a continuous distribution, a consumer with valuation v buys at time τ_v. By Lemma 1, τ_v is the surplus maximizing time at which to sell to a consumer with valuation v. Thus, the limiting outcome is fully efficient: the monopolist serves consumers sequentially as cost decreases, precisely at the point in time that maximizes total surplus. Consumers with higher valuations trade earlier, and end up paying higher prices (since $x + V_1(x)$ is strictly increasing in x).

Finally, in this setting we can think of the lowest valuation $\underline{v}$ as measuring the "gap". Note that $V_1(x) = (\underline{v} - \underline{z})(x/\underline{z})^{\lambda} \to 0$ as $\underline{v} \to 0$. Therefore, when the function describing the valuations of the consumers is continuous, the price $P_t = V_1(x_t) + x_t$ at which the monopolist sells her good converges to marginal cost x_t as $\underline{v}$ goes to zero: in the no gap case, the equilibrium outcome is competitive and the monopolist earns zero profits.

7 Conclusion

This paper studies the effect time-varying costs have on the equilibrium dynamics of an otherwise standard durable goods monopoly model. When the distribution of consumer valuations is discrete, time-varying costs provide commitment power to the monopolist. This allows the monopolist to extract more surplus from consumers with higher valuations, modifying the entire equilibrium dynamics. This commitment power disappears when the distribution of valuations is continuous. The monopolist earns zero excess profits in this case, and the equilibrium outcome is fully efficient.

Continuous time methods lead to a tractable characterization of the equilibrium. The model delivers a variety of predictions about how prices and margins relate to the different features of the environment. For instance, the model with two types of buyers predicts that prices fall at a faster rate when the monopolist's costs are lower, and that prices also fall faster in markets in which there is more heterogeneity among consumers. These and other predictions of the model could serve as a benchmark for future empirical work on durable goods pricing.

Throughout the paper, I assumed that costs follow a particular diffusion process. In applications, it might be important to have flexibility regarding the choice of the cost process, especially if we have information about how these costs actually evolve. The main the results of the paper continue to hold if costs follow a more general process of the form[12]

$$dx_t = \mu(x_t)\,dt + \sigma(x_t)\,dB_t. \tag{22}$$

For instance, suppose there are two types of buyers. If costs evolve as (22), we could still compute the lower bound $L(x,a)$ on the monopolist's profits using the procedure of Section 4, and this lower bound would still characterize the monopolist's equilibrium profits.

[12]The coefficients $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ in (22) must satisfy conditions for existence and uniqueness of a strong solution to this stochastic differential equation; see Theorem 5.2.9 in Karatzas and Shreve (1998).

Finally, the paper assumes that the process driving marginal costs is exogenous. However, in many settings firms are able to influence how their costs evolve, for instance, through investments in R&D. One way to incorporate this feature into the model is to assume that the monopolist's investment decisions affect the drift and/or volatility of the costs process. Although incorporating this feature to the model would make the analysis more complex, we could still use the methods put forward in this paper to study the dynamics of prices and sales under this environment.

A Appendix

A.1 Proofs of Lemmas 1 and 2

Let $\tau_y = \inf\{t : x_t \notin (y_1, y_2)\}$ for some $0 < y_1 < y_2$, and let $\tau_{y_1} = \inf\{t : x_t \leq y_1\}$.

Lemma A1 *Let g be a bounded function, and let W be the solution to*

$$rW(x) = \mu x W'(x) + \frac{1}{2}\sigma^2 x^2 W''(x), \tag{A.1}$$

with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. Then, $W(x) = E[e^{-r\tau_y} g(x_{\tau_y}) | x_0 = x]$ for all $x \in (y_1, y_2)$.

Proof. Let W satisfy (A.1), with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. The general solution to (A.1) is $W(x) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2}\sigma^2 q(q-1) + \mu q = r$, and where A and B are constants determined by $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$:

$$A = \frac{g(y_2)\, y_1^\kappa - g(y_1)\, y_2^\kappa}{y_1^\kappa y_2^\lambda - y_1^\lambda y_2^\kappa}, B = -\frac{g(y_2)\, y_1^\lambda - g(y_1)\, y_2^\lambda}{y_1^\kappa y_2^\lambda - y_1^\lambda y_2^\kappa} \tag{A.2}$$

Let $f(x,t) = e^{-rt} W(x)$. By Ito's Lemma, for $x_t \in (y_1, y_2)$

$$\begin{aligned} df(x_t, t) &= e^{-rt}\left(-rW(x_t) + \mu x W'(x_t) + \frac{1}{2}\sigma^2 x^2 W''(x_t)\right) dt + e^{-rt}\sigma x W'(x_t)\, dB_t, \\ &= e^{-rt}\sigma x W'(x_t)\, dB_t, \end{aligned}$$

where the second equality follows from the fact that W solves (A.1). Then,

$$\begin{aligned} E\left[e^{-r\tau_y} g\left(x_{\tau_y}\right)\middle| x_0 = x\right] &= E\left[f\left(x_{\tau_y}, \tau_y\right)\middle| x_0 = x\right] = f(x, 0) + E\left[\int_0^\tau df(x_t, t)\middle| x_0 = x\right] \\ &= W(x) + E\left[\int_0^\tau e^{-rt}\sigma x W'(x_t)\, dB_t\middle| x_0 = x\right] = W(x), \end{aligned}$$

since $\int_0^\tau e^{-rt}\sigma x W'(x_t)\, dB_t$ is a Martingale with expectation zero. ■

Corollary A1 *Let g be a bounded function, and let w be a solution to (A.1) with $w(y_1) = g(y_1)$ and $\lim_{x\to\infty} w(x) = 0$. Then, $w(x) = E[e^{-r\tau_{y_1}} g(x_{\tau_{y_1}}) | x_0 = x]$ for all $x > y_1$. Moreover, $w(x) = g(y_1)(x/y_1)^{\lambda}$ for all $x > y_1$.*

Proof. Let $\tau_{y_1} = \inf\{t : x_t \leq y_1\}$ and note that $\tau_y = \inf\{t : x_t \notin (y_1, y_2)\} \to \tau_{y_1}$ as $y_2 \to \infty$. By monotone convergence,

$$W(x) = E\left[e^{-r\tau_y} g\left(x_{\tau_y}\right) \middle| x_0 = x\right] \underset{\text{as } y_2\to\infty}{\to} E\left[e^{-r\tau_{y_1}} g\left(x_{\tau_{y_1}}\right) \middle| x_0 = x\right] = w(x).$$

By Lemma A1, $W(x) = Ax^{\lambda} + Bx^{\kappa}$ for $x \in (y_1, y_2)$, with A and B satisfying (A.2). Since $\lim_{y_2\to\infty} B = 0$ and $\lim_{y_2\to\infty} A = g(y_1)/y_1^{\lambda}$, $w(x) = \lim y_{2\to\infty} W(x) = g(y_1)(x/y_1)^{\lambda}$. ■

Proof of Lemma 1. Let $V_k(\cdot)$ be as in the statement of the Lemma, and note that $V_k(\cdot) \in C^1$. One can show that $V_k(x) > v_k - x$ for $x > z_k$. Moreover, in this range $V_k(\cdot)$ solves (A.1), with $V_k(z_k) = v_k - z_k$ and $\lim_{x\to\infty} V_k(x) = 0$. By Corollary A1, $V_k(x) = E[e^{-r\tau_k}(v_k - x_{\tau_k}) | x_0 = x]$. One can also show that

$$r(v_k - x) = rV_k(x) > \mu x V_k'(x) + \frac{1}{2}\sigma^2 x^2 V_k''(x) = -\mu x,$$

for all $x \leq z_k$. Therefore, by standard verification results $V_k(\cdot)$ is the solution to (6) (e.g., Theorem 3.17 in Shiryaev, 2008). ■

Remark A1 Since V_k is a solution to the optimal stopping problem (6), then $e^{-rt}V_k(x_t)$ is superharmonic; i.e., $V_k(x) \geq E[e^{-r\tau}V_k(x_\tau) | x_0 = x]$ for any stopping time τ (e.g., Theorem 10.1.9 in Oksendal, 2008). I will use this property repeatedly in what follows.

Proof of Lemma 2. By equation (8) and Lemma 1,

$$\begin{aligned} P(x,\alpha) - x - V_1(x) &= v_2 - x - E\left[e^{-r\tau_1}(v_2 - v_1) \middle| x_0 = x\right] - E\left[e^{-r\tau_1}(v_1 - x_{\tau_1}) \middle| x_0 = x\right] \\ &= v_2 - x - E[e^{-r\tau_1}(v_2 - x_{\tau_1}) | x_0 = x] > 0 \end{aligned}$$

for all $x \in (z_1, z_2]$, since by Lemma 1, $v_2 - x = V_2(x) > E[e^{-r\tau_1}(v_2 - x_{\tau_1}) | x_0 = x]$. ■

A.2 Proof of Lemma 3

I divide the proof of Lemma 3 in a series of Lemmas. Lemmas A2 and A3 give properties of solutions to equation (A.1). Lemma A4 characterizes the solution to the optimal stopping problem (10), while Lemmas A5 and A6 prove some properties of this solution.

Lemma A2 *Let U be a solution to (A.1) with $U(y) = (1-a)(v-y)$ and $U'(y) = -(1-a)$ for some $y \in (0, z_1)$ and $a \in [0, \alpha)$. Then, U is strictly convex for all $x > 0$.*

Proof. The general solution to (A.1) is $U(x) = Ax^{\lambda} + Bx^{\kappa}$. Using the initial conditions,

$$A = y^{-\lambda}(1-a)\frac{\kappa(v_1 - y) + y}{\kappa - \lambda} > 0 \text{ and } B = y^{-\kappa}(1-a)\frac{-(v_1 - y)\lambda - y}{\kappa - \lambda} > 0,$$

where the second inequality follows from the fact that $y < z_1 = -v_1\lambda/(1-\lambda)$. Thus, $U''(x) = \lambda(\lambda-1)Ax^{\lambda-2} + \kappa(\kappa-1)Bx^{\kappa-2} > 0$ for all $x > 0$ (since $\kappa > 1$). ■

Lemma A3 *Let U and $\widetilde{U}$ be two solutions to (A.1). If $\widetilde{U}(y) \geq U(y)$ and $\widetilde{U}'(y) > U'(y)$ for some $y > 0$, then $\widetilde{U}'(x) > U'(x)$ for all $x > y$, and so $\widetilde{U}(x) > U(x)$ for all $x > y$. Similarly, if $\widetilde{U}(y) \leq U(y)$ and $\widetilde{U}'(y) > U'(y)$ for some $y > 0$, then $\widetilde{U}'(x) > U'(x)$ for all $x < y$, and so $\widetilde{U}(x) < U(x)$ for all $x < y$.*

Proof. I prove the first statement of the Lemma. The proof of the second statement is symmetric and omitted. Suppose the claim is not true, and let $y_1 > y$ be the smallest point with $U'(y_1) = \widetilde{U}'(y_1)$. Therefore, $\widetilde{U}'(x) > U'(x)$ for all $x \in [y, y_1)$, so $\widetilde{U}(y_1) > U(y_1)$. Since U and $\widetilde{U}$ solve (A1), then

$$\widetilde{U}''(y_1) = \frac{2(r\widetilde{U}(y_1) - \mu y_1 \widetilde{U}'(y_1))}{\sigma^2 y_1^2} > \frac{2(rU(y_1) - \mu y_1 U'(y_1))}{\sigma^2 y_1^2} = U''(y_1).$$

But this implies that $U'(y_1 - \varepsilon) > \widetilde{U}'(y_1 - \varepsilon)$ for ε small enough, a contradiction. ■

For all $a \in [0, \alpha)$ and all $x > 0$, let $g(x, a) = (\alpha - a)(P(x, \alpha) - x) + \Pi(x, \alpha)$, where $\Pi(x, \alpha)$ and $P(x, \alpha)$ are given by equations (7) and (9). Let $L(x, a) = \sup_\tau E[e^{-r\tau} g(x_\tau, a) | \, x_0 = x]$.

Lemma A4 *For all $a \in [0, \alpha)$, there exists $\underline{x}(a) \in (0, z_1)$ and $\overline{x}(a) \in (z_1, z_2)$ such that $\tau(a) = \inf\{t : x_t \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]\}$ solves (10). Moreover,*

(i) for all $x \in (\underline{x}(a), \overline{x}(a)) \cup (z_2, \infty)$, $L(x, a)$ solves (A.1), with $\lim_{x\to\infty} L(x, a) = 0$.

(ii) for all $x \leq \underline{x}(a)$ and $x \in [\overline{x}(a), z_2]$, $L(x, a) = g(x, a)$.

(iii) the cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ are such that

$$L(\underline{x}(a), a) = g(\underline{x}(a), a), L(\overline{x}(a), a) = g(\overline{x}(a), a), \tag{VM}$$
$$L_x(\underline{x}(a), a) = g_x(\underline{x}(a), a), L_x(\overline{x}(a), a) = g_x(\overline{x}(a), a). \tag{SP}$$

Proof. First I show that there exists a function $L(x, a)$ satisfying conditions (i)-(iii). I start by showing that there exists a function $L(x, a)$ and unique cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ such that $L(x, a)$ solves (A1) on $(\underline{x}(a), \overline{x}(a))$ and satisfies (iii). To see this, consider solutions U to (A.1) with $U(y) = g(y, a) = (1-a)(v_1 - y)$ and $U'(y) = g_x(y, a) = -(1-a)$ for some $y < z_1$. By Lemma A2, such solutions are strictly convex. Since solutions to (A.1) are continuous in initial conditions, then the solutions I'm considering are continuous in y. If y is small enough, then $U(x)$ will remain above $g(x, a)$ for all $x > y$. On the other hand, if y close to z_1 then U will cross $g(x, a)$ at some $\widetilde{x} > y$ (see solutions I-IV in Figure A1). By Lemma A3, the point $\widetilde{x}$ moves to the right as y decreases. Let $\underline{x}(a)$ be the smallest y such that U reaches $g(x, a)$ for some $\overline{x}(a) > y$. Since a solution with $y < \underline{x}(a)$ never reaches $g(x, a)$, it follows that $U(x) \geq g(x, a)$ for all x. Thus, U is tangent to $g(x, a)$ at $\overline{x}(a)$, so

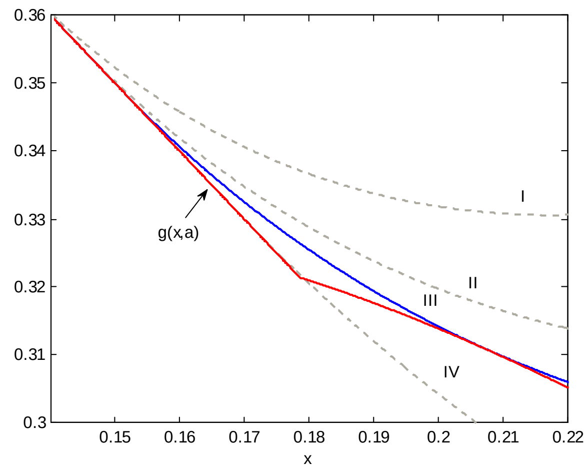

Figure A1: Solutions to equation (A.1)

$U'(\overline{x}(a)) = g_x(\overline{x}(a), a)$ (solution III in Figure A1). Note that U is the unique solution to (A1) that satisfies (VM) and (SP). Hence, $L(x,a) = U(x)$ for $x \in [\underline{x}(a), \overline{x}(a)]$.

By (ii), $L(x,a) = g(x,a)$ for $x \leq \underline{x}(a)$ and $x \in [\overline{x}(a), z_2]$. By (i), $L(\cdot, a)$ solves (A.1) for $x > z_2$, with $\lim_{x\to\infty} L(x,a) = 0$ and $L(z_2, a) = g(z_2, a)$. Corollary A1 then implies that

$$L(x,a) = E[e^{-r\tau_2} g(x_{\tau_2}, a) | x_0 = x] = g(z_2, a)(x/z_2)^\lambda \text{ for all } x > z_2.$$

For future reference, one can check that $L_x(z_2, a) = g_x(z_2, a)$; i.e., L satisfies the smooth pasting condition at z_2. Also, one can check that, for all $x > z_2$, $g(x,a) < g(z_2, a)(x/z_2)^\lambda = E[e^{-r\tau_2} g(x_{\tau_2}, a) | x_0 = x]$.

Let $L(x,a)$ be the (unique) function satisfying conditions (i)-(iii). Then, $L(x,a)$ is twice differentiable in x, with a continuous first derivative. Moreover,

$$-rL(x,a) + \mu x L_x(x,a) + \frac{1}{2}\sigma^2 x^2 L_{xx}(x,a) \leq 0, \text{ with equality on } (\underline{x}(a), \overline{x}(a)) \cup (z_2, \infty). \tag{A.3}$$

Indeed, $L(x,a)$ satisfies (A.3) with equality on $(\underline{x}(a), \overline{x}(a)) \cup (z_2, \infty)$ since it solves (A.1) in this region. One can also check that $rL(x,a) > \mu x L_x(x,a) + \frac{1}{2}\sigma^2 x^2 L_{xx}(x,a)$ for all $x \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]$. By standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008), $L(x,a) = \sup_\tau E[e^{-r\tau} g(x_\tau, a) | x_0 = x]$. By Lemma A1 and Corollary A1, $L(x,a) = E[e^{-r\tau(a)} g(x_{\tau(a)}, a) | x_0 = x]$, so $\tau(a)$ solves (10).

Finally, note that by construction it must be that $\underline{x}(a) < z_1$ and that $\overline{x}(a) > z_1$. I now show that $\overline{x}(a) < z_2$. Suppose by contradiction that $\overline{x}(a) > z_2$. In this case, $L(\overline{x}(a), a) = g(\overline{x}(a), a) < E[e^{-r\tau_2} g(z_2, a) | x_0 = \overline{x}(a)]$, contradicting the fact that $L(\overline{x}(a), a)$ solves the optimal stopping problem (10). Therefore, it must be that $\overline{x}(a) < z_2$. ■

Lemma A5 *$L(x,a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$ and all $a \in [0, \alpha)$. Moreover, $\underline{x}(a)$ and $\overline{x}(a)$ are continuous in a, with $\lim_{a\to\alpha} \underline{x}(a) = \lim_{x\to\alpha} \overline{x}(a) = z_1$.*

Proof. By Lemma A4, $L(x,a) = A(a)x^{\lambda} + B(a)x^{\kappa}$ for all $x \in (\underline{x}(a), \overline{x}(a))$, where $A(a)$, $B(a)$, $\underline{x}(a)$ and $\overline{x}(a)$ are determined by the system of equations (VM) + (SP). Denote this system of equations by $F(\underline{x}(a), \overline{x}(a), A(a), B(a)) = 0$. One can check that $F \in C^2$ and its Jacobian at $(\underline{x}(a), \overline{x}(a), A(a), B(a))$ has a non-zero determinant. By the Implicit Function Theorem, the functions $A(a)$, $B(a)$, $\underline{x}(a)$ and $\overline{x}(a)$ are all C^2 with respect to a (e.g., de la Fuente, 2000, pages 210-211). Since $L(x,a) = A(a)x^{\lambda} + B(a)x^{\kappa}$ for all $x \in (\underline{x}(a), \overline{x}(a))$, this implies that $L(x,a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$.

Next, I show that $\lim_{a\to\alpha} \underline{x}(a) = \lim_{a\to\alpha} \overline{x}(a) = z_1$. Let $\underline{x} := \lim_{a\to\alpha} \underline{x}(a)$ and $\overline{x} := \lim_{a\to\alpha} \overline{x}(a)$. Since $\underline{x}(a) < z_1$ and $\overline{x}(a) > z_1$ for all a (Lemma A4), it follows that $\underline{x} \leq z_1 \leq \overline{x}$. Let $\widehat{\tau} := \inf\{t : x_t \in [0, \underline{x}] \cup [\overline{x}, z_2]\}$, so $\tau(a_n) \to \widehat{\tau}$ for every sequence $\{a_n\} \to \alpha$. Note that $L(x,a) \geq g(x,a) \geq \Pi(x,\alpha) = (1-\alpha)V_1(x)$ for all $a \leq \alpha$, so $\lim_{a\to\alpha} L(x,a) \geq (1-\alpha)V_1(x)$.

Let $\{a_n\} \to \alpha$. Since $\lim_{a\to\alpha} g(x,a) = (1-\alpha)V_1(x)$, by Dominated Convergence

$$L(x,a_n) = E\left[e^{-r\tau(a_n)} g\left(x_{\tau(a_n)}, a_n\right) \middle| x_0 = x\right] \underset{\text{as } n\to\infty}{\rightarrow} E\left[e^{-r\widehat{\tau}}(1-\alpha)V_1(x) \middle| x_0 = x\right],$$

Suppose by contradiction that $\underline{x} < z_1$. Then, for $x \in (\underline{x}, \overline{x})$,

$$\begin{aligned} (1-\alpha)E\left[e^{-r\widehat{\tau}}V_1(x) \middle| x_0 = x\right] &= \begin{array}{l} (1-\alpha)\Pr(x_{\widehat{\tau}} = \underline{x})\, E\left[e^{-r\tau(\underline{x})}(v_1 - \underline{x}) \middle| x_0 = x\right] \\ +(1-\alpha)\Pr(x_{\widehat{\tau}} = \overline{x})\, E\left[e^{-r\tau(\overline{x})}V_1(\overline{x}) \middle| x_0 = x\right] \end{array} \\ &< (1-\alpha)V_1(x), \end{aligned}$$

where the inequality follows from Remark A1 and the fact that $E[e^{-r\tau(\underline{x})}(v_1 - \underline{x})] < V_1(x) = E[e^{-r\tau_1}(v_1 - z_1)]$ (Lemma 1). This contradicts $\lim_{a\to\alpha} L(x,a) \geq (1-\alpha)V_1(x)$, so $\underline{x} = z_1$.

Suppose next that $\overline{x} > z_1$. Let $W(x) = E[e^{-r\widehat{\tau}}(P(x_{\widehat{\tau}}, \alpha) - x_{\widehat{\tau}}) | x_0 = x]$. Since $\underline{x} = z_1$, it follows that $\widehat{\tau} = \inf\{t : x_t \in [0, z_1] \cup [\overline{x}, z_2]\}$. Let $Y_t = e^{-rt}(P(x_t, \alpha) - x_t)$. By Ito's Lemma,

$$dY_t = e^{-rt}\left(\left(-r(P(x_t,\alpha) - x_t) + \mu x_t(P_x(x_t,\alpha) - 1) + \frac{\sigma^2 x_t^2}{2}P_{xx}(x_t,\alpha)\right)dt + \sigma x_t P_x(x_t,\alpha)\,dB_t\right),$$

for all $x_t \in (z_1, \overline{x})$. Equation (9) implies $rP(x,\alpha) = rv_2 + \mu x P_x(x,\alpha) + \frac{\sigma^2 x^2}{2}P_{xx}(x,\alpha)$, so

$$dY_t = e^{-rt}(-r(v_2 - x_t) - \mu x_t)\,dt + e^{-rt}\sigma x_t P_x(x_t,\alpha)\,dB_t.$$

Therefore, for $x \in (z_1, \overline{x})$,

$$W(x) = E\left[Y_{\widehat{\tau}} | x_0 = x\right] = Y_0 + E\left[\int_0^{\widehat{\tau}} e^{-rt}(-r(v_2 - x_t) - \mu x_t)\,dt \middle| x_0 = x\right].$$

One can check that $-r(v_2 - x) < \mu x$ for all $x < \overline{x} < z_2$, so $W(x) < Y_0 = P(x,\alpha) - x$.

For each $a \in [0,\alpha)$, let $W(x,a) = E[e^{-r\tau(a)}\left(P\left(x_{\tau(a)}, \alpha\right) - x_{\tau(a)}\right) | x_0 = x]$. Pick a sequence $\{a_n\} \to \alpha$, and note that $\tau(a_n) \to \widehat{\tau}$ as $n \to \infty$. By dominated Convergence, $W(x,a_n) \to W(x)$ as $n \to \infty$. Fix $x \in (z_1, \overline{x})$. Since $W(x) < P(x,\alpha) - x$, there exists N such that $W(x,a_n) < P(x,\alpha) - x$ for all $n > N$. On the other hand, $E[e^{-r\tau(a_n)}V_1\left(x_{\tau(a_n)}\right) | x_0 =$

$x] \leq V_1(x)$ for all x (see Remark A1). Therefore, for $n > N$

$$\begin{aligned} L(x, a_n) &= E\left[e^{-r\tau(a_n)}\left((\alpha - a_n)\left(P\left(x_{\tau(a_n)}, \alpha\right) - x_{\tau(a_n)}\right) + (1-\alpha) V_1\left(x_{\tau(a_n)}\right)\right)\middle| x_0 = x\right] \\ &< (\alpha - a_n)(P(x, \alpha) - x) + (1-\alpha) V_1(x) = g(x, a_n), \end{aligned}$$

which contradicts the fact that $L(x, a_n) = \sup_\tau E\left[e^{-r\tau} g(x_\tau, a_n)\middle| x_0 = x\right]$. Thus, $\overline{x} = z_1$. ■

Proof of Lemma 3. Follows directly from Lemmas A4 and A5. ■

Lemma A6 *$L(x, a)$ is strictly convex in a for all $x \in (\underline{x}(a), \overline{x}(a))$.*

Proof. I first show that $\underline{x}'(a) > 0$ and $\overline{x}'(a) < 0$. For $a \in [0, \alpha)$, let $W(x, a) = E[e^{-r\tau(a)}(P\left(x_{\tau(a)}, \alpha\right) - x_{\tau(a)})| x_0 = x]$ and $U(x, a) = E[e^{-r\tau(a)} V_1\left(x_{\tau(a)}\right)| x_0 = x]$, so $L(x, a) = (\alpha - a) W(x, a) + (1-\alpha) U(x, a)$. By Lemma A1, $U(x, a)$ solves (A.1) with $U(\underline{x}(a), a) = v_1 - \underline{x}(a) = V_1(\underline{x}(a))$ and $U(\overline{x}(a), a) = (v_1 - z_1)(\overline{x}(a)/z_1)^\lambda = V_1(\overline{x}(a))$. Note that $U_x(\underline{x}(a), a) < -1$ and $U_x(\overline{x}(a), a) > V_1'(\overline{x}(a))$. To see this, note that $V_1(x)$ also solves (A.1) for $x \geq z_1$, with $V_1(z_1) = v_1 - z_1$ and $V_1'(z_1) = -1$. Suppose by contradiction that $U_x(\underline{x}(a), a) \geq -1$. By Lemma A2, U is strictly convex, so $U'(x) > -1$ and $U(x) > v_1 - x$ for all $x > \underline{x}(a)$. Lemma A3 then implies that $U(x, a) > V_1(x)$ for all $x > \underline{x}(a)$, a contradiction to the fact that $U(\overline{x}(a), a) = V_1(\overline{x}(a))$. Hence, $U_x(\underline{x}(a), a) < -1$. Similarly, if $U_x(\overline{x}(a), a) \leq V_1'(\overline{x}(a))$ then by Lemma A3 $U(x, a) > V_1(x)$ for all $x < \overline{x}(a)$, which contradicts $U(\underline{x}(a), a) = V_1(\underline{x}(a))$. Hence, $U_x(\overline{x}(a), a) > V_1'(\overline{x}(a))$. Since $L_x(\underline{x}(a), a) = g_x(\underline{x}(a), a) = -(1-a)$ and $L_x(\overline{x}(a), a) = g_x(\overline{x}(a), a) = (\alpha - a)(P_x(\overline{x}(a), \alpha) - 1) + (1-\alpha)V_1'(\overline{x}(a))$, it follows that $W_x(\underline{x}(a), a) > -1$ and $W_x(\overline{x}(a), a) < P_x(\overline{x}(a), \alpha) - 1$.

Let $a' < a$. The analysis above implies that,

$$\begin{aligned} (\alpha - a') W_x(\underline{x}(a), a) + (1-\alpha) U_x(\underline{x}(a), a) &> g_x(\underline{x}(a), a') \\ (\alpha - a') W_x(\overline{x}(a), a) + (1-\alpha) U_x(\overline{x}(a), a) &< g_x(\overline{x}(a), a'). \end{aligned}$$

Let $F(x) = (\alpha - a') W(x, a) + (1-\alpha) U(x, a)$. Since $W(x, a)$ and $U(x, a)$ both solve (A.1) for all $x \in (\underline{x}(a), \overline{x}(a))$ (Lemma A1), then so does F. Moreover, $F(\underline{x}(a)) = g(\underline{x}(a), a')$ and $F(\overline{x}(a)) = g(\overline{x}(a), a')$. Thus, by Lemma A1 $F(x) = E[e^{-r\tau(a)} g\left(x_{\tau(a)}, a'\right)| x_0 = x]$.

Suppose by contradiction that $\underline{x}(a') \geq \underline{x}(a)$. Let H be the solution to (A.1) with $H(\underline{x}(a)) = g(\underline{x}(a), a') = (1-a')(v_1 - \underline{x}(a))$ and $H'(\underline{x}(a)) = g_x(\underline{x}(a), a') = -(1-a')$. By Lemma A2, H is strictly convex, so $H'(x) > -(1-a')$ for all $x > \underline{x}(a)$. Since F solves (A.1) with $F(\underline{x}(a)) = g(\underline{x}(a), a')$ and $F'(\underline{x}(a)) > g_x(\underline{x}(a), a')$, it follows by Lemma A3 that $F'(x) > H'(x) = -(1-a')$ for all $x \geq \underline{x}(a)$ and $F(x) > H(x) > g(x, a')$ for all $x \in (\underline{x}(a), z_1)$. By Lemma A4, $L(x, a)$ solves (A.1) on $(\underline{x}(a'), \overline{x}(a'))$, with $L(\underline{x}(a'), a') = g(\underline{x}(a'), a')$ and $L_x(\underline{x}(a'), a') = g_x(\underline{x}(a'), a')$. The arguments above together with Lemma A3 then imply that $F(x) > L(x, a')$ for all $x > \underline{x}(a')$, a contradiction to the fact that $L(x, a') = \sup_\tau E\left[e^{-r\tau} g(x_\tau, a')\middle| x_0 = x\right]$. Thus, it must be that $\underline{x}(a') < \underline{x}(a)$.

Similarly, suppose that $\overline{x}(a') \leq \overline{x}(a)$. By a symmetric argument, one can show that $L(\overline{x}(a'), a') = g(\overline{x}(a'), a') \leq F(\overline{x}(a'))$ and $L_x(\overline{x}(a'), a') = g_x(\overline{x}(a'), a') > F_x(\overline{x}(a'))$.

Lemma A3 then implies that $F(x) > L(x, a')$ for all $x < \overline{x}(a')$, contradicting the fact that $L(x, a') = \sup_\tau E\left[e^{-r\tau} g(x_\tau, a') \middle| x_0 = x\right]$. Thus, it must be that $\overline{x}(a') > \overline{x}(a)$.

Finally, I show that $L(x, a)$ is strictly convex in a for all $x \in (\underline{x}(a), \overline{x}(a))$. Take $a' < a < \alpha$, and let $a^\gamma = \gamma a + (1-\gamma)a'$ for some $\gamma \in (0,1)$. Note that $g(x, a^\gamma) = \gamma g(x, a) + (1-\gamma) g(x, a')$. Moreover, $\underline{x}(a') < \underline{x}(a^\gamma) < \underline{x}(a)$ and $\overline{x}(a') > \overline{x}(a^\gamma) > \overline{x}(a)$. Therefore,

$$\begin{aligned} L(x, a^\gamma) &= \gamma E\left[e^{-r\tau(a^\gamma)} g(x_{\tau(a^\gamma)}, a) \middle| x_0 = x\right] + (1-\gamma) E\left[e^{-r\tau(a^\gamma)} g(x_{\tau(a^\gamma)}, a') \middle| x_0 = x\right] \\ &< \gamma L(x, a) + (1-\gamma) L(x, a'), \end{aligned}$$

for all $x \in (\underline{x}(a), \overline{x}(a))$, so $L(x, a)$ is strictly convex in a on $(\underline{x}(a), \overline{x}(a))$. ■

A.3 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. Lemmas A7 and A8 establish conditions that hold in any regular equilibrium. Using these conditions, Lemma A9 provides a partial characterization of the monopolist's equilibrium payoff. Finally, Lemma A10 establishes that in any regular equilibrium the monopolist's payoff are equal to $L(x, a)$.

Lemma A7 *Let $(\{a_t\}, \mathbf{P})$ be an equilibrium. Then,*

(i) for all t such that $x_t < z_2$ and $a_t < \alpha$, the monopolist always sells (i.e., $da_t > 0$),

(ii) for all t such that $x_t > z_2$ and $a_t < \alpha$, the monopolist doesn't sell (i.e., $da_t = 0$).

Proof. (i) Suppose that $a_t < \alpha$ and $da_t = 0$ while $x_t < z_2$. Let $\widetilde{\tau} = \inf\{s > t : da_s > 0\}$, so $\widetilde{\tau} > 0$. In this case, the price at which the marginal buyer a_t^+ is willing to buy satisfies

$$P\left(x_t, a_t^+\right) = v_2 - E\left[e^{-r(\widetilde{\tau}-t)}\left(v_2 - P\left(x_{\widetilde{\tau}}, a_{\widetilde{\tau}}\right)\right) \middle| x_t, a_t\right].$$

That is, at time t the marginal buyer a_t^+ is willing to pay a price that leaves her indifferent between buying at that price, or waiting until $\widetilde{\tau}$ and getting the good at price $P(x_{\widetilde{\tau}}, a_{\widetilde{\tau}})$. The monopolist gets a profit of $E[e^{-r(\widetilde{\tau}-t)}(P(x_{\widetilde{\tau}}, a_{\widetilde{\tau}}) - x_{\widetilde{\tau}}) | x_t, a_t]$ from selling to consumer a_t^+ at time $\widetilde{\tau}$. The monopolist gets $P\left(x_t, a_t^+\right) - x_t$ from selling to a_t^+ at time t. Note that,

$$\begin{aligned} & P\left(x_t, a_t^+\right) - x_t - E[e^{-r(\widetilde{\tau}-t)}(P(x_{\widetilde{\tau}}, a_{\widetilde{\tau}}) - x_{\widetilde{\tau}}) | x_t, a_t] \\ = \; & v_2 - x_t - E[e^{-r(\widetilde{\tau}-t)}(v_2 - x_{\widetilde{\tau}}) | x_t, a_t] > 0, \end{aligned}$$

where the last inequality follows from the fact that $v_2 - x = \sup_\tau E[e^{-r\tau}(v_2 - x_\tau) | x_0 = x]$ for all $x \leq z_2$. Thus, the monopolist is better off selling to consumer a_t^+ at t, a contradiction to the assumption that $(\{a_t\}, \mathbf{P})$ is an equilibrium.

(ii) Suppose the monopolist sells while $x_t > z_2$. Let $\tau(\alpha)$ denote the time at which consumer α buys and recall that $\tau_2 = \inf\{t : x_t \leq z_2\}$. Let $\tau = \min\{\tau(\alpha), \tau_2\}$. Since all high valuation consumers must get the same payoff in equilibrium, the price the monopolist charges at any time $s \in [t, \tau]$ must be such that

$$P(x_s, a_s) = v_2 - E\left[e^{-r(\tau - s)}(v_2 - P(x_\tau, a_\tau)) \middle| x_s, a_s\right].$$

If $\tau(\alpha) < \tau_2$, then $a_\tau = \alpha$. After time $\tau(\alpha)$, the monopolist sells to low type consumers when costs fall below z_1. Recall that $\tau_1 = \inf\{t : x_t \leq z_1\}$. By equation (5), it follows that

$$P(x_\tau, \alpha) = v_2 - E\left[e^{-r(\tau_1 - \tau)}(v_2 - v_1) \middle| x_\tau\right] = v_2 - E\left[e^{-r(\tau_2 - \tau)}(v_2 - P(x_{\tau_2}, \alpha)) \middle| x_\tau\right],$$

since $P(x_{\tau_2}, \alpha) = v_2 - E\left[e^{-r(\tau_1 - \tau_2)}(v_2 - v_1) \middle| x_{\tau_2}\right]$. By the law of iterated expectations,

$$P(x_s, a_s) = v_2 - E\left[e^{-r(\tau_2 - s)}(v_2 - P(x_{\tau_2}, \alpha)) \middle| x_s, a_s\right] = v_2 - E\left[e^{-r(\tau_2 - s)}(v_2 - P(x_{\tau_2}, a_{\tau_2})) \middle| x_s, a_s\right],$$

for $s < \tau$, since in this case $a_{\tau_2} = \alpha$. On the other hand, if $\tau(\alpha) \geq \tau_2$, then

$$P(x_s, a_s) = v_2 - E\left[e^{-r(\tau_2 - s)}(v_2 - P(x_{\tau_2}, a_{\tau_2})) \middle| x_s, a_s\right].$$

The profits that the monopolist gets from selling to high valuation consumers between time t and τ are $E_t[e^{-r(s-t)} \int_t^\tau (P(x_s, a_s) - x_s) da_s]$. If instead the monopolist waits until time τ_2 and sells to all consumers $i \in [a_t, a_{\tau_2}]$ at that instant, her profits are $E_t[e^{-r(\tau_2 - t)}(P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2})(a_{\tau_2} - a_t)]$. Note that for all $s \in [t, \tau_2)$

$$\begin{aligned}
& P(x_s, a_s) - x_s - E_s\left[e^{-r(\tau_2 - s)}(P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2})\right] \\
= \; & v_2 - E_s\left[e^{-r(\tau_2 - s)}(v_2 - P(x_{\tau_2}, a_{\tau_2}))\right] - x_s - E_s\left[e^{-r(\tau_2 - s)}(P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2})\right] \\
= \; & v_2 - x_s - E_s\left[e^{-r(\tau_2 - s)}(v_2 - x_{\tau_2})\right] < 0,
\end{aligned}$$

since τ_2 solves $\sup_\tau E\left[e^{-r\tau}(v_2 - x_\tau)\right]$. Hence, the monopolist is better off by delaying sales until time τ_2, contradiction the fact that $(\{a_t\}, \mathbf{P})$ is an equilibrium. ■

Lemma A8 *Let $(\{a_t\}, \mathbf{P})$ be a regular equilibrium and let $\Pi(x, a)$ denote the monopolist's equilibrium profits. If a_s is continuously increasing in $s \in [t, \tau]$ for some $\tau > 0$, then $P(x_s, a_s) = x_s - \Pi_a(x_s, a_s)$ for all $s \in [t, \tau]$.*

Proof. Let $(\{a_t\}, \mathbf{P})$ be a regular equilibrium. Suppose a_s is continuously increasing for $s \in [t, \tau]$, with $\dot{a}_s = da_s/ds$. Then,

$$\Pi(x_t, a_t) = E_t\left[\int_{(t,\tau]} e^{-r(s-t)}(P(x_s, a_s) - x_s)\dot{a}_s ds + e^{-r(\tau - t)}\Pi(x_\tau, a_\tau)\right].$$

By the Law of iterated expectations, the process

$$\begin{aligned}
Y_t &= \int_{[0,t]} e^{-rs}(P(x_s, a_s) - x_s) da_s + e^{-rt}\Pi(x_t, a_t) \qquad \text{(A.4)} \\
&= E\left[\int_{[0,\tau]} e^{-rs}(P(x_s, a_s) - x_s) da_s + e^{-r\tau}\Pi(x_\tau, a_\tau)\right],
\end{aligned}$$

is a martingale. The Martingale Representation Theorem implies that there exists a process $\beta_t \in \mathcal{L}^*$ such that $dY_t = e^{-rt}\beta_t dB_t$. Differentiating (A.4) with respect to t yields

$$\begin{aligned} dY_t &= e^{-rt}\left(P\left(x_s, a_s\right) - x_s\right)\dot{a}_t dt - re^{-rt}\Pi\left(x_t, a_t\right)dt + e^{-rt}d\Pi\left(x_t, a_t\right) \Rightarrow \\ d\Pi\left(x_t, a_t\right) &= r\Pi\left(x_t, a_t\right)dt - \left(P\left(x_t, a_t\right) - x_t\right)\dot{a}_t dt + \beta_t dB_t. \end{aligned}$$

One the other hand, since $\Pi \in C^{2,1}$ Ito's Lemma implies that

$$d\Pi\left(x_t, a_t\right) = \left(\mu x_t \Pi_x\left(x_t, a_t\right) + \frac{1}{2}\sigma^2 x_t^2 \Pi_{xx}\left(x_t, a_t\right)\right)dt + \Pi_a\left(x_t, a_t\right)\dot{a}_t dt + \sigma x_t \Pi_x\left(x_t, a_t\right)dB_t.$$

Combining these two equations gives

$$r\Pi\left(x_t, a_t\right) = \mu x_t \Pi_x\left(x_t, a_t\right) + \frac{1}{2}\sigma^2 x^2 \Pi_{xx}\left(x_t, a_t\right) + \left(P\left(x_t, a_t\right) - x_t + \Pi_a\left(x_t, a_t\right)\right)\dot{a}_t. \quad \text{(A.5)}$$

Suppose that $P\left(x_s, a_s\right) \neq x_s - \Pi_a\left(x_s, a_s\right)$ on a set of positive measure in $s \in [t, \tau]$, and let $\{b_s\} \in \mathcal{A}^{\mathbf{P}}_{a_t,t}$ be a process such that $\dot{b}_s = \dot{a}_s$ for all s such that $P\left(x_s, a_s\right) = x_s - \Pi_a\left(x_s, a_s\right)$, and $\dot{b}_s > \dot{a}_s$ ($\dot{b}_s < \dot{a}_s$) for all s such that $P\left(x_s, a_s\right) > x_s - \Pi_a\left(x_s, a_s\right)$ ($P\left(x_s, a_s\right) < x_s - \Pi_a\left(x_s, a_s\right)$). Let U_t denote the monopolist's profits from following strategy $\{b_s\}$ on $s \in [t, \tau]$, so

$$U_t = E_t\left[\int_t^\tau e^{-r(s-t)}\left(P\left(x_s, b_s\right) - x_s\right)\dot{b}_s ds + e^{-r(\tau-t)}\Pi\left(x_\tau, b_\tau\right)\right]. \quad \text{(A.6)}$$

By Ito's Lemma, under process $\{b_s\}$

$$de^{-r(s-t)}\Pi\left(x_s, b_s\right) = e^{-r(s-t)}\left(\begin{array}{c} \left(-r\Pi\left(x_s, b_s\right) + \mu x_s \Pi_x\left(x_s, b_s\right) + \frac{1}{2}\sigma^2 x_s^2 \Pi_{xx}\left(x_s, b_s\right)\right)ds \\ +\Pi_a\left(x_s, b_s\right)\dot{b}_s ds + \sigma x_s \Pi_x\left(x_s, b_s\right)dB_s \end{array}\right).$$

Therefore,

$$E_t\left[e^{-r(\tau-t)}\Pi\left(x_\tau, b_\tau\right)\right] = \Pi\left(x_t, a_t\right) + E_t\left[\int_t^\tau e^{-r(s-t)}\left(\begin{array}{c} -r\Pi\left(x_s, b_s\right) + \mu x_s \Pi_x\left(x_s, b_s\right) + \\ \frac{1}{2}\sigma^2 x_s^2 \Pi_{xx}\left(x_s, b_s\right) + \Pi_a\left(x_s, b_s\right)\dot{b}_s \end{array}\right)ds\right].$$

Since $P\left(x_s, a_s\right) \neq x_s - \Pi_a\left(x_s, a_s\right)$ on a set of positive measure in $s \in [t, \tau]$, the equation above together with (A.6) gives

$$U_t = E_t\left[\begin{array}{c} \Pi\left(x_t, a_t\right) + \int_t^\tau e^{-r(s-t)}\left(P\left(x_s, b_s\right) - x_s + \Pi_a\left(x_s, b_s\right)\right)\dot{b}_s ds + \\ \int_t^\tau e^{-r(s-t)}\left(-r\Pi\left(x_s, b_s\right) + \mu x_s \Pi_x\left(x_s, b_s\right) + \frac{1}{2}\sigma^2 x_s^2 \Pi_{xx}\left(x_s, b_s\right)\right)ds \end{array}\right] > \Pi\left(x_t, a_t\right),$$

a contradiction to the fact that $\left(\{a_t\}, \mathbf{P}\right)$ is an equilibrium. Thus, in equilibrium it must be that $P\left(x_s, a_s\right) - x_s + \Pi_a\left(x_s, a_s\right) = 0$ for all $s \in [t, \tau]$. ■

Corollary A2 *Let $\left(\{a_t\}, \mathbf{P}\right)$ be a regular equilibrium and let $\Pi\left(x, a\right)$ be the monopolist's profits. Then, $\Pi\left(x, a\right)$ solves (A.1) at states $\left(x, a\right)$ with $a < \alpha$ such that (i) $x > z_2$, or (ii) $\{a_t\}$ is continuously increasing at time s when $\left(x_s, a_s\right) = \left(x, a\right)$.*

Proof. (i) Let (x, a) be such that $a < \alpha$ and $x > z_2$. By Lemma A7, at such a state the monopolist will not sell until costs fall below z_2, so $\Pi(x, a) = E[e^{-r\tau_2}\Pi(z_2, a)|\, x_0 = x]$. Thus, by Corollary A1 $\Pi(x, a)$ solves (A.1).

(ii) Suppose (x, a) is such that $\{a_t\}$ is continuously increasing at time s when $(x_s, a_s) = (x, a)$. By the arguments in the proof of Lemma A8, $\Pi(x, a)$ solves (A.5). Moreover, by Lemma A8 $P(x, a) = x - \Pi_a(x, a)$, so $\Pi(x, a)$ solves (A.1). ■

Lemma A9 *Let $(\{a_t\}, \mathbf{P})$ be a regular equilibrium and let $\Pi(x, a)$ be the monopolist's profits. Let (x, a) with $a < \alpha$ be such that $\{a_t\}$ is continuously increasing at time s when $(x_s, a_s) = (x, a)$. Then, there exists $x_*(a) < x < x^*(a)$ with either $x^*(a) \le z_2$ or $x^*(a) = \infty$ such that $\{a_t\}$ jumps at t if $a_{t^-} = a$ and $x_t \in \{x_*(a), x^*(a)\}$. Moreover,*

$$\Pi(y, a) = E\left[e^{-r\tau^*(a)}\left(\left(P\left(x_{\tau^*(a)}, a_{\tau^*(a)}\right) - x_{\tau^*(a)}\right) da_{\tau^*(a)} + \Pi\left(x_{\tau^*(a)}, a + da_{\tau^*(a)}\right)\right)\Big|\, x_0 = y\right], \tag{A.7}$$

for all $y \in (x_*(a), x^*(a))$, where $\tau^*(a) = \inf\{t : x_t \notin (x_*(a), x^*(a))\}$ and where $da_{\tau^*(a)}$ denotes the jump of $\{a_t\}$ at state $\left(x_{\tau^*(a)}, a\right)$.

Proof. Note first that for every such state (x, a) there must exist $\underline{y}(a) < x$ such that a_t jumps when $x_t = \underline{y}(a)$ and $a_{t^-} = a < \alpha$. To see this, suppose by contradiction that a_t is continuous at time s when $(x_s, a_s) = (y, a)$ for every $y < x$. By Corollary A2 $\Pi(y, a)$ solves (A1) for all $y \le x$, so $\Pi(y, a) = Ay^\lambda + By^\kappa$ for some constants A and B. If $A \ne 0$ or $B \ne 0$, $\Pi(\cdot, a)$ explodes as $y \to 0$ or as $y \to \infty$, which cannot occur in equilibrium. Otherwise, $A = B = 0$ implies that $\Pi(y, a) = 0$ for all $y \le x$, which cannot occur either since $\Pi(y, a) \ge L(y, a) > 0$. Thus, there must exist $\underline{y}(a) < x$ such that a_t jumps to some $a' > a$ when $x_t = \underline{y}(a)$ and $a_{t^-} = a$. Let $x_*(a)$ denote the supremum over all such $\underline{y}(a)$. If a_t is continuous for all $y > x_*(a)$ whenever $a_t = a$, then by Corollary A2 $\Pi(y, a)$ solves (A1) for all $y > x_*(a)$. Thus, $\Pi(y, a) = Ay^\lambda + By^\kappa$ for all $y > x_*(a)$. Since $\kappa > 1$, in this case it must be that $B = 0$; otherwise, $\Pi(y, a)$ would explode as $y \to \infty$. Since $\{a_t\}$ jumps to a' when $x_t = x_*(a)$ and $a_{t^-} = a$, it follows that $\Pi(x_*(a), a) = (P(x_*(a), a') - x_*(a))(a' - a) + \Pi(x_*(a), a')$. Thus, Corollary A1 implies that $\Pi(y, a)$ satisfies (A.7) (with $x^*(a) = \infty$). Otherwise, there exists $\overline{y}(a) > x$ such that $\{a_t\}$ jumps to some $\widetilde{a} > a$ when $x_t = \overline{y}(a)$ and $a_{t^-} = a$. By Lemma A7, $\overline{y}(a) \le z_2$. Let $x^*(a)$ be the infimum over all such $\overline{y}(a)$, so $x^*(a) \le z_2$. In this case, $\Pi(y, a)$ solves (A1) for all $y \in (x_*(a), x^*(a))$, with $\Pi(y, a) = (P(y, a) - x)da_{\tau^*(a)} + \Pi\left(y, a + da_{\tau^*(a)}\right)$ whenever $y \in \{x_*(a), x^*(a)\}$. Thus, by Lemma A1 $\Pi(y, a)$ satisfies (A.7) for all $y \in (x_*(a), x^*(a))$. ■

Lemma A10 *Let $(\{a_t\}, \mathbf{P})$ be a regular equilibrium, and let $\Pi(x, a)$ denote the monopolist's profits. Then, $\Pi(x, a) = L(x, a)$ for all states (x, a) with $a < \alpha$.*

Proof. By the arguments in the main text, $\Pi(x, a) \ge L(x, a)$ for all states (x, a) with $a < \alpha$. By Lemma A9, for all (x, a) such that $\{a_t\}$ is continuously increasing at time s when $(x_s, a_s) = (x, a)$, there exists $x_*(a) < x < x^*(a)$ such that $da_t = a_t - a_{t^-} > 0$ when $a_{t^-} = a$ and $x_t \in \{x_*(a), x^*(a)\}$. Moreover, $\Pi(x, a)$ satisfies (A.7) for all $x \in (x_*(a), x^*(a))$. Suppose that $\{a_t\}$ jumps to α when $a_{t^-} = a$ and $x_t \in \{x_*(a), x^*(a)\}$, so $\Pi(x, a) = (P(x, \alpha) -$

$x)(\alpha - a) + \Pi(x, \alpha) = g(x, a)$ for $x \in \{x_*(a), x^*(a)\}$. Thus,

$$\Pi(x, a) = E[e^{-r\tau^*(a)} g\left(x_{\tau^*(a)}, a\right) \big| x_0 = x] \leq L(x, a) = \sup_{\tau} E[e^{-r\tau} g(x_\tau, a) \big| x_0 = x],$$

for all $x \in (x_*(a), x^*(a))$, so $\Pi(x, a) = L(x, a)$ for all such states (x, a).

Suppose next that $da_t = \widetilde{a} - a < \alpha - a$ when $a_{t^-} = a$ and $x_t = x_*(a)$ or $x_t = x^*(a)$. Thus, $\Pi(x_t, a) = (P(x_t, \widetilde{a}) - x_t)(\widetilde{a} - a) + \Pi(x_t, \widetilde{a})$ when $x_t \in \{x_*(a), x^*(a)\}$. By Lemma A7, the monopolist must continue selling gradually after a_t jumps (since $x_*(a) < x^*(a) \leq z_2$). By Lemma A8, it must be that $P(x_t, \widetilde{a}^+) = x_t - \Pi_a(x_t, \widetilde{a})$. Note that prices cannot jump at time t. If prices jumped down at t, then those consumers who buy at t^- would be strictly better off by delaying their purchase an instant, which cannot occur in equilibrium. Thus, it must be that $P(x_t, \widetilde{a}) = P(x_t, \widetilde{a}^+)$. By Lemma A9, $\Pi(x_t, \widetilde{a})$ satisfies (A.7), so

$$P(x_t, \widetilde{a}) - x_t = -\Pi_a(x_t, \widetilde{a}) = E[e^{-r(\tau^*(\widetilde{a}) - t)}(P\left(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}\right) - x_{\tau^*(\widetilde{a})}) \big| x_t].$$

That is, the margin that the monopolist gets from selling to consumers $[a, \widetilde{a}]$ at state (x_t, a) with $x_t \in \{x_*(a), x^*(a)\}$ is the same as the expected discounted margin she gets at state $(x, \widetilde{a})$ with $x \in \{x_*(\widetilde{a}), x^*(\widetilde{a})\}$. Since $\Pi(x_t, a) = (P(x_t, \widetilde{a}) - x_t)(\widetilde{a} - a) + \Pi(x_t, \widetilde{a})$ and since $\Pi(x_t, \widetilde{a})$ satisfies (A.7),

$$\Pi(x_t, a) = E\left[e^{-r(\tau^*(\widetilde{a}) - t)} \left(\begin{array}{c} \left(P\left(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}\right) - x_{\tau^*(\widetilde{a})}\right)\left(da_{\tau^*(\widetilde{a})} + \widetilde{a} - a\right) \\ +\Pi\left(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}\right) \end{array} \right) \Bigg| x_t \right], \quad \text{(A.8)}$$

where $da_{\tau^*(\widetilde{a})}$ denotes the jump of $\{a_t\}$ when $a_{s^-} = \widetilde{a}$ and $x_s \in \{x_*(\widetilde{a}), x^*(\widetilde{a})\}$. There are two possibilities: (i) $a_{\tau^*(\widetilde{a})} = \alpha$ with probability 1, so $da_{\tau^*(\widetilde{a})} = \alpha - \widetilde{a}$; or (ii) $a_{\tau^*(\widetilde{a})} = \widehat{a} < \alpha$, so $da_{\tau^*(\widetilde{a})} = \widehat{a} - \widetilde{a} < \alpha - \widetilde{a}$. In the first case, $da_{\tau^*(\widetilde{a})} + \widetilde{a} - a = \alpha - a$, so $(P(x_{\tau^*(\widetilde{a})}, \alpha) - x_{\tau^*(\widetilde{a})})(da_{\tau^*(\widetilde{a})} + \widetilde{a} - a) + \Pi(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}) = g(x_{\tau^*(\widetilde{a})}, a)$. Using (A.8), this implies that $\Pi(x_t, a) = E[e^{-r\tau^*(\widetilde{a})} g\left(x_{\tau^*(\widetilde{a})}, a\right) \big| x_0 = x] \leq L(x_t, a)$, so $\Pi(x_t, a) = L(x_t, a)$. In the second case, $a_{\tau^*(\widetilde{a})} = \widehat{a} < \alpha$. Since $x_{\tau^*(\widetilde{a})} \leq z_2$, by Lemma A7 the monopolist must continue selling gradually after $\{a_t\}$ jumps. Then, by Lemma A9,

$$\Pi\left(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}\right) = E\left[e^{-r(\tau^*(\widehat{a}) - \tau^*(\widetilde{a}))} \left(\begin{array}{c} \left(P\left(x_{\tau^*(\widehat{a})}, a_{\tau^*(\widehat{a})}\right) - x_{\tau^*(\widehat{a})}\right)\left(da_{\tau^*(\widehat{a})} + \widehat{a} - \widetilde{a})\right) \\ +\Pi\left(x_{\tau^*(\widehat{a})}, a_{\tau^*(\widehat{a})}\right) \end{array} \right) \Bigg| x_{\tau^*(\widetilde{a})} \right]. \quad \text{(A.9)}$$

Moreover, the same arguments as above also imply that

$$P\left(x_{\tau^*(\widetilde{a})}, a_{\tau^*(\widetilde{a})}\right) - x_{\tau^*(\widetilde{a})} = E[e^{-r(\tau^*(\widehat{a}) - \tau^*(\widetilde{a}))}(P\left(x_{\tau^*(\widehat{a})}, a_{\tau^*(\widehat{a})}\right) - x_{\tau^*(\widehat{a})}) \big| x_{\tau^*(a)}].$$

Using this equation and (A.9) in (A.8), it follows that

$$\Pi(x_t, a) = E\left[e^{-r(\widehat{\tau} - t)}\left((P(x_{\widehat{\tau}}, a_{\widehat{\tau}}) - x_{\widehat{\tau}})(da_{\widehat{\tau}} + \widehat{a} - a) + \Pi(x_{\widehat{\tau}}, a_{\widehat{\tau}})\right) \big| x_t\right],$$

for some stopping time $\widehat{\tau}$. Again, there are two possibilities: (i) $a_{\widehat{\tau}} = \alpha$ with probability 1 (so $da_{\widehat{\tau}} = \alpha - \widehat{a}$), or (ii) $a_{\widehat{\tau}} < \alpha$. In case (i), $(P(x_{\widehat{\tau}}, a_{\widehat{\tau}}) - x_{\widehat{\tau}})(da_{\widehat{\tau}} + \widehat{a} - a) + \Pi(x_{\widehat{\tau}}, a_{\widehat{\tau}}) = g(x_{\widehat{\tau}}, a)$, so $\Pi(x_t, a) = E[e^{-r(\hat{\tau}-t)} g(x_{\hat{\tau}}, a) | x_t] \leq L(x_t, a)$. Hence, $\Pi(x_t, a) = L(x_t, a)$. In case (ii), we can again repeat the same argument. Eventually, we'll get to a point at which a jumps to α, so $\Pi(x_t, a) = E[e^{-r(\tau-t)} g(x_\tau, a) | x_t]$ for some stopping time τ. Hence $\Pi(x_t, a) = L(x_t, a)$. ■

Proof of Theorem 1. Lemma A10 shows that in any regular equilibrium the monopolist's profits are given by $L(x, a)$ for all x. Thus, the monopolist sells to all high type consumers when $x_t \in [0, \underline{x}(a_t)] \cup [\overline{x}(a_t), z_2]$ (and also to low type consumers when $x_t \leq \underline{x}(a_t)$). By Lemma A7, the monopolist must sell at a positive rate while $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. The arguments in the text pin down the rate at which the monopolist sells and the price she charges when $x_t \in (\underline{x}(a_t), \overline{x}(a_t))$. ■

A.4 Proof of Theorem 2

The proof of Theorem 2 is a generalization of the proof of Theorem 1. Suppose that there are $n \geq 3$ types of consumers in the market. Here I provide a sketch of the arguments. Note that when $a \geq \alpha_3$, the only consumers left in the market are those with valuations v_1 and v_2. By Theorem 1, in any regular equilibrium the monopolist's profits are equal to $L(x, a)$ for states (x, a) with $a \geq \alpha_3$.

Consider next states (x, a) with $a \in [\alpha_4, \alpha_3)$, so there are $\alpha_3 - a$ consumers with valuation v_3 in the market (if there are only three types of consumers in the market, let $\alpha_4 = 0$). Let $P_2(x) = \sup_{i \in (\alpha_3, \alpha_2]} P(x, i)$ be the highest price a consumer with valuation v_2 is willing to pay. By equation (5), the strategy $P(x, \alpha_3)$ of consumer α_3 (the highest indexed consumer with valuation v_3) satisfies

$$P(x, \alpha_3) = v_3 - E\left[e^{-r\tau_2}(v_3 - P_2(x_{\tau_2})) \middle| x_0 = x\right],$$

where $\tau_2 = \inf\{t : x_t \leq z_2\}$ is the time at which the monopolist starts selling to consumers with valuation v_2 when the level of market penetration is α_3. By the skimming property, the monopolist can always sell to all consumers with valuation v_3 at price $P(x, \alpha_3)$. Therefore, at states (x, a) with $a \in [\alpha_4, \alpha_3)$ the monopolist's profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_3 - a)(P(x_\tau, \alpha_3) - x_\tau) + e^{-r\tau} L(x_\tau, \alpha_3)\right) \middle| x_0 = x\right]. \quad \text{(A.10)}$$

Let $\underline{x}(\alpha_3)$ and $\overline{x}(\alpha_3)$ be the cutoffs that characterize the solution to the optimal stopping problem (10) when $a = \alpha_3$ (i.e., when all consumers with valuation v_3 have left the market). The first thing to note is that the solution to (A.10) is such that it is optimal to continue when $x_t \in (\underline{x}(\alpha_3), \overline{x}(\alpha_3))$. The reason for this is that the expected payoff from delaying when $x_t \in (\underline{x}(\alpha_3), \overline{x}(\alpha_3))$ is larger when $a < \alpha_3$ than when $a = \alpha_3$, since in the former case there are more high valuation consumers to sell to.[13]

Using arguments similar to those in Lemma A4, one can show that the solution to (A.10)

[13]This can be proved formally using the arguments in the proof of Lemma A6, where I show that the cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ of the solution to (10) satisfy $\underline{x}'(a) > 0$ and $\overline{x}'(a) < 0$.

is of the form

$$\tau(a) = \inf\{t : x_t \in [0, \underline{x}_1(a)] \cup [\overline{x}_1(a), \underline{x}_2(a)] \cup [\overline{x}_2(a), z_3]\}, \tag{A.11}$$

with $\underline{x}_1(a)$, $\overline{x}_1(a)$, $\underline{x}_2(a)$, $\overline{x}_2(a)$ such that $\underline{x}_1(a) < z_1 < \overline{x}_1(a)$ and $\underline{x}_2(a) < z_2 < \overline{x}_2(a) < z_3$. That is, the solution to (A10) involves delaying when x is around z_1 or z_2 and when $x > z_3$. Using arguments similar to those in Lemma A5 the thresholds $\underline{x}_1(a), \overline{x}_1(a), \underline{x}_2(a)$ and $\overline{x}_2(a)$ are continuous in a, with $\lim_{a\to\alpha_3} \underline{x}_2(a) = \lim_{a\to\alpha_3} \overline{x}_2(a) = z_2$, and $\lim_{a\to\alpha_3} \underline{x}_1(a) = \underline{x}(\alpha_3)$ and $\lim_{a\to\alpha_3} \overline{x}_1(a) = \overline{x}(\alpha_3)$ (where $\underline{x}(\alpha_3)$ and $\overline{x}(\alpha_3)$ are the cutoffs that characterize the solution to (10) when $a = \alpha_3$). In addition, $L(x, a) \in C^{2,2}$ for all $x \in (\underline{x}_1(a), \overline{x}_1(a)) \cup (\underline{x}_2(a), \overline{x}_2(a))$.

Next, by arguments similar to those in Lemma A7 the monopolist will always sell to consumers with valuation v_3 at states (x_t, a_t) with $x_t \leq z_3$ and $a_t \in [\alpha_4, \alpha_3)$, and will never sell to them when $x_t > z_3$. Moreover, arguments similar to those in Lemma A8 imply that in any regular equilibrium, $P(x_s, a_s) = x_s - \Pi_a(x_s, a_s)$ whenever the monopolist is selling at a continuous rate (i.e., whenever $da_t = \dot{a}_t dt$). Finally, by arguments similar to those in Lemma A10, in any regular equilibrium the monopolist's profits must be equal to $L(x, a)$ at all states (x, a) with $a \in [\alpha_4, \alpha_3)$.

At states (x_t, a_t) with $a_t \in [\alpha_4, \alpha_3)$ the equilibrium dynamics are as follows. If $x_t > z_3$, the monopolist doesn't sell and waits for costs to decrease. When $x_t \in [\overline{x}_2(a_t), z_3]$, the monopolist sells immediately to all remaining consumers with valuation v_3, and then equilibrium play continuous as in the case with two consumers. When $x_t \in [\overline{x}_1(a_t), \underline{x}_2(a_t)]$, the monopolist sells immediately to all remaining consumers with valuation v_3; however, since $z_2 > \underline{x}_2(a_t)$ and $\overline{x}_1(a_t) > \overline{x}(\alpha_3)$ (where $\overline{x}(\alpha_3)$ is the cutoff that describes the solution to the optimal stopping problem $L(x, a)$ at state (x, α_3)), in this case the monopolist also sells to all consumers with valuation consumers v_2. When $x_t \leq \underline{x}_1(a_t)$ the monopolist sells to all remaining consumers at price v_1 and the market closes. Finally, when $x_t \in (\underline{x}_1(a_t), \overline{x}_1(a_t)) \cup (\underline{x}_2(a_t), \overline{x}_2(a_t))$, the monopolist sells gradually to consumers with valuation v_3 at a rate that leaves them indifferent between purchasing at t or delaying their purchase. One can derive this rate in a way similar to the derivation of equation (17) in the main text.

Consider next state (x, a) with $a \in [\alpha_5, \alpha_4)$, at which there are $\alpha_4 - a$ consumers with valuation v_4 in the market (if there are only four types of consumers in the market, let $\alpha_5 = 0$). Let $P_3(x) = \sup_{i\in(\alpha_4,\alpha_3]} P(x, i)$, and let $P(x, \alpha_4)$ be the strategy of consumer α_4 (i.e., the last consumer with valuation v_4). By equation (5), it must be that

$$P(x, \alpha_4) = v_4 - E\left[e^{-r\tau_3}(v_4 - P_3(x_{\tau_3}))\middle| x_0 = x\right],$$

where $\tau_3 = \inf\{t : x_t \leq z_3\}$ is the time at which the monopolist starts selling to consumers with valuation v_3 when $a = \alpha_4$. Since the monopolist can sell to all consumers with valuation v_4 at price $P(x, \alpha_4)$, at states (x, a) with $a \in [\alpha_5, \alpha_4)$ her profits are bounded below by

$$L(x, a) = \sup_{\tau\in T} E\left[e^{-r\tau}((\alpha_4 - a)(P(x_\tau, \alpha_4) - x_\tau) + L(x_\tau, \alpha_4))\middle| x_0 = x\right].$$

Repeating the same arguments as above, one can show that in any regular equilibrium the monopolist's profits must be given by $L(x,a)$ at all states (x,a) with $a \in [\alpha_5, \alpha_4)$. More generally, for $k \geq 5$ one can extend $L(x,a)$ for all $x \in [\alpha_{k+1}, \alpha_k)$ in a similar way, and show that in any regular equilibrium the monopolist's profits are $L(x,a)$ for all $a \geq [\alpha_{k+1}, \alpha_k)$.

A.5 Proof of Theorem 3

For each valuation v_n^k, let $z_k^n = \frac{-\lambda}{1-\lambda} v_k^n$. For $n = 2, 3, ...$, define the function $P^n(x)$ as follows. For $x \leq z_1^n$, $P^n(x) = v_1^n = \underline{v}$. For $k = 2, .., n$, and $x \in (z_{k-1}^n, z_k^n]$, let $P^n(x) = P(x, \alpha_k^n)$. That is, for all $x \in (z_{k-1}^n, z_k^n]$, $P^n(x)$ is equal to the price at which consumer α_k^n is willing to trade (where α_k^n is the highest indexed consumer with valuation v_k^n). By equation (5), for $k = 2, .., n$ and $x \in (z_{k-1}^n, z_k^n]$,

$$\begin{aligned} P^n(x) &= P(x, \alpha_k^n) = v_k^n - E[e^{-r\tau_{k-1}^n} \left(v_k^n - P\left(z_{k-1}^n, \alpha_{k-1}^n\right)\right) \big| \, x_0 = x] \\ &= v_k^n - E[e^{-r\tau_{k-1}^n} \left(v_k^n - P^n\left(z_{k-1}^n\right)\right) \big| \, x_0 = x], \end{aligned} \tag{A.12}$$

where for $k = 1, 2, ..., n$, $\tau_k^n = \inf\{t : x_t \leq z_k^n\}$ is the time at which the monopolist starts selling to buyers with valuation v_k^n when v_k^n is the highest valuation remaining in the market.

Lemma A11 *For $k = 2, ..., n$ and $x \in (z_{k-1}^n, z_k^n]$,*

$$P^n(x) = v_k^n - \sum_{j=1}^{k-1} \left(v_{j+1}^n - v_j^n\right) \left(\frac{x}{z_j^n}\right)^{\lambda}. \tag{A.13}$$

Proof. The proof is by induction. By equation (9), $P^n(x) = v_2^n - (v_2^n - v_1^n)(x/z_1^n)^{\lambda}$ for $x \in (z_1^n, z_2^n]$, so the statement is true for $k = 2$. Suppose the statement is true for $l = 2, .., k-1$. Equation (A.12), Corollary A1 and the induction hypothesis then imply that

$$P^n(x) = v_k^n - \left(v_k^n - P^n\left(z_{k-1}^n\right)\right)\left(\frac{x}{z_{k-1}^n}\right)^{\lambda} = v_k^n - \sum_{j=1}^{k-1} \left(v_{j+1}^n - v_j^n\right)\left(\frac{x}{z_j^n}\right)^{\lambda}.$$

for $x \in (z_{k-1}^n, z_k^n]$. ■

Recall that $v_1^n = \underline{v}$ and $v_n^n = \overline{v}$ for all n. Let $\overline{z} := \frac{-\lambda}{1-\lambda}\overline{v} = z_n^n$ and $\underline{z} := \frac{-\lambda}{1-\lambda}\underline{v} = z_1^n$.

Lemma A12 *$P^n(x) - x \to V_1(x)$ uniformly on $[0, \overline{z}]$ as $n \to \infty$.*

Proof. I first show that $\lim_{n\to\infty} P^n(x) = V_1(x) + x$ for all $x \in [0, \overline{z}]$. Note first that, for all n, $P^n(x) - x = v_1 - x = V_1(x)$ for all $x \leq z_1$. Next, fix $x \in (z_1, \overline{z}]$ with $x \in (z_{k-1}^n, z_k^n]$ for some $k \leq n$, and let $v(x) = \frac{1-\lambda}{-\lambda} x$. Recall that $v_{j+1}^n - v_j^n = (\overline{v} - \underline{v})/(n-1)$. Equation (A.13) and the fact that $x/z_j^n = v(x)/v_j^n$ then imply that

$$P^n(x) = v_k^n - \sum_{j=1}^{k-1} \frac{\overline{v} - \underline{v}}{n-1} \left(\frac{v(x)}{v_j^n} \right)^{\lambda}.$$

Note that $z_k^n = \frac{-\lambda}{1-\lambda} v_k^n \to x$ as $n \to \infty$, so $v_k^n \to v(x)$. Since $(v(x)/v)^{\lambda}$ is Riemann integrable,

$$\lim_{n\to\infty} P^n(x) = v(x) - \int_{\underline{v}}^{v(x)} \left(\frac{v(x)}{v} \right)^{\lambda} dv = x + (\underline{v} - \underline{z}) \left(\frac{x}{\underline{z}} \right)^{\lambda} = x + V_1(x).$$

Finally, since $P^n(x)$ is increasing in x for all $x \in [0, \overline{z}]$ and since $\lim_{n\to\infty} P^n(x) = V_1(x) + x$ in this range, it follows that $P^n(x)$ converges uniformly to $V_1(x) + x$ as $n \to \infty$. Thus, $P^n(x) - x$ converges uniformly to $V_1(x)$. ■

Proof of Theorem 3. I prove that, for all x, $L^n(x,0) \to V_1(x)$ as $n \to \infty$. The proof that $L^n(x,a) \to (1-a)V_1(x)$ as $n \to \infty$ for $a > 0$ is symmetric and omitted. Note first that $L^n(x,0) \geq V_1(x)$ for all x, since at any state $(x,0)$ the monopolist can wait until time τ_1 and sell to all consumers at price v_1, obtaining a profit of $E[e^{-r\tau_1^n}(v_1 - x_{\tau_1^n}) | x_0 = x] = V_1(x)$.

Consider next the case in which $x_0 = x \geq \overline{z}$. In this case, in equilibrium the monopolist sells to consumers with valuation v_k^n at time $\tau_k^n = \inf\{t : x_t \leq z_k^n\}$ (for $k = 1, ..., n$), at a price $P(z_k^n, \alpha_k^n) = P^n(z_k^n)$. Recall that for $k = 1, ..., n$, $\alpha_k^n = \max\{i : f^n(i) = v_k^n\}$, and let $\alpha_{k+1}^n = 0$. Thus, the monopolist's profits are

$$L^n(x,0) = E\left[\sum_{k=1}^{n} e^{-r\tau_k^n} (P^n(z_k) - z_k) \left(\alpha_k^n - \alpha_{k+1}^n\right) \middle| x_0 = x \right].$$

Since $P^n(x) - x \to V_1(x)$ uniformly on $[0, \overline{z}]$ as $n \to \infty$, for every $\eta > 0$ there exists N such that $P^n(x) - x - V_1(x) < \eta$ for all $n > N$ and all $x \in [0, \overline{z}]$. Thus, for $n > N$,

$$L^n(x,0) < E\left[\sum_{k=1}^{n} e^{-r\tau_k^n} V_1(z_k)\, d\alpha_k \middle| x_0 = x \right] + \eta = \sum_{k=1}^{n} d\alpha_k E\left[e^{-r\tau_k^n} V_1(z_k) \middle| x_0 = x \right] + \eta, \tag{A.14}$$

where $d\alpha_k = \alpha_k - \alpha_{k+1}$ (so $\sum_{k=1}^{n} d\alpha_k = 1$). Note further that for $x \geq \overline{z}$ and $k = 1, 2, .., n$,

$$\begin{aligned} E\left[e^{-r\tau_k^n} V_1(z_k^n) \middle| x_0 = x \right] &= E\left[e^{-r\tau_k^n} E\left[e^{-r\left(\tau_1^n - \tau_k^n\right)} \left(v_1 - x_{\tau_1^n}\right) \middle| x_{\tau_k^n} \right] \middle| x_0 = x \right] \\ &= E\left[e^{-r\tau_1^n} \left(v_1 - x_{\tau_1^n}\right) \middle| x_0 = x \right] = V_1(x). \end{aligned}$$

Using this and the fact $\sum_{k=1}^{n} d\alpha_k = 1$ in (A.14) gives $V_1(x) \leq L^n(x,0) < V_1(x) + \eta$ for all $n > N$. Therefore, $\lim_{n\to\infty} L^n(x,0) = V_1(x)$ for all $x \geq \overline{z}$.

Consider next the case with $x < \overline{z}$. Suppose by contradiction that there exists $x < \overline{z}$ such that $L^n(x,0) \nrightarrow V_1(x)$ as $n \to \infty$. Since $L^n(x,0) \geq V_1(x)$ for all n, there exists N and $\gamma > 0$ such that $L^n(x,0) > V_1(x) + \gamma$ for all $n > N$. Fix $y \geq \overline{z}$ and let $\tau_x := \inf\{t : x_t \leq x\}$.

Since the monopolist can always delay trade until time τ_x, for all $n > N$ it must be that

$$\begin{aligned}
L^n(y,0) &\geq E[e^{-r\tau_x} L(x,0) | x_0 = y] \\
&> E[e^{-r\tau_x} V_1(x) | x_0 = y] + E[e^{-r\tau_x}\gamma | x_0 = y] \\
&= E[e^{-r\tau_x} E[e^{-r(\tau_1-\tau_x)}(v_1 - x_{\tau_x}) | x_{\tau_x}] | x_0 = y] + \left(\frac{y}{x}\right)^\lambda \gamma = V_1(y) + \left(\frac{y}{x}\right)^\lambda \gamma,
\end{aligned}$$

a contradiction to $\lim_{n\to\infty} L^n(y,0) = V_1(y)$ (since $y \geq \overline{z}$). Therefore, $\lim_{n\to\infty} L^n(x,0) = V_1(x)$ for all $x < \overline{z}$. ■

References

[1] Ausubel, L. M., and R. J. Deneckere (1989): "Reputation in Bargaining and Durable Goods Monopoly", *Econometrica*, Vol. 57, pp. 511-531.

[2] Biehl, A. R. (2001): "Durable-Goods Monopoly with Stochastic Values", *Rand Journal of Economics*, Vol. 32, pp. 565-577.

[3] Board, S. (2008): "Durable-Goods Monopoly with Varying Demand", *Review of Economic Studies*, Vol. 75, pp. 391-413.

[4] Bulow, J. I. (1982): "Durable-Goods Monopolists", *Journal of Political Economy*, Vol. 90, pp. 314-332.

[5] Butz, D. A. (1990): "Durable-Good Monopoly and Best-Price Provisions", *American Economic Review*, Vol. 80, pp. 1062-1076.

[6] Coase, R. (1972): "Durability and monopoly", *Journal of Law and Economics*, Vol. 15, pp. 143-49.

[7] Conlon, C. T. (2010): "A Dynamic Model of Costs and Margins in the LCD TV Industry", unpublished manuscript.

[8] Daley, B., and B. Green (2011): "Waiting for News in the Market for Lemons", forthcoming *Econometrica*.

[9] de la Fuente, A. (2000): *Mathematical Methods and Models for Economists*. Cambridge University Press.

[10] Deneckere, R., and M. Liang (2006): "Bargaining with Interdependent Values", *Econometrica*, Vol. 74, pp. 1309-1364.

[11] Evans, R. (1989): "Sequential Bargaining with Correlated Values", *Review of Economic Studies*, Vol. 56, pp. 499-510.

[12] Fuchs, W., and A. Skrzypacz (2010): "Bargaining with Arrival of New Traders", *American Economic Review*, Vol. 100, 802-836.

[13] Fudenberg, D., D. Levine and J. Tirole (1985): "Infinite Horizon Models of Bargaining with One-Sided Incomplete Information", in Roth A. *Bargaining with Incomplete Information*, Cambridge University Press.

[14] Gul, F., and W. Pesendorfer (2011): "The War of Information", forthcoming *Review of Economic Studies*.

[15] Gul, F., H. Sonnenschein and R. Wilson (1985): "Foundations of Dynamic Monopoly and the Coase Conjecture", *Journal of Economic Theory*, Vol. 39, 155-190.

[16] Kahn, C. M. (1986): "The Durable Goods Monopolist and Consistency with Increasing Costs", *Econometrica*, Vol. 54 pp. 275-94.

[17] Karatzas, I., and S. E. Shreve (1998). *Brownian Motion and Stochastic Calculus*. Springer, New York.

[18] McAfee, R. P., and T. Wiseman (2008): "Capacity Choice Counters the Coase Conjecture", *Review of Economic Studies*, Vol. 75, pp. 317-332.

[19] Oksendal, B. (2007): *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag, Berlin.

[20] Ortner, J. (2011): "A Continuous Time Model of Bilateral Bargaining", unpublished manuscript.

[21] Sannikov, Y. (2007): "Games With Imperfectly Observable Actions in Continuous Time", *Econometrica*, Vol. 75, pp. 1285-1329.

[22] Sannikov, Y. (2008): "A Continuous-Time Version of the Principal-Agent Problem", *Review of Economic Studies*, Vol. 75, pp. 957-984.

[23] Shiryaev, A. (2008): *Optimal Stopping Rules*. Springer-Verlag, New York.

[24] Simon, L. K., and M. B. Stinchcombe (1989): "Extensive Form Games in Continuous Time: Pure Strategies", *Econometrica*, Vol. 57, pp. 1171-1214.

[25] Sobel, J. (1991): "Durable Goods Monopoly with Entry of New Consumers", *Econometrica*, Vol. 59, pp. 1455-1485.

[26] Stokey, N. L. (1979): "Intertemporal Price Discrimination", *Quarterly Journal of Economics*, Vol. 93, pp. 355-371.

[27] Stokey, N. L. (1981): "Rational Expectations and Durable Goods Pricing", *Bell Journal of Economics*, Vol. 12, pp. 112–28.

[28] Vincent, D. (1989): "Bargaining with Common Values", *Journal of Economic Theory*, Vol. 48, pp. 47-62.