

# Public Contracting in Delegated Agency Games

(preliminary and incomplete)

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**Abstract:** This paper studies games of delegated common agency under asymmetric information. Using tools from non-smooth analysis and optimal control, we derive best responses and equilibria under weak conditions on equilibrium schedules. Equilibrium inefficiencies arise from two sources: inefficient contracting by a given coalition of active principals and inefficient participation (insufficient activity) by principals. Particular attention is given to the continuity of the equilibrium allocation, a property which is directly related to the mode of competition that emerges among principals. Continuous allocations and contribution schedules arise when competition is soft and principals have overlapping activity sets in the agent's type space. Discontinuities exist instead when "head-to-head" competition occurs and principals battle for exclusive relationships. These findings are illustrated by means of two examples of independent economic interest: a game of voluntary contributions for a public good and a lobbying game between polarized interest groups attempting to influence the policy chosen by a political-decision maker.

**Keywords:** Delegated common agency, asymmetric information, public goods, lobbying games.

## 1 Introduction

**Overview.** This paper studies games of public delegated common agency under asymmetric information. Consider several principals offering contributions to an agent who produces a public good on their behalf and is privately informed on his production cost.

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Under public common agency, those contributions stipulate how much the agent should be paid for any level of the public good he may choose. In a delegated common agency setting, the agent can select any subset of those offers and then choose which level of public good to produce.<sup>1</sup> We are interested in characterizing equilibria of those games and their important properties.

Thanks to earlier effort,<sup>2</sup> much properties of those delegated common agency games are by now known when there is complete (or symmetric but incomplete) information on the agent's preferences. The existing literature pointed out two important lines of results. First, common agency games have efficient equilibria. Second, the principals' contributions in those equilibria are "truthful", i.e., they reflect the principals' preferences among alternatives. Intuitively, such a truthful schedule maximizes the payoff of any bilateral coalition between the agent and the corresponding principal because the former is made "residual claimant" for that coalition's payoff. A lump-sum payment can then be used to extract the agent's surplus and leave him just indifferent between accepting that principal's contract and his next best option, i.e., contracting only with all other remaining principals.

Under asymmetric information, contribution schedules not only serve to "pass" the principals' preferences onto the agent but they are also used as screening devices. All principals want now to elicit the agent's private information and they do so non-cooperatively. When designing his best response to others' contributions, a given principal trades off bilateral efficiency for the coalition he forms with the agent on the one hand and the information rent that the agent withdraws from his private information on the other hand. This idea is of course well-known from monopolistic screening environments.<sup>3</sup> However, under delegated common agency, this trade-off is modified in two important ways.

First, bilateral efficiency in a given principal-agent pair takes as given the other principals' contributions. Because of asymmetric information, those contributions are no longer truthful and also distorted for incentive reasons. This implies that bilateral efficiency in each principal-agent pair does not necessarily implies overall efficiency from the point of view of the grand-coalition.

Second, the nature of the incentive distortion induced in any bilateral relationship depends also on the competitive environment. Indeed, when the agent refuses to deal with a given principal but still contracts with others, he obtains a type-dependent reservation payoff that affects contracting with that principal. Incentives whether to exaggerate

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<sup>1</sup>This added possibility to select the set of principals whose offers are accepted distinguishes delegated from intrinsic common agency. Intrinsic common agency is studied with more care in Martimort and Stole (2009a). Public common agency should be distinguished from private common agency where different principals contract on different specific variables under the agent's control. See Martimort (2007) for a definition of private and public common agency.

<sup>2</sup>Bernheim and Whinston (1986); Grossman and Helpman (1994), Dixit and al. (1997) Laussel and Lebreton (1998, 2001), Chiesa and De Nicolo (2009) among others.

<sup>3</sup>See for instance Laffont and Martimort (2002).

or underestimate his own type in that bilateral relationship depends now both on how difficult the agent finds it to please that principal but also on how such manipulations makes the agent look easier to buy or not for that principal.

The broad objective of this paper is to study equilibrium contracts in such competitive screening environments. Doing so, the analysis of public delegated agency games unveils a whole set of new issues that cannot be touched upon in complete information models.

The first set of issues is related to the kind of inefficiencies that arise at equilibrium. How are the distortions induced by competing principals compounded at equilibrium? Given that the equilibrium outcome is inefficient, can we predict the directions of those distortions by simply looking a priori at the principals' preferences and, more generally, what are the main determinants of those distortions?

The second set of closely related issues deals with the shape of activity sets of competing principals. What is the subset of types that are targeted with positive contributions by a given principal? Under which circumstances do we have overlapping activity sets for different principals or instead activity sets that remain split apart? In which sense, if any, those patterns of activity sets depend on the principals' preferences for the agent's services and reflect the mode of competition that emerges.

**Motivating examples.** To motivate our theoretical analysis and give an overview of some of our findings, consider the following two archetypical examples.

- *Example 1: Lobbying competition.* Two interest groups (principals) are willing to influence a decision-maker (the common agent) in a highly polarized environment. Principals have conflicting preferences and want to shift the agent's policy in opposite directions. The decision-maker has private information on his bliss point. Our analysis predicts that distortions move the equilibrium policy away from the status quo to favor the closest principal in the ideological space. The equilibrium policy reflects the conflicting forces of competing principals. A given interest group might secure exclusive influence on the decision-maker when their preferences are close to each other. Instead, types with ideal points too far away are too expensive to buy for that interest group.

Importantly, different modes of competition are possible and might even coexist. In one equilibrium, lobbying groups compete "head-to-head" for the agent's services and battle for exclusive market shares in the type space. For another equilibrium, interest groups may instead have overlapping areas of influence with the agent accepting both interest groups' contributions and making policy compromises. In the first equilibrium, contribution schedules and policies are discontinuous. Instead, the second equilibrium has a continuous policy and continuously differentiable contributions.

- *Example 2: Voluntary contributions to a public good.* Consider a standard public good context where an agent who is privately informed on his marginal cost of production produces a public good on behalf of two principals. Those principals have congruent preferences since they both like more public good being produced. They non-cooperatively

offer contributions. Free-riding between those principals takes two forms in that context. First, the principals' marginal contributions are less than their marginal valuations to extract more of the agent's information rent. This induces excessive downward distortions in output compared to a cooperative contracting. Second, there might be limited participation with the weaker principal who is the less eager to contribute, not giving any contribution when the agent is too inefficient.

**Results.** These two examples illustrate several important findings of our more general analysis.

- *Compounding inefficiencies.* For a given coalition of active principals, their non-cooperative behavior implies excessive distortions compared to the cooperative outcome had they merged when contracting. The equilibrium output compounds the distortions that all principals induce. Whether distortions sum up or somewhat cancel each other depends on whether principals have *congruent* or *conflicting* preferences.
- *Non-truthful contributions.* Contributions are no longer truthful under asymmetric information. More precisely, a principal concerned by the agent's incentives to exaggerate his type distorts downward his marginal contribution to make such manipulation less attractive. Instead, a principal who is more concerned by the agent's incentives to underestimate his type contributes more at the margin.
- *Activity sets.* A given principal may find it too costly to offer any contribution for certain types. Indeed, inducing a change in the agent's output requires giving him at least his reservation payoff obtained when contracting with all other active principals. This might be too costly compared with the efficiency gains that this principal may enjoy from such change. Inefficient representation often occurs which, again, stands in contrast with the complete information environment.

Those conclusions are quite robust and hold irrespectively of the mode of competition, i.e., both for continuous equilibria and discontinuous ones when they exist. However, the characterization of those equilibria requires slightly different tools. To characterize continuous equilibria, we build on and broaden the lessons of the literature on type-dependent participation constraints and countervailing incentives in a competitive screening environment.<sup>4</sup> Under weaker technical conditions than those found earlier in the literature and using tools from optimization in non-smooth analysis,<sup>5</sup> we derive conditions under which equilibrium outputs are continuous.

The characterization of the principals' activity sets, i.e., the set of types who strictly gain from contracting with that principal, is also a key aspect of the analysis of delegated

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<sup>4</sup>Lewis and Sappington (1989), Maggi and Rodriguez-Clare (1995), Stole (1995), Jullien (2000).

<sup>5</sup>For introductions and recent developments in the mathematical tools used in non-smooth analysis, we refer to the seminal works by Clarke (1990), Loewen and Rockafellar (1997), Galbraith and Vinter (1997) and Vinter (2000) among others. This part of our analysis is of general interest beyond the characterization of best responses and may be of value for readers interested in principal-agent models with type-dependent participation constraints.

common agency games. Formally, characterizing a principal’s activity set amounts to finding where the agent’s type-dependent participation constraint binds at that principal’s best response to any aggregate offer made by others. This is done by a careful look at the necessary conditions for a best response.<sup>6</sup> At a best response in a continuous equilibrium, a “smooth-pasting” condition holds on boundaries of the principals’ activity sets: contribution schedules are zero both in value and at the margin.

Equilibria with discontinuities require a more specific treatment on boundaries of activity sets where such discontinuity arises. Although “smooth-pasting” no longer holds there, it remains true that the principal’s transfer is equal to the incremental virtual surplus that a discontinuous change in the agent’s output may generate. This condition gives us a broad and encompassing view of contributions and output both in equilibria with and without discontinuities.

**Organization of the paper.** Section 2 reviews the literature. Section 3 presents our model of delegated common agency under asymmetric information. Section 4 set describes the set of incentive feasible allocations for each principal (Section 4.1); discusses the notion of activity sets (Section 4.2) and finally sets up the best-response problem of a given principal as a generalized control problem in non-smooth analysis (Section 4.3). Section 5 provides two examples where the equilibrium policy is discontinuous: a lobbying game characterized by “head-to-head” competition and discontinuities in equilibrium schedules (Section 5.1); and a public good game of voluntary contributions with a zero-one decision on whether to produce or not that public good (Section 5.2). Clearly those two examples show that standard techniques for computing best-responses may implicitly make restrictions on strategy spaces that prevent characterization of such equilibria. Indeed, under a weak condition on contribution schedules that ensures enough convexity in each principal’s problem,<sup>7</sup> necessary conditions for a best response can be derived that imply continuity of the equilibrium schedules (Section 6). Continuous equilibria are then characterized. Section 7 illustrates our findings by deriving continuous equilibria for our public good and lobbying examples above. Our general theory gives a set of tools to straightforwardly derive continuous equilibria in those environments. Section 8 comes back on the analysis of discontinuous equilibria and characterizes contribution schedules and outputs at such discontinuities. Our final Theorem (Theorem 3) summarizes our findings and provides necessary conditions that hold for any monotonic equilibrium be it continuous or not when principals have surplus functions which are linear in the agent’s contractible action and connected activity sets. Proofs are relegated to and Appendix.

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<sup>6</sup>This step might be significantly more complex than in monopolistic screening environments. It is simplified only when there is enough structure on the principals’ preferences so that the modeler can figure out a priori the shape of activity sets. This task is rather easy for the two above examples and, more generally, when principals’ surpluses are linear in output as we shall see below.

<sup>7</sup>Of course, that condition fails in both motivating examples of Section 5. A first failure comes from discontinuity in equilibrium schedules as in Section 5.1. A second possibility is that preferences are not strictly concave and equilibrium choices are bang-bang as in Section 5.2.

## 2 Literature Review

The study of delegated common agency games under asymmetric information and public agency has been initiated in Martimort and Stole (2009b). Our focus there was on studying equilibria sets as the type distribution gets more concentrated to investigate the convergence properties of those sets as one gets closer to complete information. With a sufficiently small uncertainty on types, all principals are always active on the whole (but tiny) type space and the value of a careful study of those sets as developed below disappears. In other words, the only remaining inefficiency that arises for such delegated agency games is due to non-cooperative screening behavior and not to insufficient participation of some principals over some subsets of types. Martimort and Stole (2009c) provided a general analysis of competition with nonlinear prices under both delegated and intrinsic common agency when manufacturers selling differentiated goods may choose to target only some subsets of consumers. This earlier paper focused on the case of private contracting with each principal having a specific contracting variable (the output he sells to the agent) that he uses to screen the buyer's preferences.<sup>8</sup> Here, we are instead interested in public agency environments where all principals use the same screening variable. A second difference is that, in Martimort and Stole (2009c), manufacturers rank the agent's types in the same way, with the agent having the highest valuation for both goods being the most attractive for both manufacturers.<sup>9</sup> This also somewhat limited the study of activity sets that our present paper uncovers. Our analysis below is more general and allows for principals having conflicting preferences over whose agent's type is their most preferred one. This is exemplified by our lobbying game. Biais, Martimort and Rochet (2000) analyzed a model of competing market-makers on financial markets with traders privately informed on their willingness to buy or sell assets in a common value environment with private agency. Because of symmetry, all market-makers have similar activity sets with a hole where traders having a limited valuation for trading in either direction do not trade under asymmetric information. On top of our focus on public agency instead, a major difference is that Biais, Martimort and Rochet (2000) restricted a priori their analysis to the class of convex nonlinear prices, leaving open the question of knowing whether other less smooth equilibria exist.

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<sup>8</sup>Ivaldi and Martimort (1994) and Calzolari and Scarpa (2004) are earlier studies of delegated common agency games with private contracting but those authors restricted the analysis a priori to cases where all types are served.

<sup>9</sup>Mezetti (1997) provided a model with conflicting and differentiated principals but his focus was on an intrinsic common agency setting, putting aside the complete characterization of the activity sets of competing principals.

### 3 The Model

Consider  $n$  principals  $P_i$ , indexed with the subscript  $i \in \{1, \dots, n\} = \mathcal{N}$ . Those principals offer contribution schedules to a common agent who chooses the level of a public good on their behalf.<sup>10</sup> The feasible set for the possible levels of this public good is a closed interval  $\mathcal{Q} \subseteq \mathbb{R}$ . Let  $\mathcal{S}$  be an arbitrary coalition of principals in the power set  $2^{\mathcal{N}}$  and let  $|\mathcal{S}|$  be its cardinal.

#### 3.1 Players and Preferences

Principal  $P_i$ 's preferences are quasi-linear and defined over the level of public good  $q \in \mathcal{Q}$  and the monetary payment made to the common agent  $t_i \in \mathbb{R}$  as:<sup>11</sup>

$$V_i(q, t_i) = S_i(q) - t_i$$

where  $S_i(\cdot)$  is some gross surplus function.

The agent  $A$  has also quasi-linear preferences given by:

$$U(q, \sum_{i=1}^n t_i, \theta) = \sum_{i=1}^n t_i - \theta q + S_0(q) + u_0(\theta)$$

where again  $S_0(\cdot)$  is some gross surplus function. The function  $S_0(\cdot)$  captures the possibility that the agent might also enjoy or dislike the public good.<sup>12</sup> The agent's marginal cost of producing the public good depends on an efficiency parameter  $\theta$ .<sup>13</sup>

The following assumptions guarantee existence of some solutions to the optimization problems investigated below:

**Assumption 1**  $S_i(\cdot)$  for  $i \in \{0, \dots, n\}$  is Lipschitz-continuous on  $\mathcal{Q}$ .

**Assumption 2**  $\sum_{i=0}^n S_i(q) - \theta q$  is strictly concave in  $q$  on  $\mathcal{Q}$ .

Strict concavity ensures that a solution to the complete information problem exists and is characterized, when interior, by means of first-order conditions.

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<sup>10</sup>This public good can be viewed as a public infrastructure of variable size, or it might be given a more abstract interpretation in terms of a policy variable which would affect all principals' payoffs.

<sup>11</sup>Our model is general enough to encompass both the cases where a principal may dislike the public good over some range and where different principals may have conflicting preferences on which level of that public good should be chosen.

<sup>12</sup>At the cost of greater complexity, we could generalize preferences by introducing a type-dependent shift parameter as follows:

$$U(q, \sum_{i=1}^n t_i, \theta) = \sum_{i=1}^n t_i - \theta q + S_0(q) + u_0(\theta).$$

The function  $u_0(\theta)$  is the agent's type-dependent reservation payoff in the absence of any production of the public good.

<sup>13</sup>The quantity  $\theta q$  is a genuine cost function defined on  $\mathcal{Q} \subseteq \mathbb{R}_+$  when the model applies in a public good context. It might also be viewed as an opportunity cost of moving a public policy away from the status quo in a lobbying model.

### 3.2 Information

The agent has private information on the efficiency parameter  $\theta$ . This type is drawn from the set  $\Theta = [\underline{\theta}, \bar{\theta}]$  according to a cumulative distribution  $F(\theta)$  with an everywhere positive, atomless and bounded density  $f(\theta) = F'(\theta)$ . The hazard rates  $R(\theta) = \frac{F(\theta)}{f(\theta)}$  and  $T(\theta) = \frac{1-F(\theta)}{f(\theta)}$  satisfy the following standard assumptions.<sup>14</sup>

**Assumption 3** *Monotone hazard rate properties:*  $\dot{R}(\theta) > 0 > \dot{T}(\theta) \quad \forall \theta \in \Theta$ .

### 3.3 Benchmarks

A first-best optimal level of public good  $q^{FB}(\theta)$  is defined as:

$$q^{FB}(\theta) \in \arg \max_{q \in \mathcal{Q}} \sum_{i=0}^n S_i(q) - \theta q.$$

Assuming that  $q^{FB}(\theta)$  lies in the interior of  $\mathcal{Q}$  and that Assumption 2 holds,  $q^{FB}(\theta)$  is uniquely defined as:

$$\sum_{i=0}^n S'_i(q^{FB}(\theta)) = \theta. \quad (1)$$

For future reference, define the agent's *status quo* payoff  $U_0(\theta)$  and optimal output  $q_0(\theta)$  when principals do not contribute respectively as:<sup>15</sup>

$$U_0(\theta) = \max_{q \in \mathcal{Q}} S_0(q) - \theta q \text{ and } q_0(\theta) \in \arg \max_{q \in \mathcal{Q}} S_0(q) - \theta q.$$

### 3.4 Strategy Spaces

Let denote by  $\mathcal{T}$  the set of non-negative contribution schedules which are upper semi-continuous. Given that  $\mathcal{Q}$  is compact, those contributions are bounded and thus measurable with respect to the  $\sigma$ -field  $\mathcal{B}$  of Borel subsets of  $\mathcal{Q}$ . Unless specified otherwise, this is the strategy space available to each principal  $P_i$ .

We shall sometimes abuse slightly notations and denote the aggregate schedule offered by all principals except  $P_i$  as  $t_{-i}(q) = \sum_{j \neq i} t_j(q)$ .

### 3.5 Timing and Equilibrium

The delegated common agency game  $\Gamma = \langle (V_i(\cdot))_{1 \leq i \leq n}, U(\cdot), \Theta, F(\cdot) \rangle$  unfolds as follows:

1. The agent learns his private information  $\theta$ .

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<sup>14</sup>Bagnoli and Bergstrom (2005).

<sup>15</sup>Assuming strict concavity of  $S_0(\cdot)$  ensures uniqueness of the maximizer. Otherwise,  $q_0(\theta)$  is an arbitrary measurable selection in the correspondence.



2. Principals offer non-cooperatively contributions to the agent. The agent can accept any subset of those contributions.
3. The agent chooses how much to produce and receives accordingly the corresponding payments from the contributing principals whose offers have been accepted.

$\Gamma$  is a public common agency game since all principals observe and contract on the publicly observable decision  $q$ .

We look for pure strategy Perfect Bayesian equilibria with deterministic mechanisms of  $\Gamma$  (in short equilibria) whose definition follows.

**Definition 1** *An equilibrium of  $\Gamma$  is a vector of contribution schedules, a set and an output correspondence  $\{(\bar{t}_i)_{1 \leq i \leq n}, \mathcal{S}(t), q(\cdot|t)\}$ <sup>16</sup> such that:*

- *Given any profile of contributions  $t \in \mathcal{T}^n$ ,  $(\mathcal{S}(t), q(\theta|t))$  consists of the set of principals whose offers are accepted and an output which maximizes the agent's payoff:*

$$(\mathcal{S}(t), q(\theta|t)) \in \arg \max_{q \in \mathcal{Q}, \mathcal{S} \in 2^{\mathcal{N}}} \sum_{i \in \mathcal{S}} t_i(q) - \theta q + S_0(q);$$

- *$\bar{t}_i$  maximizes principal  $P_i$ 's expected payoff given the other principals' aggregate contribution schedules  $\bar{t}_{-i}$ :*

$$\bar{t}_i \in \arg \max_{t_i \in \mathcal{T}} \int_{\Theta} (S_i(q(\theta|t_i, \bar{t}_{-i})) - t_i(q(\theta|t_i, \bar{t}_{-i}))) dF(\theta).$$

It is straightforward to observe that the definition above can be slightly simplified. Given that  $\mathcal{T}$  contains only non-negative contributions, it is always weakly optimal for the agent to accept all contributions (eventually choosing an output that is not rewarded by a non-empty subset of inactive principals), i.e.,  $\mathcal{S}(t) \equiv \mathcal{N}$  in any continuation equilibrium. Much of the interest of the analysis below comes from determining the set of principals who contribute a positive amount for a given type of the agent.

To further simplify presentation, we will omit the dependence of the agent's output on the vector of contributions offered in continuation equilibria and denote  $q(\theta|t) \equiv q(\theta)$ .<sup>17</sup>

**Remark 1** *Of course, the agent can always refuse all contributions, choose the status quo output  $q_0(\theta)$  and get the corresponding payoff  $U_0(\theta)$  which represents his reservation payoff in any equilibrium.*

<sup>16</sup>Sometimes we refer below to those correspondences as a continuation equilibrium.

<sup>17</sup>It is worth noticing that existence of a measurable selector from the non-empty compact values correspondence  $\arg \max_{q \in \mathcal{Q}} \sum_{i \in \mathcal{S}} t_i(q) - \theta q + S_0(q)$  follows from the Measurable Maximum (Aliprantis and Border, 1999, p. 570) when  $\sum_{i \in \mathcal{S}} t_i(q)$  is continuous and will be assumed otherwise.

## 4 Setting the Stage

### 4.1 Incentive Feasible Set

We now characterize the set of incentive feasible allocations available to principal  $P_i$ .

**Definition 2** *A rent/output profile  $(U(\theta), q(\theta))$  is implementable by principal  $P_i$  under delegated common agency when the non-negative aggregate contribution offered by all other principals is  $\bar{t}_{-i} \in \mathcal{T}$  if and only if there exists a non-negative contribution schedule  $t_i \in \mathcal{T}$  such that:*

$$U(\theta) = \max_{q \in \mathcal{Q}} t_i(q) + \bar{t}_{-i}(q) - \theta q + S_0(q) \text{ and } q(\theta) \in \arg \max_{q \in \mathcal{Q}} t_i(q) + \bar{t}_{-i}(q) - \theta q + S_0(q).$$

Let define the information rent  $\bar{U}_{-i}(\theta)$  and the optimal output (or at least a selection within the best-response correspondence)  $\bar{q}_{-i}(\theta)$  that an agent with type  $\theta$  would choose when not taking  $P_i$ 's contribution respectively as:

$$\bar{U}_{-i}(\theta) = \max_{q \in \mathcal{Q}} \bar{t}_{-i}(q) - \theta q + S_0(q) \text{ and } \bar{q}_{-i}(\theta) \in \arg \max_{q \in \mathcal{Q}} \bar{t}_{-i}(q) - \theta q + S_0(q).$$

The next Lemma follows from Definition 2:

**Lemma 1** *Given the non-negative aggregate contribution offered by all other principals  $\bar{t}_{-i} \in \mathcal{T}$ , a rent/output profile  $(U(\theta), q(\theta))$  is implementable by principal  $P_i$  through an aggregate contribution schedule  $t_i \in \mathcal{T}$  if and only if*

1.  $U(\theta)$  is absolutely continuous, convex, and  $q(\theta)$  is decreasing so that both functions are a.e. differentiable with, at any differentiability point,

$$\dot{U}(\theta) = -q(\theta) \tag{2}$$

$$\dot{q}(\theta) \leq 0, \tag{3}$$

2. The agent is as least weakly better off accepting  $P_i$ 's offer

$$U(\theta) \geq \bar{U}_{-i}(\theta) \quad \forall \theta \in \Theta. \tag{4}$$

The rent/output profile  $(\bar{U}_{-i}(\theta), \bar{q}_{-i}(\theta))$  is itself implementable (when  $P_i$  offers a null contribution). Consequently, Item [1.] in Lemma 1 holds also for that profile.  $\bar{U}_{-i}(\cdot)$  is absolutely continuous, convex, and satisfies at any differentiability point

$$\dot{\bar{U}}_{-i}(\theta) = -\bar{q}_{-i}(\theta) \text{ and } \dot{\bar{q}}_{-i}(\theta) \leq 0.$$

Of course, the same remark applies as well to the status quo profile  $(U_0(\theta), q_0(\theta))$  which is obtained when principals offer no contribution.<sup>18</sup>

Because contributions are non-negative, the agent is always at least weakly better off accepting all offers in the delegated common agency game under scrutiny and Item [2.] holds. One important aspect of our analysis is nevertheless to determine precisely the subset of types where the participation constraint (4) binds.

## 4.2 Activity Sets

In this respect, let define the *activity set*  $\Omega_i$  of principal  $P_i$  as follows.

**Definition 3** *Principal  $P_i$ 's activity set is  $\Omega_i = \{\theta \in \Theta \mid U(\theta) > \bar{U}_{-i}(\theta)\}$ .*

Principal  $P_i$ 's contribution is necessarily positive on his activity set. Because with our definition, activity sets are not closed, it might nevertheless be that, at a  $\theta_i$  on the boundary of  $\Omega_i$ , principal  $P_i$ 's payment is not necessarily zero but yet (4) is binding. This would typically be the case when  $P_i$  is “becoming” an active principal with a positive transfer at some  $\theta_i$ .<sup>19</sup>

We refer to  $\Omega_i^c = \Theta \setminus \Omega_i$  as the subset of types where the participation constraint (4) binds. It might be that  $\Omega_i$  is a countably finite collection of connected intervals  $\Omega_i^j$ , namely  $\Omega_i = \bigcup_{j \in \mathcal{J}} \Omega_i^j$ , ( $j \in \mathcal{J}$  where  $\mathcal{J} \subset \mathbb{N}$ ) with the convention that each of those intervals is maximal and the notation  $\theta_i^j = \inf \Omega_i^j$ .

We denote also the set of active principals at a given type  $\theta$  and its complement as respectively  $\alpha(\theta) = \{i \in \mathcal{N} \mid \theta \in \Omega_i\}$  and  $\alpha^c(\theta) = \mathcal{N}/\alpha(\theta)$ . Finally, let  $|\alpha(\theta)|$  be the cardinal of this set.

To classify the different patterns of activity sets under scrutiny, it is useful to define the notions of *monotonic equilibrium*, *congruent* and *conflicting* principals.

**Definition 4** *An equilibrium is monotonic if either  $\Omega_i = [\underline{\theta}, \theta_i)$  or  $\Omega_i = (\theta_i, \bar{\theta}]$  for all  $i$ .*

With *monotonicity*, when a principal starts being active, he is active on the whole right- or left-interval of types.

**Definition 5** *Principals  $P_i$  and  $P_j$  have congruent objectives when  $\Omega_i = [\underline{\theta}, \theta_i)$  and  $\Omega_j = [\underline{\theta}, \theta_j)$ .*<sup>20</sup> *Principals  $P_i$  and  $P_j$  have conflicting objectives when  $\Omega_i = (\theta_i, \bar{\theta}]$  and  $\Omega_j = [\underline{\theta}, \theta_j)$ .*<sup>21</sup>

<sup>18</sup>This property is called “homogeneity” by Jullien (2000).

<sup>19</sup>This arises both in continuous equilibria with full coverage and with discontinuous equilibria. See the examples below.

<sup>20</sup>Alternatively, principals may also be congruent when  $\Omega_i = (\theta_i, \bar{\theta}]$  and  $\Omega_j = (\theta_j, \bar{\theta}]$ , i.e., both principals find it more attractive to contract with the least efficient types. This case does not strike us as being so interesting in view of the applications we develop in Section 7.

<sup>21</sup>This is without loss of generality given that we may permute subscripts for principals.

The case of congruent principals is relevant in settings like the public good example in Section 7.1 below. There, both principals contribute to increase the production of public good although they may differ in terms of their willingness to pay. The case of conflicting principals is relevant for the lobbying example of Section 5.1. There, two interest groups (the principals) influence a policy-maker (their common agent) and want to push policies in opposite directions.

We also sometimes refer to the following case:

**Definition 6** *There is full coverage of the type space when both principals' closure of activity sets cover the whole type space  $\Theta = \bar{\Omega}_i$ , for  $i = 1, 2$ .*

### 4.3 A Generalized Control Problem

Lemma 1 allows us to restate principal  $P_i$ 's optimization problem at a best response to the aggregate schedule  $\bar{t}_{-i}$  offered by other principals as:

$$(\mathcal{P}_i) : \max_{(U,q)} \int_{\Theta} (S_0(q(\theta)) + S_i(q(\theta)) + t_{-i}^*(q(\theta)) - \theta q(\theta) - U(\theta)) f(\theta) d\theta$$

subject to  $q \in \mathcal{Q}$ , (2), (3) and (4).

$(\mathcal{P}_i^r)$  is referred to as the relaxed problem obtained from  $(\mathcal{P}_i)$  by omitting the second-order condition (3). As standard in the screening literature, that latter condition will be checked ex post on the output profile obtained at equilibrium by imposing Assumption 3 on the type distribution and/or further conditions on the principals' preferences.

Introducing the auxiliary variable  $v(\theta) = -q(\theta)$ , (2) can be written as:

$$\dot{U}(\theta) = v(\theta) \tag{5}$$

where  $v(\theta) \in \mathcal{V} = -\mathcal{Q}$ . In the sequel, it is useful to define the *extended-value Lagrangian* for  $(\mathcal{P}_i^r)$ , possibly taking values in  $\mathbb{R} \cup \{+\infty\}$ , as  $L_i(\theta, u, v) = L_i^0(\theta, u, v) + \psi_{\mathcal{V}}(v)$ , where  $L_i^0(\theta, u, v) = -(S_0(-v) + S_i(-v) + \bar{t}_{-i}(-v) + \theta v - u) f(\theta)$  and  $\psi_C(\cdot)$  denotes the indicator function of a given set  $C$ , namely  $\psi_C(q) = \begin{cases} 0 & \text{if } q \in C \\ +\infty & \text{otherwise.} \end{cases}$

Using more compact notations borrowed from recent developments in non-smooth analysis,<sup>22</sup>  $(\mathcal{P}_i^r)$  can be expressed as the following minimization of a *generalized Bolza* problem over the class of absolutely continuous arcs  $U(\cdot)$ :

$$(\mathcal{P}_i^r) : \min_{(U,v)} \int_{\Theta} L_i(\theta, U(\theta), v(\theta)) d\theta \quad \text{subject to } v \in \mathcal{V}, (4) \text{ and } (5).$$

**Definition 7** *A minimizing process is a solution to  $(\mathcal{P}_i^r)$   $(\bar{U}, \bar{v}) : \Theta \rightarrow \mathbb{R} \times \mathcal{V}$  such that  $\bar{U}$  is absolutely continuous.*

<sup>22</sup>Clarke (1990), Loewen and Rockafellar (1997), Galbraith and Vinter (1997), Vinter (2000).

We shall be interested in non-trivial cases where such minimizer exists and denote accordingly by  $V_i > -\infty$  the value of  $(\mathcal{P}_i^r)$ .

We are now ready to characterize the solution to  $(\mathcal{P}_i^r)$  by means of optimal control techniques. To do so, we need a bit more notations. We first define the maximized Hamiltonian as:

$$H_i(\theta, u, p) \equiv \sup_{v \in \mathbb{R}} \{pv - L_i(\theta, u, v)\} = \max_{v \in \mathcal{V}} \{pv - L_i^0(\theta, u, v)\}.$$

That the maximum above is achieved follows from the compactness of  $\mathcal{V}$  and the lower semi-continuity of  $L_i^0(\theta, u, \cdot)$ . Note also that  $H_i(\theta, u, p)$  is convex, closed and proper.

## 5 Motivating Examples

### 5.1 Lobbying for a Public Policy

Consider two competing interest groups (principals) with conflicting preferences  $S_1(q) = -S_2(q) = q$ . The decision-maker (agent) has some status quo preferences on what should be the pursued policy. To model this, we assume that  $S_0(q) = -\frac{q^2}{2}$  where  $q \in \mathcal{Q} = [-\bar{Q}, \bar{Q}]$  with  $\bar{Q}$  being large enough to avoid corner solutions. Assume also that  $\theta$  is uniformly distributed over  $[-\delta, \delta]$  with  $\delta < 1$ . The agent's ideal policy or status quo is located at  $q_0(\theta) = -\theta$  with the corresponding payoff  $U_0(\theta) = \frac{\theta^2}{2}$ .

Since principals are symmetrically biased in opposite direction, they would just agree on letting the agent choose his status quo policy if they were cooperating. Distortions away from  $q_0(\theta)$  should thus be interpreted below as coming from non-cooperative behavior.

The nature of principals' preferences suggests that we may be able to construct an equilibrium with "head-to-head" competition for the agent's services. Indeed principal  $P_1$  enjoys higher policies and is ready to cajole types closer to  $-\delta$  since they find it more attractive to move up policy. This is the reverse for principal  $P_2$  who certainly prefers types closer to  $\delta$ .

In such equilibrium, interest groups split the type space with the agent entering into exclusive contracting with either principal on each side, i.e., accepting only the contribution of one principal at once.<sup>23</sup>

With "head-to-head" competition, the equilibrium policy might be discontinuous with a jump occurring for the type just indifferent between serving each principal. Proposition 1 below shows that such an equilibrium exists.

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<sup>23</sup>To model such "head-to-head" competition, the modeler faces two strategies. First, he might introduce a small modification of our basic model and impose *a priori* an exclusivity clause stipulating that agent can only contract with one principal. Second, such exclusive contracting may endogenously arise at equilibrium of a game where no such restriction is imposed when contributions are discontinuous and drop off to zero at outputs where the marginal agent is just indifferent between contracting with either principals. This is that latter path that we follow below. It is more compelling because, first, it does not impose any ad hoc restriction on contracts and, second, it shows that exclusivity can arise even when such clauses are banned by Law or cannot be contractually enforced.

**Proposition 1** *Assume that  $\delta < 1$ . There exists an equilibrium of the lobbying game exhibiting exclusive contracting with the following features.*

1. *The equilibrium policy  $\bar{q}(\theta)$  is decreasing with*

$$\bar{q}(\theta) = \begin{cases} \bar{q}_1(\theta) = 1 - \delta - 2\theta & \text{if } \theta \in [-\delta, 0) \\ \bar{q}_2(\theta) = -1 + \delta - 2\theta & \text{if } \theta \in (0, \delta]. \end{cases}$$

*It is discontinuous at  $\theta = 0$ , with the agent being there indifferent between choosing  $\bar{q}(0^-) = 1 - \delta > 0$  and  $\bar{q}(0^+) = -1 + \delta < 0$ .*

2. *The principals' activity sets are respectively  $\Omega_1 = [-\delta, 0)$  and  $\Omega_2 = (0, \delta]$ .*

3. *The equilibrium contributions are discontinuous and given by*

$$\bar{t}_1(q) = \begin{cases} \frac{5}{4}(1 - \delta)^2 + \frac{1-\delta}{2}q + \frac{1}{4}q^2 & \text{if } q \geq 1 - \delta \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \bar{t}_2(q) = \bar{t}_1(-q). \quad (6)$$

4. *The agent's information rent is everywhere greater than his status quo payoff:*

$$\bar{U}(\theta) = \frac{3}{2}(1 - \delta)^2 + (1 - \delta)|\theta| + \theta^2 > U_0(\theta) \quad \forall \theta \in \Theta.^{24} \quad (7)$$

This equilibrium has all the nice and intuitive features that one may expect from “head-to-head” competition between interest groups.<sup>25</sup> Each of those groups secure an area of close-by types who exclusively deal with that group. Given that this principal pays a lot for the agent's services, the competing principal who is further apart prefers not to contribute. However, competition for the marginal type who is just indifferent between pleasing either principal is fierce with principals bidding a lot for the type 0 agent's services and increasing thereby the latter's utility strictly above his status quo payoff.

The equilibrium policy is discontinuous for the marginal type who is just indifferent between serving either principal. Participation constraints are both binding at that point but  $\bar{U}$  has a kink there.

Figure 1 below displays the equilibrium output whereas Figure 2 describes the equilibrium schedules and the optimal choices of the marginal agent with type 0.

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<sup>25</sup>As uncertainty converges to zero, i.e.,  $\delta$  decreases to zero, this discontinuous equilibrium tariffs and output converge towards an equilibrium allocation of the complete information game with discontinuous schedules,  $\bar{t}_1(q) = \begin{cases} \frac{5}{4} + \frac{1}{2}q + \frac{1}{4}q^2 & \text{if } q \geq 1 \\ 0 & \text{otherwise} \end{cases}$  and  $\bar{t}_2(q) = \bar{t}_1(-q)$ . The agent located at 0 is then indifferent between choosing  $\bar{q}(0^-) = 1$  and  $\bar{q}(0^+) = -1$  and randomly choose either.

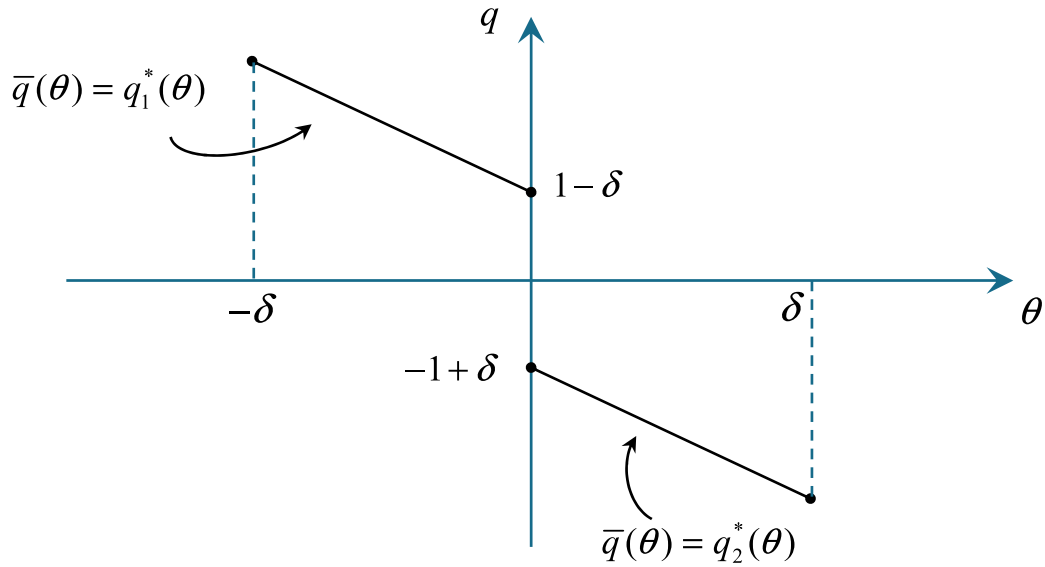


Figure 1: Policy in the “head-to-head” lobbying game.

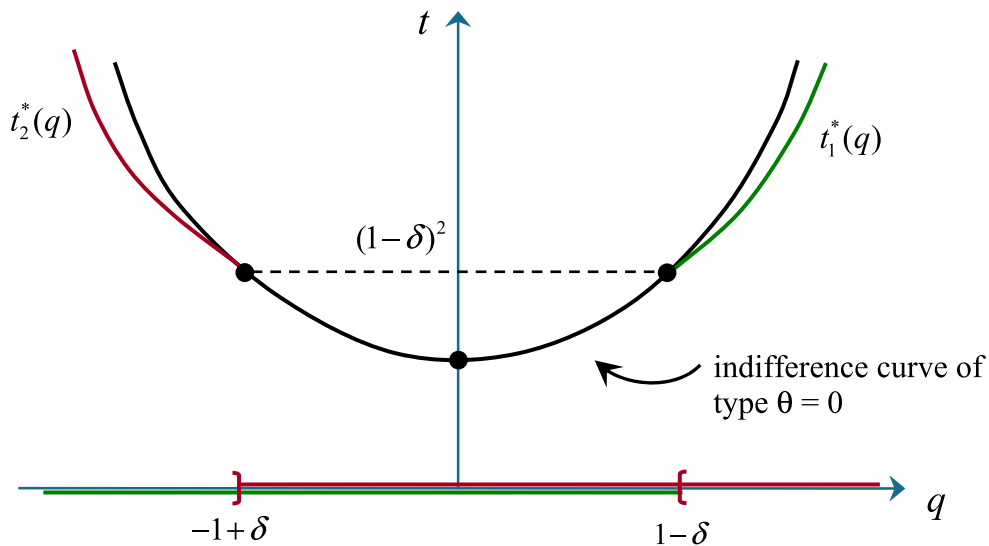


Figure 2: Contribution schedules in the “head-to-head” lobbying game.

## 5.2 Discrete Public Good

The discontinuity in the equilibrium output found in Section 5.1 above is due to the jump in contributions at an equilibrium output. Our next example of discontinuity is obtained instead when principals and the agent have linear surplus and cost functions and de facto the equilibrium has a “bang-bang” structure. The agent produces the public good for

sure if and only if his type is below some threshold.<sup>26</sup> Contributions are nevertheless continuous.

Suppose for instance that  $n = 2$  with  $S_0(q) = 0$ ,  $S_i(q) = s_i q$  with  $s_i > 0$  for  $i = 1, 2$  and that  $\theta$  is distributed on  $\Theta = [0, 1]$ . The quantity of public good is now interpreted as a probability (i.e.  $\mathcal{Q} = [0, 1]$ ). We assume also that  $s_1 + s_2 < 1$  so that even under complete information on the cost parameter, producing the public good is inefficient for the worst technologies.

**Proposition 2** *When Assumption 3 holds, there exists an equilibrium of the delegated common agency game with the following features.*

1. *Equilibrium schedules are continuous and given by:*

$$\bar{t}_i(q) = \max \left\{ 0, s_i - \frac{F(\theta^*)}{f(\theta^*)} \right\} q, \quad i = 1, 2 \quad (8)$$

where  $\theta^*$  is the unique solution in  $[0, 1]$  to the equation

$$\sum_{i=1}^2 \max \left\{ 0, s_i - \frac{F(\theta^*)}{f(\theta^*)} \right\} = \theta^*. \quad (9)$$

2. *The probability of producing the public good is:*

$$\bar{q}(\theta) = \begin{cases} 1 & \text{if } \theta \in [0, \theta^*] \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

## 6 Continuous Equilibria

In the screening literature, the “state of the art” to solve problems like  $(\mathcal{P}_i^r)$  is so far given by Jullien (2000). This paper studies screening models with type-dependent participation constraints which are particularly relevant to study public contracting under delegated agency. It provides a clear characterization of the solution to such problems when the principal’s objective is assumed to be twice continuously differentiable, strictly concave in output, and participation constraints correspond to utility profiles that can be implemented by a contract.<sup>27</sup> The second requirement certainly applies to our public delegated agency game since the agent, when refusing the offer of a given principal, makes his choice optimally in the remaining set of contracts which provides an implementable profile. Under common agency, the first of this requirement is more problematic because

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<sup>26</sup>A similar model with a bang-bang structure has been studied by Lebreton and Salanié (2003) in a lobbying context although the equilibrium is implemented with discontinuous payments as well: lobbyists (principals) contributing only if the decision-maker (their agent) is choosing the policy they prefer.

<sup>27</sup>To be complete, Theorem 1 in Jullien (2000) assumes full separation of types (i.e., no bunching) also.



it presumes a priori that equilibrium tariffs are twice differentiable. This last property is clearly at odds with our findings in Section 5.1 where equilibrium tariffs may have jumps. Even without jumps, equilibrium schedules may fail to be continuously differentiable as shown in Section 5.2. In that respect, relaxing the degree of smoothness in the data of the problem is necessary in competing screening environments.

Two features of the solution obtained by Jullien (2000) are worth to be stressed. First, output distortions away from the first-best are captured by means of a cumulative distribution function whose support is the set of types where the participation constraint binds, namely the inactivity set of a given principal in our framework. Under delegated agency, each principal's best response is thus characterized by such a distribution and the equilibrium allocation will compound the different directions in which the corresponding distortions are tilted.

Second, the solution to the contracting problems exhibited by Jullien (2000) is shown to be continuous. Again this last property, inherited from the assumption of twice continuous differentiability, is clearly at odds with our findings in Sections 5.1 and 5.2. In Section 5.1, the source of the discontinuity is clearly the jump in equilibrium tariffs. In Section 5.2, this is instead the lack of strict concavity of the principals. Unfortunately, in both cases, the avenue opened by Jullien (2000) turns out to be inapplicable to have a full description of equilibrium outcomes in delegated agency games. Nevertheless, and as a first step, we provide in this section a set of conditions, weaker than those in Jullien (2000), which ensures continuity of the solution to the contracting problem faced by each principal when computing their best-responses.

The corresponding equilibria have “pleasant” features that are studied thereafter by means of two applications.

## 6.1 Best Responses and “Smooth-Pasting”

We first look for some kind of conditions that would imply enough smoothness when computing best responses. That smoothness is a key economic ingredient to characterize equilibria where principals are no longer competing fiercely one against the other.

**Condition 1 Convexity.**  $L_i^0(\theta, u, v)$  is finite valued, continuously differentiable in  $v$ , and there exists a constant  $K > 0$  such that:

$$|\nabla_v L_i^0(\theta, u, v_1) - \nabla_v L_i^0(\theta, u, v_2)| \geq K|v_1 - v_2| \quad \forall (v_1, v_2) \in \mathcal{V}^2.$$

Condition 1 imposes a minimal amount of convexity in the maximand  $L_i^0(\theta, u, \cdot)$ . This condition will be satisfied in the examples of continuous equilibria in Section 7 below as it can be easily checked. It clearly fails in the discrete public good example of Section 5.2 because of linearity of the objectives. It also fails in our lobbying example of Section 5.1 because maximands are not even smooth there.

Contrary to the requirements in Jullien (2000), Condition 1 imposes only that the Lagrangean is continuously differentiable once. The degree of smoothness that is implicitly put on equilibrium schedules with that condition is thus a priori less stringent.<sup>28</sup>

**Theorem 1** *Assume that Condition 1 holds and  $(\bar{U}, \bar{v})$  is a minimizing process for  $(\mathcal{P}_i^r)$  with  $V_i < +\infty$  that satisfies (5) a.e.. There exists a non-negative Borel measure  $\mu_i(\cdot)$  on the Borel subsets of  $\Theta$  with  $\text{supp } \mu_i \subseteq \Omega_i^c$ , and a  $\mu_i$ -integrable function  $\gamma_i : \Theta \rightarrow \mathbb{R}$  such that for a.e.  $\theta \in \Theta$  the minimizing arc  $\bar{U}(\theta)$  satisfies the following necessary conditions.*

1.  $\gamma_i(\theta) = 0$  on  $\Omega_i$  and  $\gamma_i(\theta) = 1$  on  $\Omega_i^c$ .

2. We have a.e.:

$$r_i(\theta^-)\bar{v}(\theta) - L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) = \max_{v \in \mathcal{V}} \{r_i(\theta^-)v - L_i^0(\theta, \bar{U}(\theta), v)\} \quad (11)$$

where

$$r_i(\theta) = F(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s)\mu_i(ds) \quad \forall \theta \in (\underline{\theta}, \theta) \quad (12)$$

with  $r_i(\underline{\theta}^-) = r_i(\bar{\theta}) = 0$  and  $r_i(\cdot)$  continuous on  $(\underline{\theta}, \theta)$ .

3. The measure  $\mu_i$  is non-singular on  $(\underline{\theta}, \bar{\theta})$  and  $\bar{q}(\theta)$  is continuous.

Those necessary conditions for a minimizing process are straightforward to understand and generalize the findings in Jullien (2000) to our contexts. Items [1.] and equation (11) characterize the evolution of the costate variable for (2). Item [2.] tells us that  $-\bar{q}(\theta) = \dot{\bar{U}}(\theta) = -\bar{v}(\theta)$  maximizes the Hamiltonian. To ensure continuity (Item [3.]), one must not only look carefully to principal  $P_i$ 's objective function (Condition 1) but also avoid mass point in the singular measure  $\mu_i$ . Such singularity would induce some discontinuity in the equilibrium output at a boundary point of this principal's activity set. This absence of singularity comes to a large extent from the fact that  $\bar{U}_{-i}(\theta)$  is an implementable profile.

The continuity of  $\bar{q}(\theta)$  implies that whenever the type-participation constraint (4) starts being binding at an interior point of  $\Theta$ , it does so in a smooth-pasting way. Both  $\bar{U}(\theta) = \bar{U}_{-i}(\theta)$  and  $\dot{\bar{U}}(\theta) = \dot{\bar{U}}_{-i}(\theta) \Leftrightarrow \bar{q}(\theta) = \bar{q}_{-i}(\theta)$  at any  $\theta$  not only in the interior but also on the boundary of  $\Omega_i^c$  if that boundary lies in the interior of  $\Theta$ . As we shall see now, smooth-pasting has strong implications on the shape of equilibrium contributions close to the boundary of the activity set.

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<sup>28</sup>Of course, at equilibrium, the continuity of the solution will feed back on the property of the data of each of the principals' problems and ensure that those data are in fact continuously differentiable since equilibrium contributions turn out to be so. But, and this is a noticeable point, schedules are not assumed a priori twice continuously differentiable.

## 6.2 Equilibrium Properties

### 6.2.1 Output

Theorem 1 provides only limited information on possible output distortions because it characterizes only best responses and not yet equilibrium output. To be more explicit, we must account for how output distortions induced by different principals are compounded altogether. Next Theorem goes in that direction.

**Theorem 2** *Suppose that the conditions of Theorem 1 and Assumption 3 are satisfied.*

1. *The equilibrium output  $\bar{q}(\theta)$  satisfies*

$$S'_0(\bar{q}(\theta)) + \sum_{i \in \alpha(\theta)} \bar{t}'_i(\bar{q}(\theta)) = \theta. \quad (13)$$

and

$$S'_0(\bar{q}(\theta)) + \sum_{i \in \alpha(\theta)} S'_i(\bar{q}(\theta)) = \theta + \frac{\sum_{i \in \alpha(\theta)} r_i(\theta)}{f(\theta)} \quad (14)$$

where  $r_i(\theta)$  defined in Theorem 1 is continuous and constant over any connected subset  $\Omega_i^j$  ( $j \in \mathcal{J}$ ) of  $\Omega_i$  where principal  $P_i$  is active.

2. *The monotonicity condition (3) is satisfied a.e..*

At a best-response, principal  $P_i$  implements an output which maximizes the virtual surplus of the bilateral coalition he forms with the agent. In doing so, this principal takes into account that the agent's type  $\theta$  must be replaced by his *virtual efficiency parameter*  $\theta + \frac{r_i(\theta)}{f(\theta)}$  capturing how incentive and participation constraints interact under informational asymmetry. The presence of a type-dependent participation constraint affects this virtual type through the term  $\int_{[\underline{\theta}, \theta)} \gamma_i(s) \mu_i(ds)$ . This integral is a non-decreasing cumulative distribution function which is constant on any interval where (4) is slack. Its derivative, in the sense of distribution theory allowing possibly for mass points, is the multiplier of this state-dependent participation constraint. Such mass points necessarily arise only at boundaries of the type set for principal  $P_i$ . Otherwise, the equilibrium output is continuous with no mass point for  $\mu_i$  in the interior of the type set.

Intuitively, when an agent with type  $\theta$  behaves like a less (resp. more) efficient type  $\theta + d\theta$  (resp.  $\theta - d\theta$ ), he produces the same amount at a lower cost but he also takes into account how pretending being less (resp. more) efficient changes the payment that principal  $P_i$  offers to induce participation of this less efficient agent. The case  $r_i(\theta) > 0$  corresponds of course to settings where principal  $P_i$  finds it more attractive to induce participation by the more efficient types. Instead,  $r_i(\theta) < 0$  when principal  $P_i$  finds it more attractive to induce participation by the least efficient types.

At equilibrium, all the distortions induced by the active principals are compounded altogether (equation (14)). The point is that not all distortions may go simultaneously in the same direction and although  $r_i(\theta)$  may be positive for some principals it may instead be negative with others.

**Remark 2** *Observe that  $\bar{q}(\theta)$  being strictly decreasing it admits an inverse function, denoted thereafter by  $\bar{\theta}(q)$ .*

### 6.2.2 Activity Sets

Although Theorem 2 gives a sharp description of equilibrium outputs, it says little on where activity sets stop and start. The difficulty comes from finding a priori from the fundamental assumptions made on preferences and on the type distribution who are the active principals on a given set. This is generally a hard task towards which, guided by intuition, we devote some effort in Section 7. For the time being, let us simply investigate some general properties of the inactivity set  $\Omega_i^c$  of a given principal  $P_i$ .

**Proposition 3** *Suppose that the conditions of Theorem 1 hold and that  $\Omega_i^c$  has a non-empty interior  $\bar{\Omega}_i^c$ . For any  $\theta \in \bar{\Omega}_i^c$ , we have:*

$$S'_i(\bar{q}_{-i}(\theta)) = \frac{r_i(\theta)}{f(\theta)} \quad (15)$$

where  $r_i(\theta)$  is defined in Theorem 1.

Condition (15) suggests an algorithm for finding out activity sets. Suppose that we have already computed an equilibrium  $\bar{q}_{n-1}(\theta)$  with  $n - 1$  principals, say  $i \in \{1, \dots, n - 1\}$ . To find out principal  $P_n$ 's activity set it is enough to restrict to the subset of the type space where  $\frac{d}{d\theta}(f(\theta)S'_i(\bar{q}_{n-1}(\theta))) - f(\theta) > 0$  since, from the monotonicity of  $r_i(\cdot)$ ,  $\frac{d}{d\theta}(f(\theta)S'_i(\bar{q}_{n-1}(\theta))) - f(\theta) \leq 0$  on any activity set. In a second step, one can use the smooth-pasting conditions to get the shape of  $P_n$ 's contribution at any interior boundary point of  $\Omega_i$ . Finally, we can reconstruct from there the rest of the schedule. This algorithm can be particularly efficient with only two principals as illustrated in Section 7 below by means of examples.

**Monotonic equilibria.** If one imposes more structure on the principals' preferences, the shape of activity sets can be easily drawn from condition (15).

**Corollary 1** *Suppose that the conditions of Theorem 1 hold and that principals have linear preferences, i.e.,  $S_i(q) = s_i q$  for some  $s_i$ . Any equilibrium is then monotonic. Principal  $P_i$ 's activity set is  $\Omega_i = [\underline{\theta}, \theta_i)$  (resp.  $\Omega_i = (\theta_i, \bar{\theta}_i]$ ) where  $s_i = R(\theta_i)$  (resp.  $s_i = -T(\theta_i)$ ) and  $r_i(\theta) = F(\theta)$  (resp.  $r_i(\theta) = F(\theta) - 1$ ) if  $s_i > 0$  (resp.  $s_i < 0$ ).*

### 6.2.3 Contribution Schedules

We are now interested in deriving properties of the contribution  $\bar{t}_i(\cdot)$  that principal  $P_i$  offers on any connected interval  $\Omega_i^j$  ( $j \in \mathcal{J}$ ) of his activity set  $\Omega_i$ .

**Proposition 4** *Suppose that the conditions of Theorem 1 hold.*

1. *The equilibrium schedule  $\bar{t}_i(\cdot)$  is strictly differentiable at any equilibrium point  $\bar{q}(\theta)$ .*
2. *The following “smooth-pasting” conditions are satisfied by the best-response contribution offered by principal  $P_i$  at any boundary point  $\theta$  of his inactivity set  $\Omega_i^c$ :*

$$\bar{t}_i(\bar{q}(\theta)) = \bar{t}'_i(\bar{q}(\theta)) = 0. \quad (16)$$

*Moreover  $\bar{t}_i(\cdot)$  is convex in the neighborhood of such  $\bar{q}(\theta)$ .*

3. *The equilibrium marginal contribution for principal  $P_i$  satisfies on each connected subset of activity  $\Omega_i^j$ :*

$$\bar{t}'_i(q) = S'_i(q) - \frac{F(\bar{\theta}(q)) - M_i^j}{f(\bar{\theta}(q))} \quad (17)$$

*where  $M_i^j = \int_{\theta}^{\theta_i^j} \gamma_i(\theta) \mu_i(d\theta) \in [0, 1]$  is constant on each connected subset of activity  $\Omega_i^j$ . Moreover  $M_i^j$  increases with  $j$ .*

This result stands in contrast with the delegated common agency model under complete information due to Bernheim and Whinston (1986). These authors showed that the so-called “truthful” schedules of the form  $t_i(q) = \max\{0, S_i(q) - C_i\}$  for some  $C_i$  sustain the equilibrium outcomes. Those schedules are not smooth at a point  $q_0$  such that  $t_i(q_0) = 0$  even though they are locally convex around that point. The difference with asymmetric information comes from the fact that, around any equilibrium point, the equilibrium schedule is much more constrained. Such schedule has now to go through other equilibrium points under asymmetric information. This “extra information” implies smooth-pasting for continuous equilibria.

**Monotonic equilibria (continued).** Monotonic equilibria are such that contribution schedules are monotonically increasing (resp. decreasing) when principals value positively (resp. negatively) the public good.

**Corollary 2** *Suppose that the conditions of Theorem 1 hold and that principals have linear preferences, i.e.,  $S_i(q) = s_i q$  for some  $s_i$ . Principal  $P_i$ 's contribution schedule is:*

$$\bar{t}_i(q) = \begin{cases} \int_0^q \max\{0, s_i - R(\bar{\theta}(x))\} dx & \text{if } s_i \geq 0 \\ \int_0^q \min\{0, s_i + T(\bar{\theta}(x))\} dx & \text{if } s_i < 0. \end{cases} \quad (18)$$

Lindahl-Samuelson condition: It follows from (18) that the equilibrium output when principals have linear surplus solves:

$$S'_0(\bar{q}(\theta)) + \sum_{s_i \geq 0} \max\{0, s_i - R(\theta)\} + \sum_{s_i < 0} \min\{0, s_i + T(\theta)\} = \theta. \quad (19)$$

This equation immediately shows that principals who like the public good contribute at the margin a positive (or null) amount which is less than their marginal valuations with a discount being the hazard rate  $R(\theta)$ . Principals who dislike the public good contribute at the margin a negative (or null) amount but less (in absolute values) than their marginal valuations with a discount being again a hazard rate  $T(\theta)$ . Equation (19) can then be viewed as a modified Lindahl Samuelson condition in this common agency context. Propositions 5 and 6 illustrate how the forces of competing principals can be combined to determine the overall equilibrium output distortion both with congruent and competing principals.

## 7 Continuous Equilibria: Applications

In this Section, we show how simple economic settings give rise to different patterns of equilibria with either congruent or conflicting principals. Beyond, those examples also illustrate a few basic principles of delegated common agency models. Principles 1 and 2 below are of interest in itself as guide for readers interested in applying our methodology to other specific contexts.

### 7.1 Voluntary Provision of a Public Good

Suppose that principals have valuations for a public good given by  $S_i(q) = s_i q$  for all  $q \in \mathcal{Q} = [0, \bar{Q}]$  with  $s_i > 0$  ( $i = 1, 2$ ). We assume that  $S_0(q) = -\frac{q^2}{2}$  so that  $q_0(\theta) = U_0(\theta) \equiv 0$  and that  $\theta$  is uniformly distributed on  $\Theta = [0, \bar{\theta}]$ . Finally, principals are ordered in terms of their marginal valuations for the public good and we refer to  $P_2$  (resp.  $P_1$ ) as the *strong principal* (resp. *weak principal*), with the extra technical assumptions  $s_2 > 2s_1$  and  $s_2 < 2\bar{\theta}$  that reduce the number of cases under scrutiny.

Had both principals cooperated in designing their contributions, the *optimal cooperative output* under asymmetric information  $q^C(\theta)$  would be given by:<sup>29</sup>

$$q^C(\theta) = \max \left\{ 0, \left( \sum_{i=1}^2 s_i \right) - 2\theta \right\}. \quad (20)$$

This decision corresponds to the modified Samuelson condition where the optimal level of public good (when positive) is such that the sum of marginal valuations  $s_1 + s_2$  is just equal to the agent's virtual cost  $\theta + \frac{F(\theta)}{f(\theta)} = 2\theta$ .

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<sup>29</sup>The proof is standard and thus omitted.

Let define the *stand-alone output* that principal  $P_i$  would implement when contracting alone with the privately informed agent as:

$$q_i^*(\theta) = \max \{0, s_i - 2\theta\}. \quad (21)$$

Both  $q^C(\theta)$  and  $q_i^*(\theta)$  ( $i = 1, 2$ ) are non-increasing and  $q_1^*(\theta) \leq q_2^*(\theta) \quad \forall \theta \in \Theta$ .

Denote also by  $q^I(\theta)$  the non-increasing output schedule such that:

$$q^I(\theta) = \max \left\{ 0, \sum_{i=1}^2 s_i - 3\theta \right\}. \quad (22)$$

This output schedule is the solution to the *intrinsic* common agency game where the agent has only the choice of accepting both offers or none, i.e., the participation constraint which is the same for both principals is  $U(\theta) \geq 0$  for all  $\theta \in \Theta$ .<sup>30</sup> In particular, this output schedule entails a double distortion familiar from the intrinsic common agency literature and we get  $q^I(\theta) \leq q^C(\theta)$  with equality only at  $\theta = 0$ .<sup>31</sup> This hypothetical intrinsic setting allows to focus on one kind of distortions only, those that arise through non-cooperative contracting *taking as given* the set of active principals. The next proposition gives instead more attention to the inefficiency that comes from insufficient participation of principals. Those inefficiencies are specific to the delegated contracting setting.

**Proposition 5** *There exists an equilibrium of the delegated common agency game with the following features.*

1. *The equilibrium output  $\bar{q}(\theta)$  is continuous, decreasing in  $\theta$  and satisfies*

$$\bar{q}(\theta) = \begin{cases} q_I(\theta) & \text{if } \theta \in [0, s_1] \\ q_2^*(\theta) & \text{otherwise} \end{cases}$$

*with an inverse function defined as*

$$\bar{\theta}(q) = \begin{cases} \frac{1}{3}(\sum_{i=1}^2 s_i - q) & \text{if } s_2 - 2s_1 \leq q \\ \max \{0, \frac{1}{2}(s_2 - q)\} & \text{otherwise.} \end{cases}$$

2. *Activity sets are respectively given by  $\Omega_1 = [\underline{\theta}, s_1)$  and  $\Omega_2 = [\underline{\theta}, \frac{s_2}{2})$  with  $\Omega_1 \subset \Omega_2 \subset \Theta$ .*

3. *Equilibrium contributions are piecewise quadratic, and continuously differentiable with*

$$\bar{t}_1(q) = \begin{cases} \int_{s_2-2s_1}^q (s_1 - \bar{\theta}(x))dx & \text{if } s_2 - 2s_1 \leq q \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{t}_2(q) = \int_0^q (s_2 - \bar{\theta}(x))dx. \quad (23)$$

---

<sup>30</sup>Martimort and Stole (2009a).

<sup>31</sup>Note of course that  $q^I(0) > q_2(0) > q_1(0)$ . This simply means that the grand-coalition made of both principals and the agent produces always more than any simple bilateral coalition between one of those principals and the agent when the latter produces at zero cost.

4. The agent's information rent is

$$\bar{U}(\theta) = \int_{\theta}^{\bar{\theta}} \bar{q}(x) dx.$$

Several remarks are in order. First, the equilibrium output in the delegated common agency game is always greater than in the intrinsic game. This captures the fact that the non-zero participation constraints under delegated agency force principals to give more rent than under intrinsic agency, raising thereby equilibrium output.

Second, the weak principal does not reward the agent for levels of output which are small enough. Indeed, only the strong principal does so. Not all principals contribute under asymmetric information, only those who are able to pay the corresponding agency cost do so. As a result, the equilibrium output reflects the existing set of active principals.<sup>32</sup> More precisely, this equilibrium is obtained by piecing together the contribution that the strong principal would offer alone for the least efficient types with an intrinsic equilibrium allocation that would arise when both principals find it worth to contribute, i.e., for the most efficient types. Smooth-pasting at the threshold type  $s_1$  allows to recover the analytical expressions of contributions beyond that point.

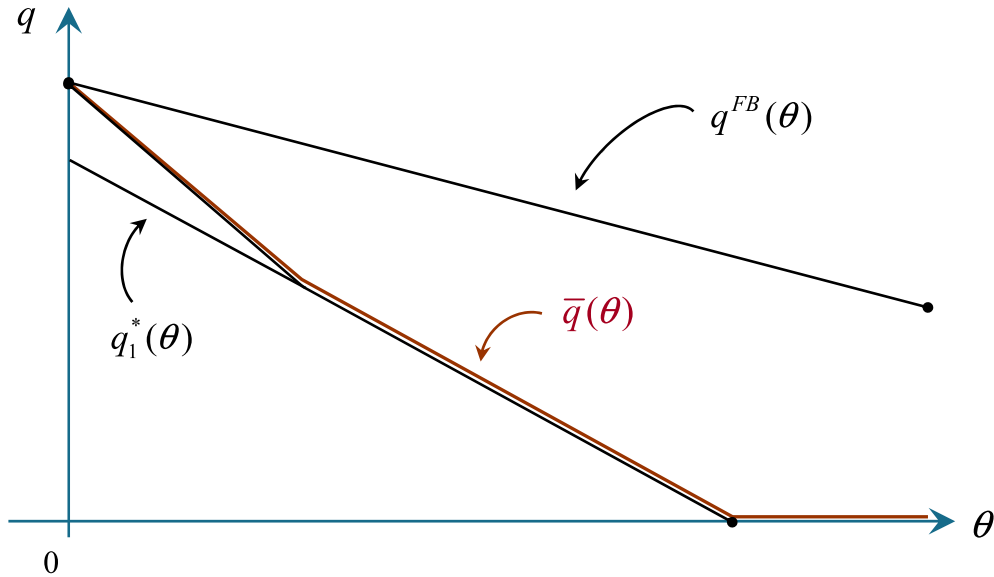


Figure 3: Output in the public good game.

Third, *free-riding* in contributing for the public good takes a double form here. On the one hand, for the most efficient types, (i.e., if the marginal cost is small enough), both principals contribute but the equilibrium output is lower than if they were cooperating

<sup>32</sup>Note that this feature of the equilibrium was already present in Proposition 2 above.



( $q^I(\theta) \leq q^C(\theta)$  at all  $\theta$ ). This is so because each principal reduces his marginal contribution below his marginal valuation for the public good without internalizing the fact that the other principal does so as well. On the other hand, under asymmetric information, the set of active contributors may be a strict subset of what it would be under complete information. This arises for the least efficient types for which the weak principal finds it optimal to stop contributing.

This double failure of the Coase Theorem under asymmetric information is a crucial finding that we state in a less formal way below:

**Principle 1** *In a game of voluntary contributions to a public good under asymmetric information, decentralized bilateral contracting under the common agency institution induces inefficiently low output and inefficient representation of active principals compared with cooperative contracting.*

**Remark 3** *Interestingly, the equilibrium output is clearly not invariant with respect to redistributions of the principals' surplus such that the overall marginal surplus  $\sum_{i=1}^2 s_i$  is kept fixed. The intuition is that such redistribution may affect the set of active principals. This result is of course reminiscent of the literature on voluntary contributions (Bergstrom, Blume and Varian 1986, among others) although this literature obtains that result by making restriction in the set of instruments (fixed contributions instead of schedules) and assumes complete information.*

## 7.2 Lobbying for a Public Policy (Continued)

Let us come back to the lobbying setting of Section 5.1. One may wonder whether discontinuous equilibria and continuous ones may coexist. The response is positive as shown below by exhibiting a continuous equilibrium.

**Proposition 6** *Assume that  $\delta < 1$ . There exists an equilibrium of the lobbying game with the following features.*

1. *The equilibrium policy  $\bar{q}(\theta)$  is continuous, decreasing in  $\theta$  and satisfies*

$$\bar{q}(\theta) = \begin{cases} \begin{cases} q_1^*(\theta) = 1 - \delta - 2\theta & \text{for } \theta \in [-\delta, -1 + \delta) \\ -3\theta & \text{for } \theta \in [-1 + \delta, 1 - \delta] \\ q_2^*(\theta) = -1 + \delta - 2\theta & \text{for } \theta \in (1 - \delta, \delta] \end{cases} & \text{if } \delta \geq \frac{1}{2} \\ -3\theta & \text{if } \delta \leq \frac{1}{2} \end{cases}$$

*with an inverse function defined as*

$$\bar{\theta}(q) = \begin{cases} \begin{cases} \frac{1}{2}(1 - \delta - q) & \text{if } 3(1 - \delta) \leq q \\ -\frac{q}{3} & \text{if } 3(1 - \delta) \geq q \geq 3(\delta - 1) \\ \frac{1}{2}(-1 + \delta - q) & \text{if } q \leq 3(\delta - 1) \end{cases} & \text{if } \delta \geq \frac{1}{2} \\ -\frac{q}{3} & \text{if } \delta \leq \frac{1}{2}. \end{cases}$$

2. Activity sets are  $\Omega_1 = [-\delta, 1 - \delta]$  and  $\Omega_2 = (-1 + \delta, \delta]$  when  $\delta \geq \frac{1}{2}$  or there is full coverage by both principals when  $\delta \leq \frac{1}{2}$ .

3. The principals' contributions are given by

$$\bar{t}_1(q) = \begin{cases} \int_{-3(1-\delta)}^q \max\{0, (1 - \delta - \bar{\theta}(x))\} dx & \text{and } \bar{t}_2(q) = t_1^*(-q) & \text{if } \delta \geq \frac{1}{2} \\ (1 - \delta)q + \frac{q^2}{6} + \frac{3}{4} - \frac{3}{2}\delta^2; & \bar{t}_2(q) = \bar{t}_1(-q) & \text{if } \delta \leq \frac{1}{2}. \end{cases} \quad (24)$$

4. The agent's information rent is given by the following expressions

$$\bar{U}(\theta) = \begin{cases} \frac{3}{2}(1 - \theta^2) - 3\delta^2 & \text{if } \delta \leq \frac{1}{2} \\ 3(1 - \delta)^2 - \int_0^\theta \bar{q}(x) dx & \text{if } \delta \geq \frac{1}{2}. \end{cases} \quad (25)$$

When there is enough uncertainty on the agent's type, i.e.,  $\delta \geq \frac{1}{2}$ , each principal can secure himself an area of influence where he is the only one dealing with the agent. This area is a neighborhood of the "most-preferred" type of the agent from that principal's viewpoint. For instance, that area for principal  $P_1$  contains type  $-\delta$  which is the most eager to push policies up.<sup>33</sup> Instead, in the middle of the type space, both principal do contribute with overlapping areas of influence. Principals stop contributing for types which are too "far away" in the type space.

Figure 4 describes the equilibrium policy in that case.

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<sup>33</sup>There is an interesting phenomenon that arises when  $\delta = 1$ . Proposition 6 shows that there exists an equilibrium where *both* principals offer zero contribution both in absolute terms but also at the margin for the marginal type at  $\theta = 0$ . There is no overlap in their contributions and the equilibrium outcome. This is the same equilibrium outcome as in Proposition 1 (taking there limits as  $\delta$  converges towards 1).

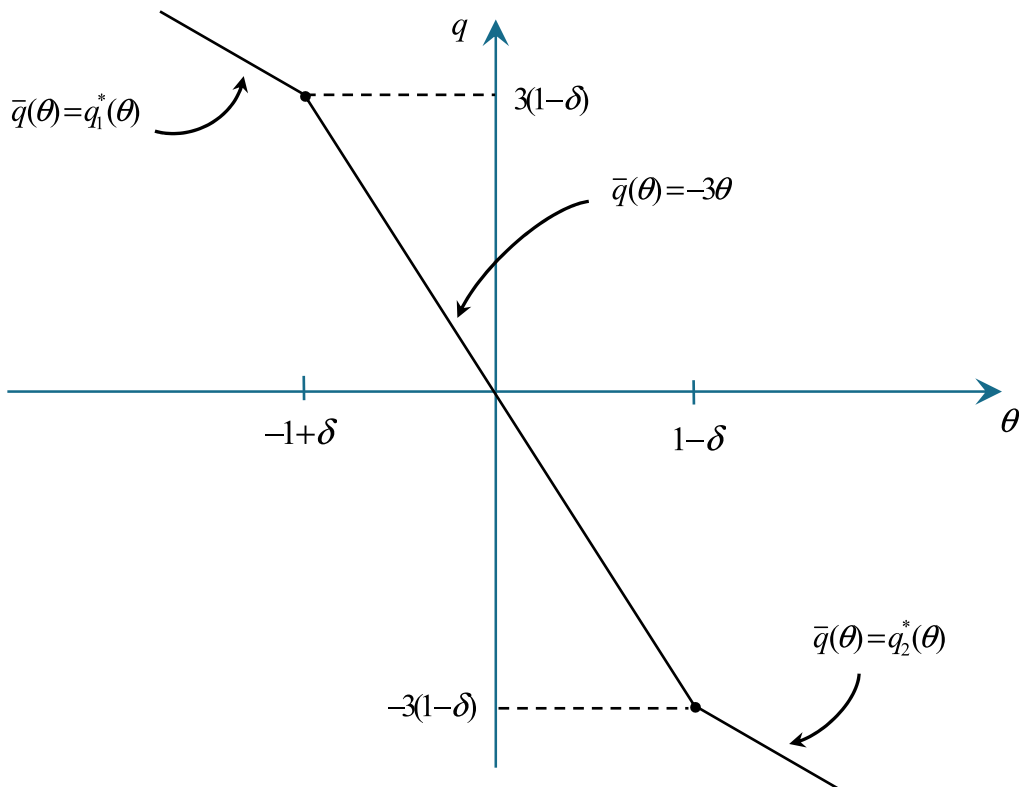


Figure 4: Policy in the continuous equilibrium of the lobbying game  $\delta \geq \frac{1}{2}$ .

Of course, as there is less uncertainty on the agent's type, i.e.,  $\delta \leq \frac{1}{2}$ , the area where principals have overlapping influence covers now the whole type space. The equilibrium policy reflects then the preferences of both principals which each offering a positive contribution everywhere even though they want to push policies in opposite directions.<sup>34,35</sup>

Again this lobbying model confirms that the Coase Theorem under asymmetric information fails on both sides: Policy is inefficient and interest groups may fail to be represented. We state this result in a “loose” form as:

**Principle 2** *In a lobbying game under asymmetric information, common agency induces inefficient policy choices and inefficient representation of active interest groups with some groups possibly securing unchallenged influence on some subset of the type space.*

<sup>34</sup>For more general results along those lines and further classifications of the patterns of contributions in a lobbying game, we refer to Martimort and Semenov (2008).

<sup>35</sup>As  $\delta$  decreases to zero, the continuous equilibrium above converges towards an allocation where the agent located at 0 chooses  $\bar{q}(0) = 0$ , getting payoff  $\bar{U}(0) = \frac{3}{2}$  which is the same as in the discontinuous equilibrium seen in Section 5.1. Contributions converge towards  $\bar{t}_1(q) = q + \frac{q^2}{6} + \frac{3}{4}$  and  $\bar{t}_2(q) = \bar{t}_1(-q)$ . Those contributions form an equilibrium of the complete information game. Clearly, those contributions are not “truthful” although they implement the same allocation as the “truthful” ones that would be defined as  $t_1^{tr}(q) = q$  and  $t_2^{tr}(q) = t_1^{tr}(-q)$ . See Martimort and Stole (2009b) for more results on the limits of equilibria of asymmetric information games as uncertainty converges towards zero.

## 8 More on Discontinuous Equilibria

So far, our main focus was on characterizing continuous equilibria. The examples developed respectively in Sections 5.1 and 5.2 show that relevant and intuitive equilibria with important economic insights may be lost from making such restriction.

Those two examples however differ significantly. In the discrete public good example of Section 5.2, equilibrium contributions remain continuous which implies continuity of the Lagrangean that principals maximize at their best responses. The discontinuity follows therefore from the fact that Condition 1 fails. Strict convexity does not hold when all players have linear preferences.

The equilibrium with head-to-competition found in the lobbying model of Section 5.1 exhibits a discontinuity of a different nature. There, such discontinuity arises when one principal stops contributing. Coming back on the specification of the equilibrium transfer  $\bar{t}_2(q)$ , we observe that principal  $P_1$  should pay a positive extra amount  $\bar{t}_2(-1+\delta) = (1-\delta)^2$  to convince types in a left-neighborhood of zero to accept his own contribution and choose the positive policies that he favors instead of the negative policies that please principal  $P_2$ . This positive fee is a fixed-cost that principal  $P_1$  has to pay to convince the agent to please him. This principal chooses thus not to contribute unless that fixed-cost is covered by a significant benefit coming from the agent changing discontinuously his policy.

**Proposition 7** *Consider a monotonic equilibrium with a discontinuity at some  $\theta_0$  with principal  $P_i$  being active (resp. inactive) on a left-neighborhood  $[\underline{\theta}, \theta_0)$  and inactive (resp. active) on a right-neighborhood  $(\theta_0, \bar{\theta}]$  and such that the conditions of Theorem 1 hold on that interval. The following properties hold:*

$$\bar{t}_i(\bar{q}(\theta_0^-)) = [S_i(q) - R(\theta_0)q]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)}, \quad (26)$$

$$\left( \text{resp. } \bar{t}_i(\bar{q}(\theta_0^+)) = [S_i(q) + T(\theta_0)q]_{\bar{q}(\theta_0^-)}^{\bar{q}(\theta_0^+)} \right), \quad (27)$$

$$\left[ \sum_{i=1}^n \bar{t}_i(q) + S_0(q) - \theta_0 q \right]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)} = 0. \quad (28)$$

Those conditions are readily understood. Consider a principal  $P_i$  active on a left-neighborhood  $[\theta_0 - \epsilon, \theta_0)$  but inactive on the right of  $\theta_0$ . At a best response, that principal must be indifferent between inducing types in that neighborhood to accept his transfer and move the policy from  $\bar{q}(\theta_0^-)$  to  $\bar{q}(\theta_0^+)$  for all those types. Paying that extra amount of money for agents in such neighborhood yields a net gain in surplus for that principal worth  $[S_i(q)]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)} f(\theta_0)\epsilon$ . At the same time, it also increases the agent's rent for all types less than  $\theta_0$  for which principal  $P_i$  is active. This has a cost worth  $[F(\theta_0^-)q]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)} \epsilon$ . Equation

(26) captures this optimality condition.<sup>36</sup> Finally, (28) reflects the agent with type  $\theta_0$ 's indifference condition at this discontinuity.

**Remark 4** *It is interesting to note the strong similarity of those conditions with those that define the equilibrium marginal contributions in the case of a continuous equilibrium. In both cases, the “incremental” contribution of a principal just covers his “incremental” virtual surplus from changing the equilibrium quantity. With continuity, that increment is marginal whereas it corresponds to a discrete jump at a discontinuity.*

**Corollary 3** *Assume that  $n = 2$  and that principals have linear surplus, i.e.,  $S_i(q) = s_i q$  for  $i = 1, 2$ .*

1. *A discontinuity (if any) arises at  $\theta_0 \in \Theta$  if principals have conflicting preferences (i.e.,  $s_1 > 0 > s_2$ ) and*

$$\left[ S_0(q) + \left( \sum_{i=1}^2 s_i - \theta_0 \right) q - \frac{2F(\theta_0) - 1}{f(\theta_0)} q \right]_{\bar{q}_2(\theta_0)}^{\bar{q}_1(\theta_0)} = 0. \quad (29)$$

where  $S'_0(\bar{q}_1(\theta)) + s_1 = \theta + R(\theta)$  and  $S'_0(\bar{q}_2(\theta)) + s_2 = \theta - T(\theta)$ .

2. *There is no equilibrium discontinuity when principals are congruent and  $S_0(0) = 0$ .*

The discontinuity at zero in the lobbying equilibrium with “head-to-head” characterized in Section 5.1 fulfills condition (29) as it can be readily seen. However, this condition is certainly more general and allow to find out the possible discontinuity in any equilibrium with non-overlapping activity sets with principal  $P_1$  dealing with the most efficient types  $\Omega_1 = [\underline{\theta}, \theta_0)$  and offering them his stand-alone output  $q_1^*(\theta)$  whereas principal  $P_2$  deals with the least efficient types and offers them his own stand-alone output  $q_2^*(\theta)$ .

Interestingly, discontinuities are due to the principals having conflicting preferences. When principals are instead congruent, no such discontinuity arises.

Finally, it is worth noticing that the property (19) holds both for continuous equilibria but also for discontinuous ones at any discontinuity point. Together with condition (29) that treats discontinuities this yields a full description of equilibrium candidates.

Gathering all our findings above, we can state our final result that characterize all possible equilibrium patterns.

**Theorem 3** *Assume principals have linear preferences (i.e.,  $S_i(q) = s_i q$ ). Any monotonic equilibrium of the delegated common agency game entails an output that satisfies the following conditions.*

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<sup>36</sup>Similarly, (27) captures the optimal choice of a principal active only on a right-neighborhood of  $\theta_0$ .

1. The Lindahl-Samuelson condition (19) holds on any interval of continuity.
2. Possible discontinuities at points  $\theta_0$  are such that:

$$\left[ S_0(q) - \theta_0 q + \sum_{s_i \geq 0} \max\{0, s_i - R(\theta_0)\}q + \sum_{s_i < 0} \min\{0, s_i + T(\theta_0)\}q \right]_{q(\theta_0^-)}^{q(\theta_0^+)} = 0. \quad (30)$$

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## Appendix

• **Proof of Lemma 1:** The proof is standard and is thus omitted. See for instance Champsaur and Rochet (1989) and Milgrom and Segal (2004) among others. ■

• **Proof of Proposition 1:** We look for principal  $P_1$ 's best response to the schedule  $\bar{t}_2(q)$  described in (6). This contribution induces the following rent/policy profile  $(\bar{U}_2(\theta), \bar{q}_2(\theta))$ :

$$\bar{U}_2(\theta) = \max_q \left\{ \bar{t}_2(q) - \theta q - \frac{q^2}{2} \right\} = \max \left\{ \frac{\theta^2}{2}, \max_q \left\{ \frac{5}{4}(1 - \delta)^2 - \frac{1 - \delta + 2\theta}{2}q - \frac{1}{4}q^2 \right\} \right\}$$

where the maximand on the right-hand side is achieved for

$$\bar{q}_2(\theta) = \arg \max_q \left\{ \bar{t}_2(q) - \theta q - \frac{q^2}{2} \right\} = -1 + \delta - 2\theta. \quad (\text{A1})$$

Hence, we find

$$\bar{U}_2(\theta) = \frac{3}{2}(1 - \delta)^2 + (1 - \delta)\theta + \theta^2.$$

and thus

$$\dot{\bar{U}}_2(\theta) = -\bar{q}_2(\theta) = 1 - \delta + 2\theta \geq 0. \quad (\text{A2})$$



Note in particular that  $\bar{U}_2(\cdot)$  is increasing in the neighborhood of  $\theta = 0$  since  $1 > \delta$ .

We conjecture an equilibrium with non-overlapping activity sets. The contribution  $t_1(q)$  offered by principal  $P_1$  induces a rent/policy profile  $(U_1(\theta), q_1(\theta))$  on the interior of the activity set  $\Omega_1$  such that

$$U_1(\theta) = \max_q \left\{ t_1(q) - \theta q - \frac{q^2}{2} \right\} \text{ and } q_1(\theta) = \arg \max_q \left\{ t_1(q) - \theta q - \frac{q^2}{2} \right\}. \quad (\text{A3})$$

Moreover, we conjecture that  $\Omega_1$  is of the form  $[-\delta, \theta_1^*]$ . Indeed, we have:

$$\dot{U}_1(\theta) = -q_1(\theta) \quad (\text{A4})$$

and thus  $U_1(\cdot)$  is decreasing when  $q_1(\theta) \geq 0$  (an assertion checked below). In that case, the participation constraint

$$U_1(\theta) \geq \bar{U}_2(\theta) \quad (\text{A5})$$

can be binding only on an interval of the form  $\Omega_1^c = [\theta_1^*, \delta]$ .

Consider the exclusive contracting game where principal  $P_1$  would be alone contracting with the agent on  $\Omega_1$ .  $P_1$ 's best response must solve the following problem (neglecting again the second-order monotonicity condition for the agent's problem, namely  $q(\cdot)$  decreasing, and checking it ex post):

$$(\mathcal{P}_1^E) : \max_{\{\theta_1^*, U_1, q_1\}} \int_{\Omega_1} \left( q_1(\theta)(1 - \theta) - \frac{q_1^2(\theta)}{2} - U_1(\theta) \right) \frac{d\theta}{2\delta} + \int_{\Omega_1^c} \bar{q}_2(\theta) \frac{d\theta}{2\delta}$$

subject to (A4) and (A5).

Using (A4) and (A5), we get

$$U_1(\theta) = \bar{U}_2(\theta_1^*) + \int_{\theta}^{\theta_1^*} q_1(\theta') d\theta'.$$

Inserting into the integrand of  $(\mathcal{P}_1^E)$  and integrating by parts yields the following expression of the maximization problem:

$$(\mathcal{P}_1^E) : \max_{\{\theta_1^*, q_1\}} \int_{\Omega_1} \left( q_1(\theta)(1 - \delta - 2\theta) - \frac{q_1^2(\theta)}{2} \right) \frac{d\theta}{2\delta} + \int_{\Omega_1^c} \bar{q}_2(\theta) \frac{d\theta}{2\delta} - U_2^*(\theta_1^*) \frac{(\theta_1^* + \delta)}{2\delta}.$$

Optimizing with respect to  $q_1$  over  $\Omega_1$  yields

$$q_1^*(\theta) = 1 - \delta - 2\theta \quad (\text{A6})$$

which is positive in the neighborhood of  $\theta = 0$ . From (A6), we deduce that for any  $\theta \in \Omega_1$ , principal  $P_1$ 's marginal contribution is

$$\dot{t}_1(q_1^*(\theta)) = \theta + \bar{q}_1(\theta) = \frac{1 - \delta + \bar{q}_1(\theta)}{2}. \quad (\text{A7})$$

Finally, optimizing with respect to  $\theta_1^*$  yields a first-order condition.<sup>37</sup>

$$\bar{q}_1(\theta_1^*)(1 - \delta - 2\theta_1^*) - \frac{(\bar{q}_1(\theta_1^*))^2}{2} - \bar{q}_2(\theta_1^*)(1 - \delta - \theta_1^*) = \bar{U}_2(\theta_1^*) = \frac{3}{2}(1 - \delta)^2 + (1 - \delta)\theta_1^* + (\theta_1^*)^2.$$

Solving this equation yields  $\theta_1^* = 0$  so that principals share equally the market. From the equality  $\bar{U}_1(0) = \bar{U}_2(0)$ , we deduce  $\bar{t}_1(0) = \frac{5}{4}(1 - \delta)^2$ . Finally, using (A7), we get the expression of  $\bar{t}_1(q)$  given in (6).

Finally, the equilibrium profile  $(\bar{U}(\theta), \bar{q}(\theta))$  is obtained by piecing together the best response of each principal on their respective activity sets  $\Omega_1$  and  $\Omega_2$ . ■

• **Proof of Proposition 2:** Type  $\theta^*$  is just indifferent between producing or not given the contributions defined in (8). Types below (resp. above) that threshold produce (resp. do not produce) the public good for sure which gives us condition (10).

Contributions defined in (8) are non-negative and linear in the probability of producing the public good so that the agent always accept those contributions.

To check that those schedules are best responses to each other, observe that one can rewrite  $(\mathcal{P}_i)$  as:

$$(\mathcal{P}_i) : \max_{(U, q)} \int_{\Theta} \left( \left( s_i + \max \left\{ 0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)} \right\} - \theta \right) q(\theta) - U(\theta) \right) f(\theta) d\theta$$

subject to  $q \in [0, 1]$ , (2), (3) and (4).

Again, the second-order condition (3) is suppressed and only checked ex post.

Assuming first that (4) binds at  $\bar{\theta}$  (and possibly on an non-empty interval including that boundary), we get

$$U(\theta) = \bar{U}_{-i}(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} q(x) dx.$$

Inserting this expression into the integrand above yields a more compact expression of  $(\mathcal{P}_i)$  as:

$$(\mathcal{P}_i) : \max_{q \in [0, 1]} \int_{\Theta} \left( \left( s_i + \max \left\{ 0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)} \right\} - \theta \right) q(\theta) - U(\theta) \right) f(\theta) d\theta.$$

Optimizing pointwise, the solution is  $\bar{q}(\theta) = \begin{cases} 1 & \text{if } \theta \in [0, \hat{\theta}] \\ 0 & \text{otherwise} \end{cases}$  where  $\hat{\theta}$  is uniquely defined when Assumption 3 holds as the solution to:

$$s_i + \max \left\{ 0, s_{-i} - \frac{F(\theta^*)}{f(\theta^*)} \right\} = \hat{\theta} + \frac{F(\hat{\theta})}{f(\hat{\theta})}. \quad (\text{A8})$$

Two cases might arise. If  $\theta^*$  that solves (9) is such that  $s_i \geq \frac{F(\theta^*)}{f(\theta^*)}$ , then the solution to (A8) is  $\hat{\theta} = \theta^*$ . If instead  $\theta^*$  that solves (9) is such that  $s_i < \frac{F(\theta^*)}{f(\theta^*)}$ , then  $\hat{\theta} < \theta^*$ , which

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<sup>37</sup>It can be easily checked that the second-order condition holds at  $\theta_1^*$  solving the first-order condition.

implies that  $\bar{q}(\theta) = 0 < \bar{q}_{-i}(\theta) = 1$  on  $(\hat{\theta}, \theta^*)$  (with also  $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$  for all  $\theta \in \Theta / (\hat{\theta}, \theta^*)$ ). Hence, we get that the slope of  $U(\theta)$  is less than that of  $\bar{U}_{-i}(\theta)$ . A contradiction with our starting assumption that (4) binds at  $\bar{\theta}$ . In that case, the best-strategy for principal  $P_i$  is to offer no contribution at all and (9) still holds.

Gathering everything, the contribution schedules that implement this equilibrium outcomes are given by (8). ■

• **Proof of Theorem 1:** We apply the necessary conditions for optimality of a minimizing process  $(\bar{U}, \bar{v})$  for the state-constrained Maximum Principle (Theorem 10.2.1. in Vinter (2000)) when  $V_i < +\infty$ . Checking that latter condition is immediate. Observe indeed that offering the null contribution  $t_i(q) \equiv 0$  yields a payoff  $\int_{\Theta} S_i(\bar{q}_{-i}(\theta)) dF(\theta) = - \int_{\Theta} L_i(\theta, \bar{U}_{-i}(\theta), -\bar{q}_{-i}(\theta)) > -\infty$  to principal  $P_i$ .

• **Necessary conditions.** Theorem 10.2.1. in Vinter (2000) applies when a number of conditions that we now check are satisfied.

[a.] *Measurability and lower semi-continuity in  $(\theta, v)$  of  $L_i(\cdot, u, \cdot)$ .* Condition 1 implies those weaker requirements.

[b.] *Lipschitz continuity in  $u$  of  $L_i(\theta, \cdot, v)$ .* Observe that we have  $|L_i(\theta, u', v) - L_i(\theta, u, v)| \leq \max_{\theta \in \Theta} |f(\theta)| |u' - u|$  so that  $L_i(\theta, \cdot, v)$  is Lipschitz continuous.

[c.] *Boundedness of  $L_i(\theta, \bar{U}, v)$ .* Strategies lie in  $\mathcal{T}$  and are thus bounded, and  $\mathcal{V}$  is compact so that the Lagrangean is bounded below.

[d.] *Lipschitz condition on participation:* The function  $u \rightarrow h_i(\theta, u) = \bar{U}_{-i}(\theta) - u$  is Lipschitz continuous in  $u$  and lower semi-continuous in  $(\theta, u)$  since  $\bar{U}_{-i}(\theta)$  is absolutely continuous (see comments following Lemma 1).

When Conditions [a.] to [d.] above hold, Theorem 10.2.1 in Vinter (2000) shows that there exist an arc  $p_i$  which is absolutely continuous on  $\Theta$ , a non-negative real number  $\lambda_i \geq 0$ , a measure  $\mu_i$  and a  $\mu_i$ -integrable function  $\gamma_i : \Theta \rightarrow \mathbb{R}$  such that the following conditions hold.<sup>38</sup>

$$\lambda_i + \|p_i\|_{L^\infty} + \|\mu_i\|_{T.V.} = 1. \quad (\text{A9})$$

$$\dot{p}_i(\theta) \in \text{co} \left\{ \eta : \left( \eta, p_i(\theta) - \int_{[\underline{\theta}, \theta]} \mu_i(ds) \right) \in \lambda_i \partial L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) \right\} \text{ a.e.} \quad (\text{A10})$$

$$\left( p_i(\underline{\theta}), -p_i(\bar{\theta}) + \int_{\Theta} \gamma_i(s) \mu_i(ds) \right) = (0, 0). \quad (\text{A11})$$

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<sup>38</sup>Let us briefly add some useful notations. For  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , the epigraph of  $f$  is the set  $\text{epi} f \equiv \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}$ . The following definitions of normal subgradients, subgradients and limiting subgradients are used in the sequel:

$$\hat{\partial} f(\bar{x}) = \{v | (v, -1) \in \hat{N}_{\text{epi} f}(\bar{x}, f(\bar{x}))\}, \partial f(\bar{x}) = \{v | (v, -1) \in N_{\text{epi} f}(\bar{x}, f(\bar{x}))\}, \partial^\infty f(\bar{x}) = \{v | (v, 0) \in N_{\text{epi} f}(\bar{x}, f(\bar{x}))\}.$$

$$\begin{aligned}
& \left( p_i(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds) \right) \bar{v}(\theta) - \lambda_i L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) \\
&= \max_{v \in \mathbb{R}} \left( p_i(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds) \right) v - \lambda_i L_i(\theta, \bar{U}(\theta), v).
\end{aligned} \tag{A12}$$

$$\gamma_i(\theta) = \begin{cases} 0 & \text{if } h_i(\theta, \bar{U}(\theta)) < 0 \\ 1 & \text{if } h_i(\theta, \bar{U}(\theta)) = 0. \end{cases} \tag{A13}$$

Several remarks help us to rewrite those necessary conditions.

- Since  $\partial_u L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) = \nabla_u L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) = f(\theta)$ , (A10) yields

$$\dot{p}_i(\theta) = \lambda_i f(\theta) \text{ a.e.} \tag{A14}$$

- Since  $\partial_v L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) = \nabla_v L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))$ , (A12) implies:

$$p_i(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds) = \lambda_i \nabla_v L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)) \text{ a.e.} \tag{A15}$$

- Taking into account the expression of the extended-value Lagrangean  $L_i(\cdot)$ , the right-hand side of (A12) can be rewritten as:

$$\max_{v \in \mathcal{Q}} \left( p_i(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds) \right) v - \lambda_i L_i^0(\theta, \bar{U}(\theta), v). \tag{A16}$$

- From condition (A13), we deduce that  $\text{supp } \mu_i \subseteq \Omega_i^c$ .

Those preliminary remarks being made, we propose the following definition:

**Definition 8** *The minimizing process  $(\bar{U}, \bar{v})$  is a normal extremal when conditions (A9) to (A13) above are obtained for  $\lambda_i = 1$ .*

We are now ready to state:

**Lemma 2** *A minimizing process  $(\bar{U}, \bar{v})$  is a normal extremal.*

**Proof:** From (A14) and (A11), we get:

$$p_i(\theta) = \lambda_i F(\theta). \tag{A17}$$

Then define

$$r_i(\theta^-) = \lambda_i F(\theta) - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds). \tag{A18}$$

Observe that  $r_i(\underline{\theta}^-) = 0$ .

Suppose that  $\lambda_i = 0$ , then  $p_i(\theta) = 0$  (thus  $\|p_i\|_{L^\infty} = 0$ ) and  $r_i(\theta^-) = - \int_{[\underline{\theta}, \theta]} \gamma_i(s) \mu_i(ds)$ . Using (A11) yields

$$\int_{\Theta} \gamma_i(s) \mu_i(ds) = 0 \tag{A19}$$

where  $\gamma_i(s) = 1$  on  $\text{supp } \mu_i$ . Since  $\mu_i$  is a measure, (A19) implies that this is a zero measure (and thus  $\|\mu_i\|_{T.V.} = 0$ ). Inserting the corresponding values of  $\lambda_i$ ,  $\|p_i\|_{L^\infty}$  and  $\|\mu_i\|_{T.V.} = 0$  yields a contradiction with (A9).  $\blacksquare$

From Lemma 2, we may as well use the normalization  $\lambda_i = 1$  and get

$$r_i(\theta^-) = F(\theta) - \int_{[\underline{\theta}, \theta)} \gamma_i(s) \mu_i(ds). \quad (\text{A20})$$

Provided that the measure  $\mu_i$  is non-singular at any point in  $(\underline{\theta}, \bar{\theta})$  (see the proof below), we obtain (12) with the extra condition  $r_i(\bar{\theta}) = 0$  coming from (A11). Finally, (11) follows from using the definition (A18) into (A12).

• **Continuity and non-singular measure.** To prove this continuity, we adapt the arguments in Galbraith and Vinter (2004) to our strategic setting. Condition (11) implies by the rules governing subdifferentials of convex functions:

$$\dot{U}(\theta) = \bar{v}(\theta) = -\bar{q}(\theta) \in \partial_p H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) \quad (\text{A21})$$

where  $H_i(\theta, u, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is the Legendre-Fenchel dual transform of  $L_i(\theta, u, \cdot)$ .

**Lemma 3** *The following properties hold:*

- For each  $(\theta, u, p) \in \Theta \times \mathbb{R}^2$ ,  $\partial_p H_i(\theta, u, p)$  is single-valued, continuously differentiable and  $\partial_p H_i(\theta, u, p) = \nabla_p H_i(\theta, u, p)$ ;
- Fix  $(\theta, u)$ ,  $p \rightarrow \nabla_p H_i(\theta, u, p)$  is locally Lipschitz continuous.

**Proof:** By the representation of the subdifferential we have:

$$\partial_p H_i(\theta, u, p) = \{\xi : p\xi - L_i^*(\theta, u, \xi) = \max_{v \in \mathcal{V}} pv - L_i^0(\theta, u, v)\}.$$

The max above is achieved because  $\mathcal{V}$  is compact. Hence  $\partial_p H_i(\theta, u, p)$  is non-empty.

Take now two triplets  $(\theta, u, p) \in \Theta \times \mathbb{R}^2$  and  $(\theta, u, p') \in \Theta \times \mathbb{R}^2$  and choose  $v$  and  $v'$  such that  $v \in \partial_p H_i(\theta, u, p)$  and  $v' \in \partial_{p'} H_i(\theta, u, p')$ . By the fundamental property of convex subdifferentials (see Rockafellar and Wets, 2004, p. 511),  $p \in \partial_v L_i^*(\theta, u, v)$  and  $p' \in \partial_{v'} L_i^*(\theta, u, v')$ . By Condition 1,  $\partial_v L_i^0(\theta, u, v) = \nabla_v L_i^0(\theta, u, \cdot)$  is continuously differentiable, we have:  $\partial_v L_i^*(\theta, u, v) = \nabla_v L_i^*(\theta, u, v) + N_{\mathcal{V}}(v)$  and  $\partial_{v'} L_i^*(\theta, u, v') = \nabla_{v'} L_i^*(\theta, u, v') + N_{\mathcal{V}}(v')$  where  $N_{\mathcal{V}}(v) = \{x | (v, x - \bar{x}) \leq 0\}$  (resp.  $N_{\mathcal{V}}(v')$ ) is the normal cone at  $v \in \mathcal{V}$  (resp.  $v'$ ). Therefore, we have  $p = \nabla_v L_i^0(\theta, u, v) + e$  and  $p' = \nabla_{v'} L_i^0(\theta, u, v') + e'$  where  $e \in N_{\mathcal{V}}(v)$  and  $e' \in N_{\mathcal{V}}(v')$ . From this, we deduce:

$$\begin{aligned} |p - p'| |v - v'| &= |\nabla_v L_i^0(\theta, u, v) - \nabla_{v'} L_i^0(\theta, u, v') + e - e'| |v - v'| \\ &= (\nabla_v L_i^0(\theta, u, v) - \nabla_{v'} L_i^0(\theta, u, v'))(v - v') + e(v - v') + e'(v' - v). \end{aligned} \quad (\text{A22})$$

Because  $e \in N_{\mathcal{V}}(v)$  and  $e' \in N_{\mathcal{V}}(v')$ , we have  $e(v - v') \geq 0$  and  $e'(v' - v) \geq 0$ . Inserting into (A22), we get:

$$|p - p'| |v - v'| \geq (\nabla_v L_i^0(\theta, u, v) - \nabla_v L_i^0(\theta, u, v'))(v - v') \geq K |v - v'|^2$$

and

$$|v - v'| = |\partial_p H_i(\theta, u, p) - \partial_p H_i(\theta, u, p')| \leq \frac{1}{K} |p - p'|. \quad (\text{A23})$$

where the last but one equality follows from strict convexity of  $L_i^0(\theta, u, \cdot)$  (Condition 1).

From this, we immediately deduce that  $\partial_p H_i(\theta, u, p)$  is single-valued (take  $p = p'$  in the above inequality and observe that any pair  $(v, v') \in \partial_p H_i(\theta, u, p)^2$  is such that  $v = v'$ ), we thus denote  $\partial_p H_i(\theta, u, p) = \nabla_p H_i(\theta, u, p)$ . Inequality (A23) tells us also that  $p \rightarrow \nabla_p H_i(\theta, u, p)$  is locally Lipschitz continuous with Lipschitz modulus  $\frac{1}{K}$ . ■

From Lemma 3, the representation (A21), and the condition (A11) from the Maximum Principle, we deduce that  $\bar{v}(\theta)$  has left- and right-hand limits at all points  $\theta \in \Theta$  and one-sided limits at the end-points. Finally, we deduce that  $\bar{v}(\theta)$  is bounded, and  $\bar{U}(\theta)$  is Lipschitz continuous.

**Lemma 4**  $\mu_i$  has no atoms in  $(\underline{\theta}, \bar{\theta})$ ;  $r_i(\theta)$  is continuous on  $(\underline{\theta}, \bar{\theta})$ , has one-sided limits at end-points and  $\bar{v}(\theta)$  is continuous on  $\Theta$

**Proof:** Fix  $\theta \in \Theta$  and observe that  $\mu_i(\{\theta\}) = 0$  if  $h_i(\theta, \bar{U}(\theta)) = \bar{U}_{-i}(\theta) - \bar{U}(\theta) < 0$ . Suppose on the other hand that  $h_i(\theta, \bar{U}(\theta)) = 0$ . Then, we get:

$$\frac{1}{\epsilon} (h_i(\theta + \epsilon, \bar{U}(\theta + \epsilon)) - h_i(\theta, \bar{U}(\theta))) \leq 0 \leq \frac{1}{\epsilon} (h_i(\theta, \bar{U}(\theta)) - h_i(\theta - \epsilon, \bar{U}(\theta - \epsilon))).$$

Passing to the limit as  $\epsilon \downarrow 0$  and taking into account the right- and left-hand side differentiability of  $\bar{U}_{-i}(\cdot)$  at any point  $\theta$  yields:

$$\bar{U}_{-i}(\theta^+) - \bar{v}(\theta^+) \leq 0 \leq \bar{U}_{-i}(\theta^-) - \bar{v}(\theta^-) \text{ or } \bar{v}(\theta^-) \leq \bar{U}_{-i}(\theta^-) \leq \bar{U}_{-i}(\theta^+) \leq \bar{v}(\theta^+)$$

where the inequality  $\bar{U}_{-i}(\theta^-) \leq \bar{U}_{-i}(\theta^+)$  follows from the convexity of the implementable profile  $\bar{U}_{-i}(\cdot)$ . From this, we get  $\bar{v}(\theta^-) = \nabla_p H_i(\theta, \bar{U}(\theta), r_i(\theta^-)) \leq \bar{v}(\theta^+) = \nabla_p H_i(\theta, \bar{U}(\theta), r_i(\theta^+))$ . Because,  $H_i(\theta, \bar{U}(\theta), \cdot)$  is convex in  $p$ , we get  $r_i(\theta^-) \leq r_i(\theta^+)$ . Using (12), we obtain:

$$r_i(\theta^-) - r_i(\theta^+) = \gamma_i(\theta) \mu_i(\{\theta\}) \leq 0. \quad (\text{A24})$$

Given that  $\gamma_i(\theta) = 0$  for  $\theta \in \Omega_i^c$ , it follows that  $r_i(\cdot)$  is continuous at any such  $\theta \in \Omega_i^c$ .

Consider now  $\theta \in \Omega_i$ . We know then that  $\gamma_i(\theta) = 1$ . Henceforth, (A24) implies  $\mu_i(\{\theta\}) = 0$ , i.e.,  $\mu_i$  has no atom  $(\underline{\theta}, \bar{\theta})$ ;  $r_i(\cdot)$  is continuous; and  $\bar{v}(\theta^-) = \nabla_p H_i(\theta, \bar{U}(\theta), r_i(\theta)) = \bar{v}(\theta^+)$  so that  $\bar{v}(\cdot)$  is continuous as well. ■

This finally ends the proof of Theorem 1. ■

• **Proof of Theorem 2:** We first prove the following preliminary result.

**Proposition 8** *Under the assumptions of Theorem 1, in any pure strategy equilibrium with deterministic mechanisms, the equilibrium output  $\bar{q}(\theta)$  satisfies  $\forall i \in \mathcal{N}$ :*

$$0 \in \partial_q co \left\{ -\widetilde{W}_i \right\} (\bar{q}(\theta), \theta), \quad (\text{A25})$$

and

$$-co \left\{ -\widetilde{W}_i \right\} (\bar{q}(\theta), \theta) = \widetilde{W}_i(\bar{q}(\theta), \theta) \quad (\text{A26})$$

where  $\widetilde{W}_i(q, \theta) = S_0(q) + S_i(q) + \bar{t}_{-i}(q) - \left( \theta + \frac{r_i(\theta)}{f(\theta)} \right) q$ . and  $r_i(\theta)$  is defined in Theorem 1.

**Proof:** Let define a *stochastic mechanism* for principal  $P_i$  as a pair  $\{t_i(m_i), \mu_i(\cdot|m_i)\}_{m_i \in \mathcal{M}_i}$  where  $\mathcal{M}_i$  is an arbitrary message space for the agent to communicate with  $P_i$ . Conditionally on a message  $m_i$ , such stochastic mechanism stipulates a payment  $t_i(m_i)$  to the agent and recommends him to choose outputs according to the distribution function  $\mu_i(\cdot|m_i)$  whose support is included in  $\mathcal{Q}$ .<sup>39</sup> When  $\mathcal{M}_i \equiv \Theta$ , the stochastic mechanism is *direct*. Denote by  $\Delta\mathcal{T}$  the corresponding strategy space of such direct stochastic mechanisms.<sup>40</sup>

By the Legendre-Fenchel Transform Theorem<sup>41</sup>, the biconjugate  $L_i^c(\theta, u, v)$  of  $L_i(\theta, u, v)$  is itself convex, closed and proper and  $epi L_i^c(\theta, u, \cdot) = co\{epi L_i(\theta, u, \cdot)\}$ .<sup>42</sup> Hence, we get:

$$L_i^c(\theta, u, v) \equiv \max_{p \in \mathbb{R}} \{pv - H_i(\theta, u, p)\} \text{ and } H_i(\theta, u, p) \equiv \max_{v \in \mathbb{R}} \{pv - L_i^c(\theta, u, v)\} \equiv \max_{v \in \mathbb{R}} \{pv - L_i(\theta, u, v)\}$$

where the first and second equality follows from dualization and the last one is just the definition of  $H_i(\theta, u, p)$ . Condition (11) becomes

$$r_i(\theta^-) \in \partial_v L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta)) \quad (\text{A27})$$

where  $\partial_v L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta)) = \{\xi : L_i^c(\theta, \bar{U}(\theta), v) - L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta)) \geq \xi(v - \bar{v}(\theta))\}$  because  $L_i^c(\theta, \bar{U}(\theta), \cdot)$  is convex. This condition can alternatively be rewritten as (A25).

Observe that, in any pure-strategy equilibrium with deterministic mechanisms, we must have a.e.:

$$L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta)) = L_i(\theta, \bar{U}(\theta), \bar{v}(\theta)).$$

To see that, suppose the contrary, then, by definition, we have:

$$L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta)) < L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))$$

<sup>39</sup>Because of quasi-linearity of the agent's utility function, there is no value of using stochastic payments for incentive reasons.

<sup>40</sup>Considering deviations within such larger strategy space might increase principal  $P_i$ 's payoff. We will find below conditions so that such deviations are suboptimal in any equilibrium with deterministic mechanisms on which we focus.

<sup>41</sup>See Rockafellar and Wets (2004, p.474).

<sup>42</sup>Take any function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and define the convex envelope of  $f$  as  $co\{f\} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ :  $co\{f\}(x) = \min_{\alpha_i \geq 0, \sum_{i \in \mathcal{I}} \alpha_i = 1, x_i \in \mathcal{R}^n} \left\{ \sum_{i \in \mathcal{I}} \alpha_i f(x_i) \mid x = \sum_{i \in \mathcal{I}} \alpha_i f(x_i) \right\} = \min_{\alpha \geq 0, (x_1, x_2) \in \mathcal{R}^n \times \mathcal{R}^n} \{\alpha f(x_1) + (1-\alpha)f(x_2)\}$  where the last equality follows from Caratheodory Theorem (Rockafellar and Wets, 2004, Theorem 2.29).

for a set  $I$  of non-zero measure of types  $\theta$ . From Caratheodory Theorem, there exists  $(v_1(\theta), v_2(\theta)) \in \mathcal{V}^2$  and  $\alpha(\theta) \in (0, 1)$  such that  $\bar{v}(\theta) = \alpha(\theta)v_1(\theta) + (1 - \alpha(\theta))v_2(\theta)$ . Consider now the new mechanism obtained by replacing for any type  $\theta \in I$  the deterministic mechanism implementing  $(\bar{U}(\theta), \bar{v}(\theta))$  by a direct stochastic mechanism that recommends to the agent to randomize between  $v_1(\theta)$  and  $v_2(\theta)$  with probabilities  $\alpha(\theta)$  and  $1 - \alpha(\theta)$ . This stochastic mechanism entails a payment

$$t_i(\theta) = \bar{U}(\theta) - \theta \bar{v}(\theta) - \alpha(\theta)(\bar{t}_{-i}(-v_1(\theta)) + S_0(-v_1(\theta)) - (1 - \alpha(\theta))(\bar{t}_{-i}(-v_2(\theta)) + S_0(-v_2(\theta)))).$$

On the complementary set  $I^c$ , the mechanism is unchanged and remains deterministic. Incentive compatibility is preserved by definition since we have still  $\dot{U}(\theta) = \bar{v}(\theta)$  a.e. both on  $I$  and  $I^c$ . This direct stochastic mechanism allows principal  $P_i$  to reach a payoff  $\int_{\Theta} L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta < \int_{\Theta} L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta$  since  $\int_I L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta < \int_I L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta$  and  $\int_{I^c} L_i^c(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta = \int_{I^c} L_i(\theta, \bar{U}(\theta), \bar{v}(\theta))d\theta$ . This new mechanism would be a valuable deviation with respect to what that principal can get with a deterministic mechanism. Therefore, this cannot arise in any pure equilibrium with deterministic mechanisms. Finally, we get (A26). ■

**Strict differentiability of  $t_i^*(\cdot)$  at equilibrium point:**<sup>43</sup> Because  $L_i^0(\theta, u, \cdot)$  is strictly convex in  $q$  and continuously differentiable,  $-\widetilde{W}_i(q, \theta)$  is also strictly convex in  $q$ , so that  $co\{-\widetilde{W}_i\} = -\widetilde{W}_i$  and  $\partial_q\{-\widetilde{W}_i\} = \nabla_q\{-\widetilde{W}_i\}$ . Condition (A25) can thus be rewritten as:

$$\begin{aligned} 0 &= \partial_q \left\{ - \left( S_0(q) + S_i(q) + \bar{t}_{-i}(q) - \left( \theta + \frac{r_i(\theta)}{f(\theta)} \right) q \right) \right\} \Big|_{q=\bar{q}(\theta)} \\ &= \nabla_q \left\{ - \left( S_0(q) + S_i(q) + \bar{t}_{-i}(q) - \left( \theta + \frac{r_i(\theta)}{f(\theta)} \right) q \right) \right\} \Big|_{q=\bar{q}(\theta)} \\ &\Leftrightarrow 0 = -S'_0(\bar{q}(\theta)) - S'_i(\bar{q}(\theta)) + \theta + \frac{r_i(\theta)}{f(\theta)} + \partial_q(-t_{-i}^*)(\bar{q}(\theta)) \end{aligned}$$

where the last equality comes from observing that  $-S_0(q) - S_i(q) + \left( \theta + \frac{r_i(\theta)}{f(\theta)} \right) q$  is strictly differentiable in  $q$  and  $\partial_q(h + g)(q) = \partial_q h(q) + \partial_q g(q)$  when  $h$  is strictly differentiable.<sup>44</sup>

The fact that  $L_i^0(\theta, u, \cdot)$  is finite, convex and continuously differentiable implies that

$$0 = \partial_q^\infty \left\{ - \left( S_0(q) + S_i(q) + \bar{t}_{-i}(q) - \left( \theta + \frac{r_i(\theta)}{f(\theta)} \right) q \right) \right\} \Big|_{q=\bar{q}(\theta)}.$$

By Rockafellar and Wets, 2004, Exercice 8.8, this implies also that  $\partial_q^\infty(-\bar{t}_{-i})(\bar{q}(\theta)) = 0$ . Since  $\bar{t}_{-i} \in \mathcal{T}$ ,  $-\bar{t}_{-i}$  is lower semi-continuous. Hence, it is strictly differentiable at  $\bar{q}(\theta)$  by Rockafellar and Wets (2004, Theorem 9.18.c). Because strict differentiability is preserved

<sup>43</sup>A function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is strictly differentiable at a point  $\bar{x}$  if  $f(\bar{x})$  is finite and there exists a vector  $v$ , which is the gradient  $\nabla f(\bar{x})$  such that  $f(x') = f(\bar{x}) + \langle v, x' - \bar{x} \rangle + o(|x' - \bar{x}|)$ .

<sup>44</sup>See Rockafellar and Wets, 2004, Exercice 8.8 or 10.10.



by summation and subtraction, a simple induction argument immediately tells us that  $\bar{t}_i(q)$  is strictly differentiable at any equilibrium point  $\bar{q}(\theta)$ .

**Conditions on equilibrium output:** Condition (A25) can finally be rewritten as:

$$S'_0(\bar{q}(\theta)) + S'_i(\bar{q}(\theta)) + \bar{t}'_{-i}(\bar{q}(\theta)) = \theta + \frac{r_i(\theta)}{f(\theta)}. \quad (\text{A28})$$

Summing those equalities over  $i$  yields:

$$S'_0(\bar{q}(\theta)) + \sum_{i=1}^n S'_i(\bar{q}(\theta)) + (n-1) \left( S'_0(\bar{q}(\theta)) + \sum_{i=1}^n \bar{t}'_i(\bar{q}(\theta)) - \theta \right) = \theta + \frac{\sum_{i=1}^n r_i(\theta)}{f(\theta)}. \quad (\text{A29})$$

Observe that  $\bar{t}_i(q)$  being strictly differentiable at any equilibrium point  $\bar{q}(\theta)$  implies

$$0 \in \partial_q \left\{ -S_0(q) - \sum_{i=1}^n \bar{t}_i(q) + \theta q \right\} \Big|_{q=\bar{q}(\theta)}$$

which itself implies

$$S'_0(\bar{q}(\theta)) + \sum_{i=1}^n \bar{t}'_i(\bar{q}(\theta)) = \theta. \quad (\text{A30})$$

Now, note that at any  $\theta \in \bar{\Omega}_i^c$ , it must be that

$$\bar{t}_i(\bar{q}_{-i}(\theta)) = \bar{t}'_i(\bar{q}_{-i}(\theta)) = 0. \quad (\text{A31})$$

So that, finally, (13) holds. Inserting those findings into (A29) yields

$$S'_0(\bar{q}(\theta)) + \sum_{i=1}^n S'_i(\bar{q}(\theta)) = \theta + \frac{\sum_{i=1}^n r_i(\theta)}{f(\theta)} \quad (\text{A32})$$

which, taking again (A31) into account yields (14).

**Monotonicity of  $\bar{q}(\theta)$ :** We already know that  $\bar{q}(\cdot)$  is continuous on  $(\underline{\theta}, \bar{\theta})$ . This implies that it is enough to show monotonicity on each interval  $I$  where  $\alpha(\theta)$  is fixed (denote  $\bar{I}$  the interior of such interval). Note that on any such interval  $\sum_{i \in \alpha(\theta)} r_i(\theta)$  is constant and such that  $\sum_{i \in \alpha(\theta)} r_i(\theta) \leq |\alpha(\theta)|$ . Differentiating (14) w.r.t.  $\theta$  on such interval  $I$  yields:

$$\dot{\bar{q}}(\theta) \left( \sum_{i \in \alpha(\theta) \cup \{0\}} S''_i(\bar{q}(\theta)) \right) = 1 + \frac{d}{d\theta} \left( \frac{|\alpha(\theta)| F(\theta) - \sum_{i \in \alpha(\theta)} r_i(\theta)}{f(\theta)} \right). \quad (\text{A33})$$

The derivative of the right-hand side becomes:

$$\frac{d}{d\theta} \left( \frac{\left( |\alpha(\theta)| - \sum_{i \in \alpha(\theta)} r_i(\theta) \right) F(\theta) - \left( \sum_{i \in \alpha(\theta)} r_i(\theta) \right) (1 - F(\theta))}{f(\theta)} \right)$$

$$= \left( |\alpha(\theta)| - \sum_{i \in \alpha(\theta)} r_i(\theta) \right) \frac{d}{d\theta} \left( \frac{F(\theta)}{f(\theta)} \right) - \left( \sum_{i \in \alpha(\theta)} r_i(\theta) \right) \frac{d}{d\theta} \left( \frac{1 - F(\theta)}{f(\theta)} \right) > 0 \quad (\text{A34})$$

where the last inequality follows from Assumption 3. Inserting into (A33) yields the requested monotonicity  $\dot{q}(\theta) < 0 \quad \forall \theta \in \bar{I}$ .  $\blacksquare$

• **Proof of Proposition 3:** On any connected interval included in  $\Omega_i^c$ , equation (A31) holds both in the interior but also on the boundaries from Proposition 4. Taking into account (A28) and the agent's first-order condition for optimality (13) yields (15).  $\blacksquare$

• **Proof of Proposition 4:** We prove each point in turn.

**Strict differentiability of equilibrium schedules.** We have shown in passing in the proof of Theorem 2 that  $t_i^*(q)$  is strictly differentiable at any equilibrium point  $\bar{q}(\theta)$ .

**Smooth-pasting.** Consider  $\theta$  on the boundary of  $\Omega_i^c$ , i.e., such that  $\bar{U}(\theta) = \bar{U}_{-i}(\theta)$ , or

$$\bar{t}_i(\bar{q}(\theta)) + \bar{t}_{-i}(\bar{q}(\theta)) - \theta \bar{q}(\theta) + S_0(\bar{q}(\theta)) = \bar{t}_i(\bar{q}_{-i}(\theta)) + \bar{t}_{-i}(\bar{q}_{-i}(\theta)) - \theta \bar{q}_{-i}(\theta) + S_0(\bar{q}_{-i}(\theta))$$

where this equality follows from the fact that  $\bar{q}(\theta)$  is continuous at such  $\theta$  and  $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$  on  $\Omega_i^c$ . This yields  $\bar{t}_i(\bar{q}(\theta)) = 0$ .

Finally, (A30) above tells us that any  $\theta \in \Omega_i^c$  is such that:

$$\bar{t}'_i(\bar{q}(\theta)) + \bar{t}'_{-i}(\bar{q}(\theta)) + S'_0(\bar{q}(\theta)) = \theta = \bar{t}'_{-i}(\bar{q}_{-i}(\theta)) + S'_0(\bar{q}_{-i}(\theta)).$$

Together with the continuity property  $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$ , we get  $\bar{t}'_i(\bar{q}(\theta)) = 0$ .

**Marginal contributions.** First, observe that (A28) can be rewritten as

$$S'_0(\bar{q}(\theta)) + S'_i(\bar{q}(\theta)) + \sum_{j \in \alpha(\theta)/i} \bar{t}_j(\bar{q}(\theta)) = \theta + \frac{r_i(\theta)}{f(\theta)} \quad (\text{A35})$$

since inactive principals contribute zero both at the margin and in value. Using then (13), we get the expression of the following marginal contribution:

$$\bar{t}'_i(\bar{q}(\theta)) = S'_i(\bar{q}(\theta)) - \frac{r_i(\theta)}{f(\theta)}. \quad (\text{A36})$$

Now observe that, on a connected subset  $\Omega_i^j$  of the activity set  $\Omega_i$ , we have  $\int_{\underline{\theta}}^{\theta} \mu_i(d\theta) = \int_{\underline{\theta}}^{\theta^j} \mu_i(d\theta) = M_i^j \in [0, 1]$ .  $\blacksquare$

• **Proof of Corollary 1:** When  $S_i(q) = s_i q$  for some  $s_i > 0$  (the proof for  $s_i < 0$  is similar and omitted), (15) amounts to

$$F(\theta) - f(\theta)s_i = \int_{[\underline{\theta}, \theta]} \gamma_i(\theta) \mu_i(d\theta) \quad \forall \theta \in \bar{\Omega}_i^c. \quad (\text{A37})$$

We want to prove that the inactivity set  $\Omega_i^c$  is of the form  $[\theta_i^*, \bar{\theta}]$  where  $s_i = R(\theta_i)$ . Take  $\mu_i(\theta) = f(\theta) - \dot{f}(\theta)s_i$  for  $[\theta_i^*, \bar{\theta}]$ . Observe that:

$$\frac{\mu_i(\theta)}{f(\theta)} = 1 - \frac{\dot{f}(\theta)}{f(\theta)}s_i > 1 - \frac{f(\theta)}{F(\theta)}s_i \geq 0$$

where the first inequality follows from  $s_i > 0$  and Assumption 3 (since then  $1 \geq \frac{F(\theta)\dot{f}(\theta)}{f^2(\theta)}$ ) and the second inequality follows from Assumption 3 and the definition of  $\theta_i^*$ . ■

• **Proof of Corollary 2:** Take the case  $s_i > 0$  (the proof for  $s_i < 0$  is similar and omitted). Using the specification of the activity set  $\Omega_i = [\underline{\theta}, \theta_i^*]$  given in Corollary 1 and the smooth-pasting condition (16) yields immediately (18). ■

• **Proof of Proposition 5:** We are looking for an equilibrium with a continuous output schedule  $\bar{q}(\theta)$ . We follow the steps of the general analysis in characterizing such an equilibrium through its activity sets, contributions and output.

• **Activity sets.** First, observe that  $q^I(\theta) \geq q_2(\theta)$  if and only if  $\theta \leq s_1$ . We now prove that  $\Omega_1 = [0, s_1]$ . To do so, we use (15) to derive the non-singular measure  $\mu_1$  on  $\Omega_1^c = [s_1, \bar{\theta}]$ . We have:

$$r_1(\theta) = \frac{\theta}{\bar{\theta}} - \int_0^\theta \mu_1(d\theta) = \frac{S'_1(q_2(\theta))}{\bar{\theta}} = \frac{s_1}{\bar{\theta}}.$$

Reminding that  $\mu_1$  is non-singular for a continuous equilibrium and denoting  $\mu_1(d\theta) = \eta_1(\theta)\frac{d\theta}{\bar{\theta}}$ , we compute immediately  $\eta_1(\theta) = 1$  if  $\theta \in [s_1, \bar{\theta}]$ .

We now prove that  $\Omega_2 = [0, s_2]$ . We use again (15) to derive the non-singular measure  $\mu_2$  on  $\Omega_2^c = [s_2, \bar{\theta}]$ . We have:

$$r_2(\theta) = \frac{\theta}{\bar{\theta}} - \int_0^\theta \mu_2(d\theta) = \frac{S'_2(0)}{\bar{\theta}} = \frac{s_2}{\bar{\theta}}.$$

Reminding that  $\mu_2$  is non-singular and denoting  $\mu_2(d\theta) = \eta_2(\theta)\frac{d\theta}{\bar{\theta}}$ , we get  $\eta_2(\theta) = 1$  if  $\theta \in [s_2, \bar{\theta}]$ .

• **Outputs.** The formula for computing the equilibrium output follows immediately from using (14) and the definition of the activity sets above.

• **Contributions.** Principals have congruent preferences and thus marginal contributions, on their activity respective sets, are of the form

$$\bar{t}'_i(q) = S'_i(q) - R(\bar{\theta}(q)) = s_i - \frac{\bar{\theta}(q)}{\bar{\theta}}.$$

Moreover, the binding participation constraint for principal  $P_2$ , namely  $\bar{U}(s_2) = \bar{U}_1(s_2) = 0$  determines completely  $\bar{t}_2(q)$  as in (23). In turn, the binding participation constraint for principal  $P_1$ , namely  $\bar{U}(s_1) = \bar{U}_2(s_1) = \max_q \left\{ \bar{t}_2(q) - \theta q - \frac{q^2}{2} \right\} = \int_{\frac{s_2}{2}}^{s_1} (s_2 - 2\theta)d\theta > 0$

determines completely  $\bar{t}_1(q)$  as in (23). ■

• **Proof of Proposition 6:** We are again looking for an equilibrium with a continuous output schedule  $\bar{q}(\theta)$ . We follow the steps of the general analysis in characterizing such an equilibrium through its activity sets, output and contributions.

• **Activity sets.** First, observe that  $-3\theta \geq \bar{q}_1(\theta)$  if and only if  $\theta \leq -1 + \delta$ . We now prove that  $\Omega_1 = [-\delta, 1 - \delta)$  when  $\delta \geq \frac{1}{2}$  and  $\Omega_1 = [-\delta, \delta)$  when  $\delta \leq \frac{1}{2}$ .

Let us begin with  $\delta \geq \frac{1}{2}$ . We use (15) to derive the non-singular measure  $\mu_1$  on  $\Omega_1^c = [1 - \delta, \delta]$ . We have on that interval :

$$r_1(\theta) = \frac{\theta + \delta}{2\delta} - \int_{1-\delta}^{\theta} \mu_1(d\theta) = \frac{S'_1(\bar{q}_2(\theta))}{2\delta} = \frac{1}{2\delta}.$$

Reminding that  $\mu_1$  is non-singular for a continuous equilibrium and denoting  $\mu_1(d\theta) = \eta_1(\theta) \frac{d\theta}{2\delta}$ , we compute  $\eta_1(\theta) = 1$  if  $\theta \in \Omega_1^c$ . The non-singular measure  $\mu_2$  on  $\Omega_2^c = [-\delta, -1 + \delta]$  is obtained by symmetry.

Consider now the case  $\delta \leq \frac{1}{2}$ . We have then full coverage with  $\mu_1(\{\delta\}) = 1$  and by symmetry  $\mu_2(\{\delta\}) = 1$ .

• **Outputs.** The formula for computing the equilibrium output follows immediately from using (14) and the definition of the activity sets above.

• **Contributions.** Principals have conflicting preferences and thus marginal contributions on their activity sets are of the form

$$\bar{t}_1(q) = S'_1(q) - R(\bar{\theta}(q)) = \frac{\delta - \bar{\theta}(q)}{2\delta} \text{ and } \bar{t}_2(q) = S'_2(q) + T(\bar{\theta}(q)) = -\frac{\delta + \bar{\theta}(q)}{2\delta}.$$

When  $\delta \geq \frac{1}{2}$ , the binding participation constraint for principal  $P_1$ , namely  $\bar{U}(1 - \delta) = \bar{U}_2(1 - \delta)$ , determines completely  $\bar{t}_1(q)$  as in (24). In particular, note that  $\bar{t}_1(\bar{q}(1 - \delta)) = \bar{t}'_1(\bar{q}(1 - \delta)) = 0$ , i.e., the smooth-pasting condition holds. By symmetry, we obtain  $\bar{t}_2(q)$ .

When  $\delta \leq \frac{1}{2}$ , there is full coverage and the binding participation constraint for principal  $P_1$  becomes  $\bar{U}(\delta) = \bar{U}_2(\delta)$ . It again determines completely  $\bar{t}_1(q)$  as in (24). Note in particular that  $\bar{t}_1(\bar{q}(\delta)) > 0$  because of full coverage. A symmetry argument gives  $\bar{t}_2(q)$ . ■

• **Proof of Proposition 7:** Consider a discontinuity at some  $\theta_0$  in the interior of  $\Theta$  where principal  $P_i$  ceases to be active. For simplicity, we consider only the case where  $P_i$  is active on  $\Omega_i = [\underline{\theta}, \theta_0)$  (the case where  $P_i$  is active on  $\Omega_i = (\theta_0, \bar{\theta}]$  can be treated symmetrically and is omitted). Provided that the conditions of Theorem 1 hold on  $\Omega_i$ , the optimality conditions [1.], [2.] and [3.] still hold on  $\Omega_i$  with the extra condition  $r_i(\theta) = F(\theta)$ .

**Optimality condition for principal  $P_i$ .** Consider first the optimality condition for principal  $P_i$  with respect to the set of types that this principal is ready to target with a positive contribution. Observe first that  $U(\theta) = \bar{U}_{-i}(\tilde{\theta}_0) + \int_{\tilde{\theta}_0}^{\theta} q(x)dx$  where  $\Omega_i = [\underline{\theta}, \tilde{\theta}_0)$  is the activity set of that principal in a monotonic equilibrium.

Consider a monotonic equilibrium with outputs  $\bar{q}(\theta)$  on  $\Omega_i = [\underline{\theta}, \theta_0)$  and  $\bar{q}(\theta) = \bar{q}_{-i}(\theta)$  on  $\Omega_i^c$ . It must be that:

$$\theta_0 \in \arg \max_{\tilde{\theta}_0 \in \Theta} \int_{\underline{\theta}}^{\tilde{\theta}_0} (S_i(\bar{q}(\theta)) + S_0(\bar{q}(\theta)) - (\theta + R(\theta))\bar{q}(\theta) - \bar{U}_{-i}(\tilde{\theta}_0)f(\theta))d\theta + \int_{\tilde{\theta}_0}^{\bar{\theta}} S_i(\bar{q}_{-i}(\theta))f(\theta)d\theta.$$

Taking into account that  $\dot{\bar{U}}_{-i}(\theta) = -\bar{q}_{-i}(\theta)$ , the first-order condition for this optimality condition yields (26).

All principals must agree on choosing that discontinuity  $\theta_0$  and condition (26) must hold for all principals starting being active on the left-neighborhood of  $\theta_0$ . Similarly, (27) applies for all principals starting being active on the right-neighborhood of  $\theta_0$ .

**The agent's indifference.** Condition (28) immediately follows from incentive compatibility and the fact that the agent with type  $\theta_0$  is just indifferent between moving to the right or to the left of the discontinuity. ■

• **Proof of Corollary 3:** Summing conditions (26), (27) and (28) at a discontinuity point yields (29) in the case of conflicting principals.

Consider now the case of congruent principals (with  $s_i > 0$  for  $i = 1, 2$ ). At a discontinuity, we should have:

$$\left[ S_0(q) + \left( \sum_{i=1}^2 s_i - \theta_0 \right) q - 2R(\theta_0)q \right]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)} = 0. \quad (\text{A38})$$

Suppose w.l.o.g that principal  $P_1$ 's activity set is  $\Omega_1 = [\underline{\theta}, \theta_0)$ .

*Case 1: Principal  $P_2$  active on  $\Omega_2 = [\underline{\theta}, \theta'_0)$  with  $\theta'_0 > \theta_0$ .* Then we must have  $\bar{q}(\theta_0^+) = \bar{q}_2(\theta_0)$  where  $\bar{q}_2(\theta)$  is such that  $S'_0(\bar{q}_2(\theta)) + s_2 = \theta + R(\theta)$  on  $[\underline{\theta}, \theta'_0]$ . For  $\theta \in \Omega_1$ , we have instead  $\bar{q}(\theta)$  such that

$$S'_0(\bar{q}(\theta)) + \sum_{i=1}^2 s_i = \theta + 2R(\theta).$$

Clearly, we have then:

$$\left[ S_0(q) + \left( \sum_{i=1}^2 s_i - \theta_0 \right) q - 2R(\theta_0)q \right]_{\bar{q}(\theta_0^+)}^{\bar{q}(\theta_0^-)} > 0$$

and (A38) cannot hold.

*Case 2: Principal  $P_2$  active on  $\Omega_1 = [\underline{\theta}, \theta_0)$ .* We would have  $\bar{q}(\theta_0^+) = 0 < \bar{q}(\theta_0^-)$  and again (using  $S(0) = 0$ )

$$\left[ S_0(q) + \left( \sum_{i=1}^2 s_i - \theta_0 \right) q - 2R(\theta_0)q \right]_0^{\bar{q}(\theta_0^-)} = S_0(\bar{q}(\theta_0^-)) + \left( \sum_{i=1}^2 s_i - \theta_0 \right) \bar{q}(\theta_0^-) - 2R(\theta_0)\bar{q}(\theta_0^-) > 0$$

where the last inequality follows from strict concavity of  $S_0(q) + (\sum_{i=1}^2 s_i - \theta_0)q - 2R(\theta_0)q$  and the fact that this function is maximized at  $\bar{q}(\theta_0^-)$  and worth zero at  $q = 0$ . Hence, (A38) again cannot hold. ■

• **Proof of Theorem 3:** Item [1.] is obvious given that the necessary conditions for optimality that hold for our characterization of continuous equilibria must also hold for discontinuous ones on any continuity interval.

Item [2.] follows from adding up conditions (26), (27) and (28) at any discontinuity point  $\theta_0$  once one has noticed that contributions have to remain positive and that principals have linear surpluses. ■