Max-Stable Processes and Their Applications

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Abstract: Environmental problems such as floods require statistical analysis that takes into account the complex nature of the data, namely observations are sampled at different spatial points in a given region for a certain time. Thus the spatial dependence structure cannot be ignored. Extreme statistics for the design of structures for flood protection, for the study of the structural failures such as bridges, dams, etc., for the prediction of heat waves and others should be based on a solid theoretical framework. Max-stable processes provide a such theory and in the last decade have emerged as fertile ground for research and a common tool for the statistical modeling of spatial extremes. This entry provides a summary of max-stable processes.

Keywords: Brown-Resnick processes; Coefficient of tail dependence; Ergodicity; Extremes; Extremal coefficient function; Fréchet distributions; Gaussian processes, Gumbel distributions; Hüsler–Reiss model, Mixing; Multivariate extremes; Pointwise Maximum; Poisson point processes; Stationary max-stable processes.

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Introduction

Extreme value theory is the part of probability and statistics that studies and develops mathematical models and statistical methods for describing the stochastic behavior of extremes of processes. Extreme values of a phenomenon are those events that occur with low frequency but can have a large impact on real life. Relevant examples include the floods of "biblical proportions" that are affecting many countries in the world: Pakistan in July 2010, Italy in October 2010, south-eastern Brazil, Queensland (Australia) and Sri Lanka in January 2011, and Mississippi (USA) in May 2011, all of them causing much damage and many deaths.

The first theoretical results related to univariate random variables date back to the early twentieth century. Over time, in the univariate case many mathematical results have been obtained, concerning the domain of attraction (e.g. [29], Chapter 0; [16], Chapter 1), regular varying functions (e.g. [29], Chapter 0; [16], Appendix B), etc. But also several extreme distributions have been derived, for the maximum (or minimum) of independent and identically distributed (iid) random variables, intermediate order statistics, exceedances over (below) high (low) thresholds, etc. See for instance [29], Chapters 1,3 and 4; [7], Chapters 3, 4 and 7; [16], Chapters 1–5.

Given the multivariate nature of many applications, after the seventies the attention of the scientific community moved to characterize multivariate extreme values. Imagine problems where multiple variables are available and the data on one variable may inform us about the others. In the univariate case "the extreme value" is a clear concept, however in high dimensions this is no longer the case since many definitions are possible. A simple way to proceed is to extend the concept of the maximum to "componentwise" maxima (e.g. [29], Chapter 5; [16], Chapter 6). We refer to Chapter 8 of [7] and Chapters 5 and 7 of [18] for discussions on thresholds and point processes approaches.

Environmental problems such as floods require statistical analyses that take into account the complex nature of the data, namely observations are sampled at different spatial points in a given region for a certain time. Thus the spatial (or spatial-temporal) dependence structure cannot be ignored. Space-time statistical analysis, based on the theory of continuous stochastic processes, allow us to study this type of data properly. Clearly, the incidents previously listed are spatial (or spatial-temporal) events belonging to extreme value theory. In order to study the stochastic behavior of these events a theory of extremes of continuous stochastic processes is needed. Recently, such a theory has been developed (e.g. [13]; [30]; [16], Chapter 9; [18], Chapter 7.4). This is also based on the simple notion of maximum that in the context of continuous processes (in time or space or both) becomes "pointwise" maximum. This entry, which summarizes this theory, is organized in the following way: in the first section basic definitions and main results are introduced. In the second section the spectral representation useful to derive and simulate max-stable processes is described, and in the third a discussion on applications is provided.

Max-Stable Processes

A simple extension of the extreme value theory to the infinite-dimensional space is provided by max-stable processes. Their definition is based on a concept of pointwise-maximum that is analogous to that used from the theory in finite-dimensional spaces for the univariate and multivariate max-stable distributions.

Definitions and main results

Let $\{Y(x)\}_{x\in\mathcal{X}}$ be a stochastic process defined on $\mathcal{X} \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, with continuous sample paths. Assume that n iid copies of it are available, denoted by Y_i with $i = 1, \ldots, n$ and $n \in \mathbb{N}$. We stress, that with x one may indicate a spatial (or temporal) index while i denotes the independent replications. We define the process $\{M_n(x)\}_{x\in\mathcal{X}}$ as the pointwise maximum of the underlying processes Y_i, that is $M_n(x) := \max_{i=1,\ldots,n} Y_i(x)$ for every $x \in \mathcal{X}$. Therefore, the interest is in studying the limiting process $M_n(x)$, for $n \to \infty$, because it may provide an approximate model in order to describe the behavior of extremes. In particular, extreme value theory says that if there exist continuous functions $a_n(x) > 0$ and $b_n(x) \in \mathbb{R}$ for all $x \in \mathcal{X}$, such that

$$Z(x) = \lim_{n\to\infty} \left\{ \frac{M_n(x) - b_n(x)}{a_n(x)} \right\} \tag{1}$$

has non-degenerate marginal distributions for all $x \in \mathcal{X}$, then this defines an *extreme-value process* (e.g. [16], Chapter 9, p. 293–294). Any process Y satisfying the limit (1) is said to lie in the *domain of attaction* of Z. See [16], Chapter 9, p. 311–313, for necessary and sufficient conditions on the law of Y such that Y is in the domain of attraction of Z. Of particular interest are those limiting processes that are *max-infinitely divisible* (max-id) and *max-stable* because they are strongly connected with the class of limits (1). A stochastic process Z is named max-id if for every $n \in \mathbb{N}$, it can be represented as the pointwise maximum of n iid stochastic processes (e.g. [2]). A stochastic process Z is named max-stable if for every $n \in \mathbb{N}$ the rescaled pointwise maximum of n iid replicates of Z, has the same law as Z. Formally, if there are sequences of functions $a_n(x) > 0$ and $b_n(x) \in \mathbb{R}$ such that for any $n \in \mathbb{N}$

$$\max_{i=1,\ldots,n} \left\{ \frac{Z_i(x) - b_n(x)}{a_n(x)} \right\}_{x \in \mathcal{X}} \stackrel{\mathcal{D}}{=} \{Z(x)\}_{x \in \mathcal{X}},$$

then Z is max-stable, where Z_i are iid copies of it and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Loosely speaking, the probability of Z is invariant under the maximum operation apart from location and scale factors. Clearly every max-stable process is max-id. Max-stable processes possess the following important features (see [13]):

(i) All its univariate marginal distributions belong to *the generalized extreme-value* (GEV) class of distributions, that is

$$\mathbb{P}(Z \leq y) = \exp\left\{ - \left[1 + \xi \left(\frac{y - \eta}{\varphi} \right) \right]_+^{-1/\xi} \right\}, \quad -\infty < y, \eta, \xi < \infty, \, \varphi > 0, \tag{2}$$

where $(x)_+ = \max(x, 0)$ and η, σ and ξ represent respectively the location, the scale and the shape of the distribution (see [16], Chapter 1, p. 6–12).

(ii) All its p-dimensional distributions ($p \geq 2$) are multivariate max-stable distributions. These distributions with common unit Fréchet margins, that is $\mathbb{P}(Y \leq y) = \exp(-1/y)$ for $y > 0$, admit the representation

$$H(y) = \mathbb{P}(Z_1 \leq y_1, \ldots, Z_p \leq y_p) = \exp\{-L(y)\}, \quad L(y) = \int_{S_{p-1}} \max_{j \in I} \left(\frac{w_j}{y_j} \right) d\nu(\mathbf{w}) \tag{3}$$

for all $y = (y_1, \ldots, y_p) \in \mathbb{R}_+^p$, where $I := 1, \ldots, p$ is the index set. L named the *exponent measure* function represents the dependence between the p components. The exponent function depends on an arbitrary finite measure ν, named the *spectral measure*, defined on the $(p-1)$-dimensional simplex $S_{p-1} = \{w = (w_1, \ldots, w_p) : w_1 + \cdots + w_p = 1, w_j \geq 0, j \in I\}$ but with the constraint that it must satisfy the p moment conditions $\int_{S_{p-1}} w_j \, d\nu(w_1, \ldots, w_p) = 1$, with

$j \in I$. This guarantees that the margins are unit Fréchet and implies that $\nu(S_{p-1}) = p$. This characterization stems from the fact that max-stable distributions are max-id and therefore they admit a compound Poisson representation, see [29], Chapter 5, p. 251–274 and [18], Chapter 4, p. 140–147. Assuming different types of common margins (e.g., exponential) leads to slightly different characterizations of (3), see [27], [18], Chapter 4, p. 147–157 and [16], Chapter 6, p. 221–226.

(iii) If Z is a max-stable process then it is an extreme-value process, indeed in the definition of M_n taking $Y = Z$ then (1) holds.

We stress that a max-stable process with common margins can always be obtained by rescaling it. Because

$$\lim_{n\to\infty} n \left\{ 1 - \mathbb{P}\left(\frac{M_n(x) - b_n(x)}{a_n(x)} \le y \right) \right\} = \left(1 + \frac{\xi(x)(y - \eta(x))}{\varphi(x)} \right)^{1/\xi(x)}$$

uniformly for $x \in \mathcal{X}$ and locally uniformly for y with $\xi(x)(y - \eta(x))/\varphi(x) > 0$, then in the case of unit Fréchet margins, because these are members of the GEV family, with the transformation

$$\{Z'(x)\}_{x\in\mathcal{X}} := \left\{ \left(1 + \frac{\xi(x)(Z(x) - \eta(x))}{\varphi(x)} \right)_+^{1/\xi(x)} \right\}_{x\in\mathcal{X}},$$

we obtain a max-stable process with such margins, where $\eta(x)$, $\xi(x) \in \mathbb{R}$ and $\varphi(x) > 0$ are continuous functions in x. Similarly, one can get max-stable processes with Gumbel margins or of other types. Essentially, the class of limiting processes (1) is characterized by the continuous functions $\eta(x)$, $\xi(x)$ and $\varphi(x)$, concerning the marginal distributions, and the spectral measure ν that controls (separately) the dependence structure.

Example 1. Let $\{Y(x)\}_{x\in\mathcal{X}}$ be a Gaussian process with mean zero and unit-variance. Consider $Y_1, \ldots, Y_n$, iid copies of it. Assume that for every pair of points $x_j, x_j + h \in \mathcal{X}$, with $h = x_k - x_j$, the stationary correlation $\rho_n(h) := \mathbb{E}\{Y_n(x_j)Y_n(x_j + h)\}$ becomes stronger with the increasing of the sample size n. This in order to avoid getting a trivial limit distribution as shown by [32]. Specifically we suppose that $\rho_n(h) \to 1$ for $n \to \infty$, so that $4 \log n(1 - \rho_n(h)) \to \lambda(h) \in [0, \infty]$. For all $x \in \mathcal{X}$, select constants $a_n(x) \equiv a_n$, $b_n(x) \equiv b_n$ with $a_n = 1/b_n$ and $b_n = \sqrt{2 \log n} - ((1/2) \log \log n + \log(2\sqrt{\pi}))/\sqrt{2 \log n}$ (e.g. [22]). Then $\{Z(x)\}_{x\in\mathcal{X}}$, the limit in (1), is a stationary max-stable process with univariate Gumbel marginal distributions, that is $\mathbb{P}(Y \le y) = e^{-e^{-y}}$, and

p-dimensional distribution

$$H(y_1,\ldots,y_p)=\exp\left[-\sum_{j=1}^{p}e^{-y_j}\Phi_{p-1}\left\{\left(\frac{\lambda(h)}{2}+\frac{y_k-y_j}{\lambda(h)}\right)_{k\in I_j};\bar{\Lambda}_j\right\}\right]. \tag{4}$$

In particular $I_j := I\setminus\{j\}$, Φ_{p-1} is a $p-1$-dimensional Gaussian distribution with zero mean and $(p-1)\times(p-1)$ partial correlation matrix $\bar{\Lambda}_j$, whose entries (k,k), (k,r) for all $k,r\in I_j$ are equal to $\lambda_{kk|j}=1$ and

$$\lambda_{kr,|j}=\frac{\lambda_{kj}^2+\lambda_{rj}^2-\lambda_{kr}^2}{2\lambda_{kj}\lambda_{rj}},\quad k\neq j, r\neq j.$$

The parameters $\lambda(h)$ control the pairwise dependence between the couples $(Z(x_j),Z(x_j+h))$ with $j\in I$ and $h\in\mathcal{X}$. Complete dependence is given by the boundary case $\lambda(h)=0$, the dependence decreases for increasing values of λ and independence is obtained for the boundary case $\lambda(h)=\infty$.

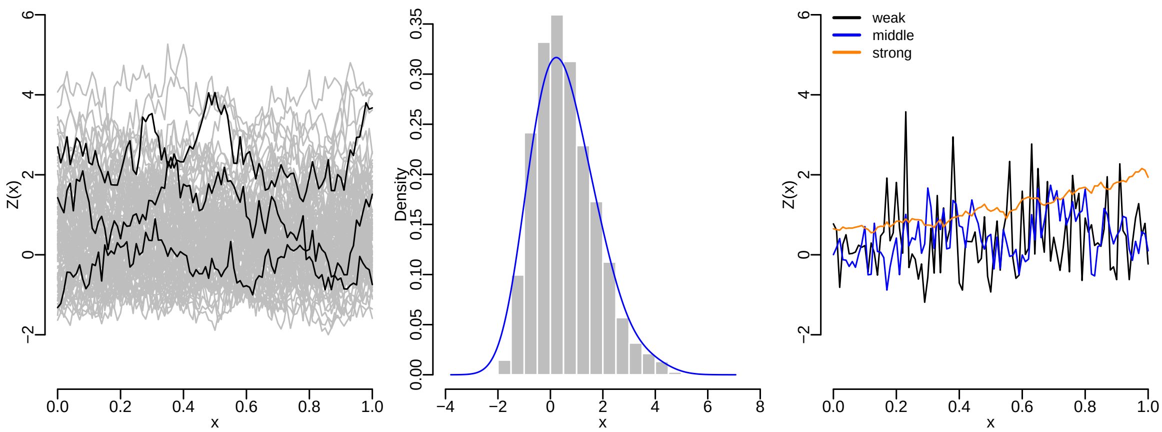

Figure 1 Max-stable realizations based on underlying Gaussian processes, as described in Example 1. The left panel illustrates the behavior of the rescaled maximum process (100 realizations) based on a sample size $n=10^5$. The middle panel shows the histogram of the marginal observations and the solid line is the kernel estimate of the density. The right panel displays three realizations of a max-stable process corresponding to different levels of the dependence structure (driven by λ): weak, middle and strong.

This result was first proved by [20] (see also [18], Chapter 4, Example 4.14) assuming that $\mathcal{X}$ is finite, $\lambda(h)<\infty$ for all $x_j, x_j+h\in\mathcal{X}$ and with a slightly different parameterization. Subsequently it was extended by [22] to a more general case, see also [24] and [4] for related results. Note, that in (4) we have adopted the simplified representation suggested by [28]. Figure 1 illustrates some aspects of the process considered in Example 1 but defined for simplicity on $\mathcal{X} = [0, 1]$. In particular, the left panel shows 100 simulations (gray lines) of the rescaled maximum of underlying iid Gaussian processes with sample sizes $n = 10^5$ and mild dependence structure. The black lines highlight the paths for three selected realizations. The middle panel depicts the distribution of the margins which is consistent with the limiting standard Gumbel distribution, indicating that convergence has taken place. The right panel shows the path of the max-stable process for different levels of the dependence. We can see that when the dependence is weak (solid black line), then the path is rough because the process assumes quite different values also between neighboring points. Conversely, when the dependence is strong (orange solid line), then the path is very smooth because the process takes similar values over all the domain.

Stationary max-stable processes may be ergodic and mixing. General criteria are provided by [23] (see also [34]) in order to verify if these properties are satisfied. Specifically, consider a max-stable process Z with unit Fréchet margins and let $\chi(h) = \lim_{y\to\infty} \mathbb{P}(Z(x_j + h) > y|Z(x_j) > y)$ be the pairwise coefficients of upper tail dependence (e.g. [32];[14]). If the couples $(Z(x_j), Z(x_j + h))$ are asymptotically independent for large enough h, that is $\lim_{\|h\|\to\infty} \chi(h) = 0$, then Z is mixing, and if $t^{-1} \lim_{t\to\infty} \int_0^t \chi(h)dh = 0$ then Z is ergodic.

Finally, [21] has extended the result of Example 1 for spatial-temporal Gaussian processes, taking into account also time dependence. In this case, $Y(x, t)$ is a zero-mean, unit-variance space-time Gaussian process with stationary correlation $\rho_n(h, u) := \mathbb{E}\{Y_n(x_j, t_j)Y_n(x_j + h, t_j + u)\}$, where $t > 0$ is a continuous time index and $u = t_k - t_j$ with $t_j, t_k > 0$. Because we no longer have iid copies of a spatial process $Y_i(x)$, the pointwise maximum is over a dependent sequence, that is $M_n(x) := \max_{0\le t\le n} Y(x, t)$. However, restricting the correlation for large gaps ($|u| \to \infty$), assuming appropriate conditions and considering suitable normalizing sequences, [21] has shown that the rescaled maximum process has the same limiting process as Example 1 (in the sense of the finite-dimensional distributions).

Spectral representation and simulations

The spectral representation is a powerful tool for understanding the features of sample paths of stationary processes; an example widely discussed is the case of stationary Gaussian processes (e.g.

[1]). A useful aspect of such a representation is that realizations from stationary processes can be simulated with a low computational cost. de Haan in his seminal work [13] has provided a spectral representation for max-stable processes (see also [16], Chapter 9).

Specifically, consider the discrete and counting measures defined by

$$\mathbb{I}_{\{P_i\}}(A) = \begin{cases} 1, & \text{if } P_i \in A, \\ 0, & \text{if } P_i \notin A, \end{cases} \qquad N(\cdot) := \sum_{i=1}^{\infty} \mathbb{I}_{\{P_i\}}(\cdot),$$

so that $N(A)$ is the random number of points falling in a bounded set $A \subset \mathcal{A}$. Let $P_i := \{W_i, U_i\}_{i\geq 1}$ be an enumeration of points of a non-homogeneous Poisson process N, on the product space $\mathcal{A} := \mathcal{X} \times (0, \infty)$, with intensity measure $d\mu(w, u) = \nu(dw) \times u^{-2}du$, where ν is a positive finite measure. Let also $f(w; x)$ be a family of measurable functions, named *spectral functions*, such that: $\forall\, w \in \mathcal{X},\ f(w; x) : \mathcal{X} \to [0, \infty)$,

$$\int_{\mathcal{X}} f(w; x)\nu(dw) = 1, \quad \forall\, x \in \mathcal{X}, \qquad \int_{\mathcal{X}} \sup_{x\in\mathcal{X}} f(w; x)\nu(dw) < \infty.$$

Then, the stochastic process

$$Z(x) := \max_{i=1,2,\ldots} \{U_i\, f(W_i, x)\}, \qquad x \in \mathcal{X}, \tag{5}$$

is a max-stable with unit Fréchet margins (see [13];[16], Chapter 9, pp. 302–306 and pp. 314–320). The finite-dimensional distribution of (5) concerns the probability of the event $\{Z(x_j) \leq y_j; y_j > 0, \forall j \in I\}$ and it corresponds to $\mathbb{P}(N(\bar{A}) = 0)$, where $A := \{(u_i, w_i) : u_i f(w_i; x_j) \leq y_j \,\forall\, j \in I\}$. Hence, to derive such probability it comes to finding the product measure $\mu(\bar{A}) = \int_{\mathcal{X}} \int_0^{\infty} \mathbb{I}\{u > \min_j y_j/f(w, x_j)\}\, u^{-2}du\, \nu(dw)$ of the Poisson process and from which we get in a few steps, see [13], the following result

$$\mathbb{P}(Z(x_j) \leq y_j, \forall j \in I) = \exp\left[-\int_{\mathcal{X}} \max_{j\in I} \left\{\frac{f(w; x_j)}{y_j}\right\} \nu(dw)\right]. \tag{6}$$

For practical purposes in order to obtain manageable models, we can restrict ourselves to the subclass of stationary max-stable processes. For simplicity, in formula (6) let us consider the Lebesgue measure for ν and some standard classes of density functions like the Gaussian, Student-t or Laplace for f (see [15]).

Example 2. Define $f(w; x_j) := \phi_d(w - x_j; \Omega)$, the d-dimensional Gaussian density with zero-mean and $d \times d$ covariance matrix Ω. We name the resulting stationary process the *Gaussian extreme value* process. For $p \leq d + 1$, then the distribution (6) takes the expression

$$H(y_1, \ldots, y_p) = \exp\left[-\sum_{j=1}^{p} \frac{1}{y_j} \Phi_{p-1}\left\{\left(\frac{\lambda(h)}{2} + \frac{\log(y_k/y_j)}{\lambda(h)}\right)_{k \in I_j}; \bar{\Omega}_j\right\}\right], \tag{7}$$

where $\lambda(h) = \sqrt{h^T \Omega^{-1} h}$ is a dependence parameter, $\bar{\Omega}_j = \mathrm{diag}(\Omega_j)^{-1}\Omega_j$ is a $d \times (p-1)$ matrix with $\Omega_j = (x_j 1_{p-1}^T - X_j)^T \Omega^{-1}(x_j 1_{p-1}^T - X_j))$, $X_j = \{x_k\}_{k \in I_j} \in \mathbb{R}^{d \times (p-1)}$ and $1_{p-1} = (1, \ldots, 1)^T$ is the vector of $p - 1$ ones, see [19]. In this example given the specific form of $\lambda(h)$, the dependence between the components $Z(x_1), Z(x_2), \ldots$ of the process is controlled by the covariance matrix Ω of the Gaussian density. The bivariate case of distribution (7) was derived originally from [33] who also suggested the use of this process for modeling spatial extremes. Within this setting, several models can be obtained for different types of applications: if $d = 2$ we get bivariate or trivariate spatial or bivariate spatial-temporal models, if $d = 3$ we get up to quadrivariate spatial or trivariate spatial-temporal models. However, to date the analytical expression of the associated density function is known at the most for a trivariate spatial or bivariate spatial-temporal model, see [19].

Example 3. Define $f(w; x_j) := t_2(w - x_j; \Omega)$, the 2-dimensional standard Student-t density with $2(\alpha - 1)$ $(\alpha > 1)$ degrees of freedom and scale matrix Ω, where $\omega_{11} = \omega_{22} = \omega$ and $\omega_{12} = 0$. We name the resulting stationary process the *Student-t extreme value* process. Then, [15] showed that (6) in the bivariate case takes the expression,

$$H(y_j, y_k) = \exp\{-L(y_j, y_k)\}, \qquad L(y_j, y_k) = \begin{cases} 1/y_k, & 0 < y_k < r_+^{-\alpha} y_j, \\ \frac{\mathbb{P}\{T \in B_1\}}{y_j} + \frac{\mathbb{P}\{T \in B_1^c\}}{y_k}, & r_+^{-\alpha} y_j \leq y_k < y_j, \\ \frac{2\mathbb{P}\{T_1 \leq \|h\|/2\}}{y}, & y_j = y_k =: y, \\ \frac{\mathbb{P}\{T \in B_v^c\}}{y_j} + \frac{\mathbb{P}\{T \in B_v\}}{y_k}, & y_j < y_k < r_-^{-\alpha}, \\ 1/y_j, & y_k \geq r_-^{-\alpha} y_j, \end{cases}$$

where $T := (T_j, T_k)$ is a random vector with t-distribution and

$$r_{\pm} = 1 + z/2 \pm \sqrt{z(1+z/4)}, \qquad z = \frac{\|h\|^2}{2(\alpha-1)\omega^2},$$

$$B_v = \left\{ u \in \mathbb{R}^2 : \|z_v\|^2 \leq \frac{v\|h\|^2}{(1-v)^2} - 2(\alpha-1)\omega^2 \right\}, \qquad z_v = u - \frac{hv}{1-v}, \quad v = \left(\frac{y_j}{y_k}\right)^{1/\alpha}.$$

In this example, the dependence between the components of the process is controlled by the scale matrix Ω and the degrees of freedom α of the t density.

We refer to [15] for another example based on the Laplace density. The dependence structure of these max-stable processes is based on the dependence of the underlying spectral functions, which in these examples are probability density functions, where their contribution in (5) is deterministic. However, definition (5) can be extended considering a measurable random spectral function, see [30]. More precisely, let $\{W(x)\}_{x\in\mathcal{X}}$ be a non-negative random process on $\mathcal{X}$, such that for a suitable measure ν we have

$$\int W(x)\,\nu(dw) = \tau \in (0,\infty),\ \forall\, x \in \mathcal{X}, \qquad \int \sup_{x\in\mathcal{X}} W(x)\,\nu(dw) < \infty,$$

and consider $W_1, W_2, \ldots$, iid copies of it. Let $P_i := \{U_i\}_{i\geq 1}$ be points of a non-homogeneous Poisson process on $(0,\infty)$ with intensity measure $d\mu(u) = \tau^{-1}u^{-2}du$, then

$$Z(x) := \max_{i=1,2,\ldots} \{U_i\, W_i(x)\}, \qquad x \in \mathcal{X}, \tag{8}$$

is a stationary max-stable process with unit Fréchet margins, see also [16], Chapter 9, pp. 307–308. In definition (8) the max-stable process is based on independent processes with the same correlation, which control the dependence structure of the process.

Example 4. Define $W(x) := \max\{0, Y(x)\}$, the positive part of a stationary Gaussian process, with mean zero , unit-variance and correlation $\rho(h)$. In this case $\tau = 1/\sqrt{2\pi}$. With this particular choice [30] has defined, applying (8), the stationary process $Z(x)$ named the *extremal Gaussian* and has shown that the finite-dimensional distribution (6) in the bivariate case takes the expression,

$$H(y_j, y_k) = \exp\left[-\frac{1}{2}\left(\frac{1}{y_j}+\frac{1}{y_k}\right)\left\{1+\left(1-\frac{2y_jy_k(\rho(h)+1)}{(y_j+y_k)^2}\right)^{1/2}\right\}\right].$$

The dependence of $Z(x)$ is controlled by the correlation of the underlying Gaussian process $Y(x)$. Specifically, the random pair $(Z(x_j), Z(x_k))$ for $\rho(h) = -1$ is independent, instead for $\rho(h) > -1$ is positively dependent and finally for $\rho(h) = 1$ is completely dependent.

Example 5. Define $W(x) := \exp\{Y(x) - \omega(x)/2\}$, the exponent of a Gaussian process with mean zero, variance $\omega(x) = \mathrm{Var}(x)$, stationary increments and from which has been subtracted a suitable drift term. Hence, in this case $\tau = 1$. Stationarity of the increments $\{Y(x_j+h)-Y(x_j)\}_{h\in\mathcal{X}}$ (meaning that their distribution depends only on h and not on the choice of x_j) implies that the distribution of $Y(x)$ is completely characterized by the variogram $2\gamma(h) = \mathbb{E}(Y(x_j+h)-Y(x_j))^2$. With this setting [24] have named *Brown–Resnick* (associated to the variogram 2γ) the resulting family of stationary processes (8). They have shown that the distribution (6) also depends only on the variogram γ, taking the form

$$H(y_1,\dots,y_p) = \exp\left[-\sum_{j=1}^{p}\frac{1}{y_j}\Phi_{p-1}\left\{\left(\frac{\sqrt{2\gamma(h)}}{2}+\frac{\log(y_k/y_j)}{\sqrt{2\gamma(h)}}\right)_{k\in I_j};\bar{\Lambda}_j\right\}\right]. \qquad (9)$$

Observe, that taking the transformation $y \mapsto \log(y)$ we can switch from (8) to the representation used by [24], who considered a Poisson process on $\mathbb{R}$ with intensity $e^{-u}du$, hence deriving a max-stable process with Gumbel margins and distribution (4), but where $\lambda(h) = \sqrt{2\gamma(h)}$. So (4) and (9) are equivalent but the former is stated on the Gumbel scale and the latter on the Fréchet. The Brown–Resnick is quite a wide class of processes. Indeed, if Y is also stationary and γ is bounded, then we obtain the family presented in Example 4. If Y is a fractional Brownian motion with $Y(0)=0$ and $\gamma(h) = \|h\|^\alpha$ for some $\alpha \in (0, 2]$, then we obtain a family of isotropic processes. In particular, when $\alpha = 1$ and $x \in \mathbb{R}$ then Y is a standard Brownian motion and we obtain the process introduced by [4] (see also [17]). When $\alpha = 2$, then we obtain the family described in Example 2 for the particular case that the matrix Ω is equal to the identity. Finally, [24] (see also [22]) have shown that a Brown–Resnick process (associated to the variogram 2γ) can be obtained, as described in Example 1, taking a zero-mean, unit-variance Gaussian process $\{Y(x)\}_{x\in\mathcal{X}}$ with correlation function

$\rho_n(h) = \exp(-2\gamma(h)/4\log n)$, $h \in \mathcal{X}$. Thus, considering $\gamma(h) = \|h\|^\alpha$ then Y has a stable type correlation or in other words a Gaussian process with stable correlation $\rho_n(h)$ has approximately the semivariogram $\|h\|^\alpha$. Finally, [23] have provided conditions for the ergodicity and mixing of Brown–Resnick processes. The latter is satisfied iff $\lim_{\|h\|\to\infty}\gamma(h) = \infty$ given that $\chi(h) = \Phi(-\sqrt{\gamma(h)/2})$. Furthermore, [23] have also shown that there are ergodic, but not mixing Brown–Resnick processes.

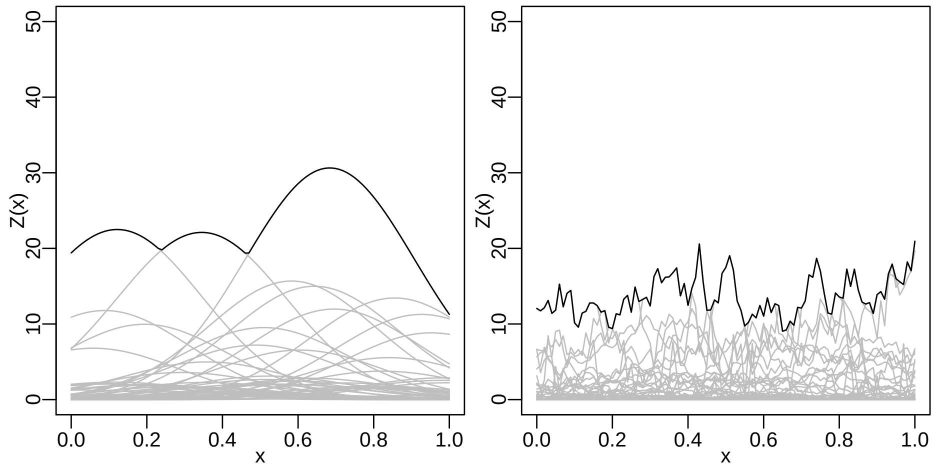

Figure 2 Graphical illustration of the spectral representation mechanism. The left panel shows the maximum of finitely many Gaussian densities (see Example 2) obtained according to construction (5). The right panel displays the maximum of finitely many iid Gaussian processes (see Example 4) obtained according to construction (8).

Figure 2 illustrates graphically the spectral representation in the simple case that $\mathcal{X} = [0, 1]$. The left panel displays a max-stable path obtained by (5) and with the spectral function of Example 2. The grey solid lines symbolize many evaluations of the Gaussian density with variance $\omega = 0.05$ rescaled by Fréchet random factors and the solid black line depicts the pointwise maximum computed over all these replications. Observe that increasing ω we obtain a smoother path of the process because of its stronger dependence, whereas decreasing ω decreases the dependence of the process and its path appears to be much rougher. The right panel shows a max-stable path obtained with (8) and the spectral function of Example 4. The grey lines display the positive part of Gaussian process replications with stable correlation function (with scale and power parameters equal to 1 and 0.8) rescaled by Fréchet random factors. The resulting max-stable process is delineated by the black lines. In this case increasing or decreasing the parameters of the correlation function we can obtain processes with smoother or rougher paths and hence with stronger or weaker dependence. However, with definition (8), we obtain more flexible processes as a result of the more complex dependence structure allowed from the stochastic spectral functions. Simulations of max-stable processes based

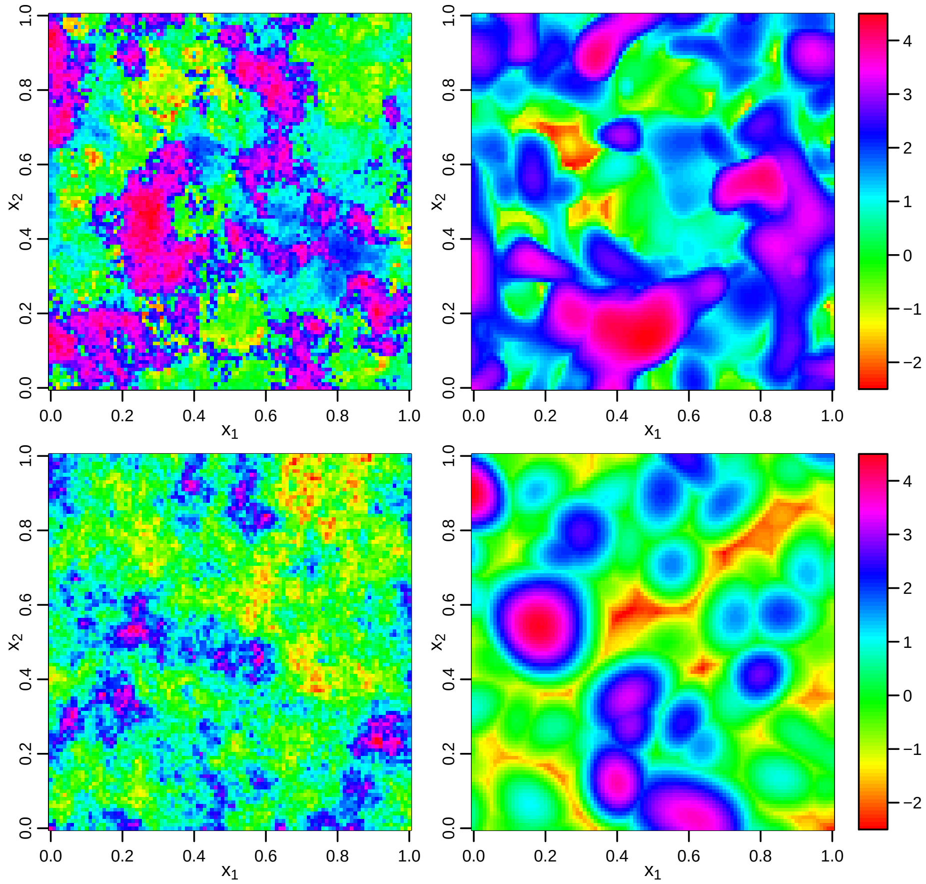

Figure 3 Simulations of max-stable processes on $\mathcal{X} = [0,1]^2$. The panels display realizations attained from an extremal Gaussian process (top panels) and a Brown–Resnick process (bottom panels). The top panels are obtained using a stable correlation function $\exp(-(\|h\|/\beta)^\alpha)$. The bottom are obtained using a semivariogram $(\|h\|/\beta)^\alpha$. The parameters are set to $\alpha = 1$ for the left panels, $\alpha = 2$ for the right panels and $\beta = 0.5$. The processes are plotted with standard Gumbel margins

on (5) and (8), as those of Figure 2, in practice are necessarily obtained by a finite number of points of the Poisson process N. Because N with intensity measure $d\mu(u) = u^{-2}du$ can be generated by the points $U_i = (\sum_{m=1}^{i} E_i)^{-1}$, with E_i exponentially distributed, that is $\mathbb{P}(Y \leq y) = 1 - e^{-y}$, and since these form an increasing sequence so that U_i goes to zero rapidly for increasing i, then the maximum of only finitely many terms may be sufficient. For practical purposes one can follows this simple algorithm based on the stopping rule proposed by [30]: (i) generate $\{E_i, W_i\}$, (ii) compute $Z(x) = \max_{m=1}^{i}\{U_m W_m(x)\}$ (or $Z(x) = \max_{m=1}^{i}\{U_m f(W_m, x)\}$), repeat steps (i) and (ii) while the condition $Z(x) \geq CU_i$ is not satisfied, where C is a positive constant. If Y is uniformly bounded by C then the simulation is exact, otherwise an approximate simulation can be obtained selecting C such that $\mathbb{P}(Y(x) > C)$ is small. We refer to [25] for a discussion on the simulation of the Brown–Resnick process.

Figure 3 shows some realizations from the max-stable processes of Examples 4 and 5. Those on the top panels are generated from an extremal Gaussian process and those on the bottom from a Brown-Resnick process. Specifically, the simulations of the top left and right panels are attained using a stable correlation function $\exp(-(\|h\|/\beta)^{\alpha})$ while for the bottom left and right panels using the semivariogram $(\|h\|/\beta)^{\alpha}$. In both cases we set respectively the parameters $\alpha = 1$, $\alpha = 2$ and $\beta = 0.5$ We can see comparing the top with the bottom panels that for the same parameter values the extremal Gaussian process involves a stronger dependence than the Brown–Resnick. Instead comparing the left with the right panels we can see that for increasing values of α the dependence structure in both cases increases.

Applications

For practical purposes the utility of max-stable processes is that apart from the parameters that characterize the upper tail of the marginal distributions, they allow, through the parameters of some dependence functions, the modeling of the spatial dependence at extreme levels. We focus our attention on the case where $\mathcal{X} \subset \mathbb{R}^2$. For instance, Examples 2-5 are concrete max-stable processes that provide simple models for statistics of spatial extremes that can be used for applications. These models depend on few parameters that measure the strength of the tail dependence as a function of the distance between locations. A simple way to summarize the dependence between extremes is through the *extremal coefficient* (e.g. [33];[31]). In particular, let Z be a stationary max-stable process with unit Fréchet margins. Then a measure of extremal dependence for Z at a pair of locations separated by h is given by the extremal coefficient function, $\theta(h) \in [1,2]$, that comes from the condition

$$\mathbb{P}(Z(x_j) \leq y, Z(x_j+h) \leq y) = \exp\left(-\frac{\theta(h)}{y}\right), \quad \theta(h) = \int_{\mathcal{X}} \max\left\{f(w;x_j), f(w;x_j+h)\right\} \nu(dw), \quad \forall y > 0,$$

where f is the spectral function either of (5) or (8). We recall that the coefficient of upper tail dependence is related to the extremal coefficient function by $\chi(h) = 2 - \theta(h)$. For every fixed h, $(Z(x_j), Z(x_j+h))$ are asymptotically independent if $\theta(h) = 2$ ($\chi(h) = 0$) and completely dependent if $\theta(h) = 1$ ($\chi(h) = 1$), instead they may become asymptotically independent as $\|h\| \to \infty$. With the models of Examples 2 and 5 we obtain $\theta(h) = 2\Phi(\lambda(h)/2)$, where respectively $\lambda(h) = \sqrt{h^T\Omega^{-1}h}$ and $\lambda(h) = \sqrt{2\gamma(h)}$, and thus $\theta(h) \to 2$ as $\|h\| \to \infty$ iff $\lambda(h) \to \infty$, where in the latter case this means that the variogram needs to be unbounded. Whereas with the model of Example 4 we obtain $\theta(h) = 1 + \sqrt{(1-\rho(h))/2}$ and thus $\lim_{\|h\|\to\infty} \theta(h) \leq 1 + 1/\sqrt{2}$, then the asymptotic independence can not be reached.

Therefore, if daily or hourly data of rainfalls, temperatures or other environmental quantities are recorded at several sites spread over a region and for a certain time-period, then sequences of block-maxima (the maximum on a time-window such as a month, a year, etc) can be calculated, one for each site, and one of the models here described can be fitted to the data (e.g., annual maxima). Once the parameters that control the dependence have been estimated, such as the matrix Ω for model (7), the coefficients of some correlation function $\rho(h)$ for model (4) or the coefficients of the semivariogram $\gamma(h)$ for model (5), the extremal coefficient function can be determined and the tail dependence as a function of the distance can be assessed. If the hypothesis of stationarity in space is reasonable then the dependence for distances between locations for which data have not been observed can be interpolated.

However, extreme value problems also typically concern the extrapolation of larger values of those observed from the data of the available period. This has to do with the computation of the *return level*, $y_T(x)$, associated with the *return period*, $T > 0$, namely the value that satisfies the equation $\mathbb{P}(Z(x) > y_T(x)) = 1/T$ for each $x \in \mathcal{X}$ (e.g [7], p. 49). Since the margins are GEV distributed, then the functions $\eta(x)$, $\psi(x)$ and $\xi(x)$ need to be estimated in order to derive the return levels. [26] have discussed an estimation method for the joint estimate of the marginal and dependence parameters with max-stable processes. Also they have proposed using the information on the dependence to calculate the *conditional* return values, so that given a fixed extreme event in a particular site one can estimate extreme events for other sites (possibly more extreme than the observed maxima). Alternatively, [5] and [11] proposed to determine extreme quantiles using max-stable processes and simulations. Estimation methods for max-stable processes, based on the likelihood have been discussed by [26] and [11], whereas other approaches have been discussed by [33], [31], [15] and [10]. A review of different methods has been provided by [12]. Applications of max-stable processes to rainfall problems have been discussed by [33], [6], [8], [31], [5] and [26], while on extreme temperatures by [11], extreme snow depths by [3] and windspeeds by [9].

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