Contractible Contracts in Common Agency Problems

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October 30, 2009

Abstract

This paper analyzes contactual situations between many principals and many agents. Agents have private information and principals take actions. Principals can contract not only on the reports of the agents but also on the contracts offered by the other principals. Contracts are required to be representable in a formal language. The main result of the paper is a full characterization of the allocations that can be implemented as equilibria in our contracting game.

Equilibrium contracts are shown to be *incomplete*, in general. That is, a contract only restricts the action space of a principal but does not necesarrily determines a single action. Finally, certain environments are identified where the contractibility of contracts can only decrease social welfare.

1 Introduction

The most famous example of a common agency problem is the *meet the competition clause*. There are many firms (principals) selling a good to a consumer (agent). Each firm posts a price but can credibly promise to sell its product at the lowest price posted by its competitors. The ability to make such a promise makes it possible to sustain any price between the monopoly and the competitive prices. However, it is essential in this example that the posted prices are publicly observable and can be contracted upon. If the prices were not observable and the firms had to rely on the report of the consumer, the competitive price would be the only sustainable market price.[1]

Although this example is used as motivation in a large number of papers dealing with common agency models, these papers all assume that the contracts cannot be contracted directly, only

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[1] Other examples for contractible contracts are the reciprocical trade agreements, like the GATT. A reciprocical contract specifies to set a low tariff against one country if the other country also has such a reciprocical contract. Finally, tax treaties sometimes have this flavor – for example, out of state residents who work in Pennsylvania are exempt from Pennsylvania tax as long as they live in a state that has a reciprocal agreement that exempts out of state residents from Pennsylvania from state taxes. See http://www.revenue.state.pa.us/revenue/cwp/view.asp?A=238&Q=244681.

through the reports of the agents. This paper departs from the existing literature in two ways. First, the contracts are assumed to be publicly observable and contractible. That is, a principal can credibly make his action contingent on the contracts of the other principals, and there is no need for the agents to communicate what kind of contracts they received from the other principals. Second, the contracts are required to be describable in a formal language. Below, we elaborate on both of these assumptions.

These two assumptions lead to a characterization of the allocations that can be implemented as equilibria of our contracting game. Indeed, the main result of this paper is a *Folk Theorem*. We show that the possibility of contracting on contracts enables the principals to collude and implement various outcomes, much like in repeated games. The collusion is accomplished by writing contracts which punish a principal if his contract is not the one which the others expected from him in equilibrium. Since contracts are contractible, a principal is able to commit to punish a deviator although the interaction is not repeated. We identify environments where the contractibility of contracts can offset any efficiency gain generated by competition among the principals. Therefore, a policy implication of our results is that contracting on contracts should be prohibited in certain environments.

In the specific model analyzed in this paper there are several principals and several agents. The agents have types and principals can take actions. Each principal wishes to enter into a contractual relationship with each agent. Following the usual approach of the literature, we analyze equilibria in *communication games*. In a communication game, the agents are endowed with message spaces, and the game has three stages. At the first stage, principals offer contracts to the agents simultaneously. A contract of a principal is a mapping from message profiles of the agents and contract profiles of the principals to the subsets of the action space of the principal.[2] During the second stage, agents simultaneously send messages to each principal. In the final stage, principals take actions from the subsets of their actions that were determined by both the first-stage contracts, and the second-stage message profiles. Our goal is to characterize the set of equilibrium outcomes of these games.

The *infinite regress problem* is one of the difficulties of solving these models. It can be observed when there are two principals and the payoff of each principal depends on the action of the other principal. This action is contracted on by the other principal and the agents, therefore, both principals want to offer contracts to the agents which are contingent on the contracts offered by the other principal. A typical contract of a principal will be contingent on the contract of the other principal, which, in turn, is contingent on the contract of the other principal etc. It is not clear how to construct a contract space which allows for this kind of hierarchical dependency. Therefore, perhaps the most important feature of a common agency model is the set of contacts available to the principals. Next, we describe the contract space of our model in more detail.

[2] The literateture on common agency models usually assumes that the contracts determine a single action for the principal as opppsed to a subset. We show that this assumption is with the loss of generality.

The Contract Space.— We endow each market participant with a formal language. We require each contract offered by a principal and each message sent by an agent to be a text written in this language, where a text is a finite string of symbols. It is well known that there are bijections from the set of texts into the set of integers. One such a mapping is called the *Godel Coding*. This implies that any contract and message uniquely corresponds to an integer. As we mentioned earlier, a contract of a principal is a mapping from contract profiles and message profiles to subsets of the action space of the principal. Since the contracts and the messages correspond to integers, one can think of such a contract as a description of an arithmetic correspondence from the codes of contracts and messages to the codes of the names of the actions of the principals. There is a well-known set of arithmetic correspondences, called the *definable correspondences,* which can be precisely described in the formal language by using finitely many characters. (We shall formally define this set later.) Hence, one can think of the contract space as the set of definable functions from $\mathbb{N}^{n+k} \to 2^{\mathbb{N}}$, where n is the number of principals and k is the number of agents. The first n arguments of these functions are the codes of the contracts of the principals, and the last k arguments are the codes of the messages of the agents. The range of these functions is the subsets of the codes of the names of the actions of a principal. We identify the contracts space of a principal with the set of definable correspondences.

The introduction of arithmetics and Godel Coding into our model deserves some explanation. Take it as given that contracts and messages must be expressed in a formal language. Then a contract of a principal must give *precise instructions* how to restrict his action space as a function of the texts submitted by the other market participants. In order to describe the contracting game, one must carefully define what "precise instruction" means and the set of those texts which give these instructions. Any such definition would lead to a definition of a set of arithmetic correspondences which can be described as finite texts. To see this, suppose that there is a text which gives instructions how to pin down a subset of a principal's action space as a function of the other texts. Then, there is also a text that gives the same instructions as a function of the Godel Codes of texts of the other market participants instead of their texts. This is because the Godel Coding and its inverse are definable functions, that is, they can be described as texts. This implies that this new text describes an arithmetic correspondence. In this paper we adopt the definition of definable functions from arithmetics instead of introducing a new definition. In fact, the set of definable functions is the largest set of arithmetic functions which can be described in a first-order language. Therefore, our contract space is the largest given the restriction to messages and contracts which can be expressed as texts. Implicitly, our approach makes it possible for players to offer any finite text as a contract. We simply identify the original text with the corresponding definable mapping.

The contracting game we consider is the following: First, the principals offer contracts simultaneously. These contracts are publicly observed. Second, the agents send messages to each principal privately. Compute the Godel Code of each contract and message. Fit these codes into each principal's contract to determine a unique set of actions available to every principal. Finally, once the sets of actions of each principal have been specified, principals choose from these sets simultaneously. Our goal is to characterize the set of equilibria of this game.

The Main Result.— We prove a *folk theorem* for our environment. We show that an allocation can be implemented as an equilibrium in the contractible contracting game if and only if the allocation is *strongly incentive compatible* and the payoff of each principal induced by the allocation is larger than his minmax value. Below, we define the notion of strong incentive compatibility and explain what the minmax value of a principal is.

A allocation in our model is a mapping from type profiles of the agents to action profiles of the principals. Each coordinate of an allocation is a mapping from the type profiles of the agents to the action space of a certain principal. Suppose that each principal offers a direct mechanism implementing his coordinate of the allocation simultaneously. An allocation is said to be strongly incentive compatible if truth-telling by the agents constitute an equilibrium in the product of these direct mechanisms. That is, no agent can increase his payoff by misreporting his type to the principals.[3]

It remains to explain what Principal q's minmax value is. In order to do so call a contract of a principal *ordinary* if it does not condition on the contracts of the other principals.[4] Modify our contracting game so that each principal has to offer an ordinary contract at the first stage. Suppose that Principal j's goal at the first stage is to minimize the payoff of Principal q for all $j \neq q$. Define the minmax value of Principal q as his lowest equilibrium payoff in this game. This minmax value is similar to the standard definition, except that the principals can only punish Principal q in the contracting stage, but each player behaves strategically in the subgame generated by the contract profile. In the subgame, they can only punish Principal q by playing an equilibrium which is the worst for him.

The difficult part of the statement of our folk theorem is the minmax values of the principals. Since contracts are contractible, one might imagine that the punishment inflicted on a deviating principal can potentially depend on the actual deviation. If punishments could be contingent on the deviator's contract then one might suspect that a deviator can be pushed below his minmax value, perhaps even to his maxmin value. This argument turns out to be false. Despite the contractibility of the contracts, the punishments of the principals can only depend on the deviator's identity but not on his contract. In other words, when the principals punish a deviator they use ordinary contracts. This fact is due to a result in mathematical logic stated in Proposition 2.

We emphasize that we were able to characterize the set of equilibrium allocations without any reference to the contractibility of the contracts. Indeed, the set of strongly incentive compatible allocations is defined in terms of the preferences of the agents and the minmax values of the

[3] This definition is stronger than the standard definition of incentive compatibility becase an agent can report different types to different principals.

[4] That is, an ordinary contract is a mapping from the message profile of the agents to the subsets of the action space of the principal.

principals are defined in terms of ordinary contracts. We also show that with one additional assumption even the minmax values can be characterized in terms of the physical environment only, in particular, in terms of the preferences of the principals. The assumption is that there exists an action profile used to minmax a principal which does not depend on the types of the agents. Essentially, this means that each principal can be prevented from interacting with the agents by the other principals. Such a type-independent punishment exists in many economic applications. For example, if the principals are sellers and the agents are buyers, then the sellers can set prices so low then every seller is best off if he exists the market. If the principals are employers and the agents are workers then the employers can offer so high wages to the workers that every employer would maximize his payoff by not competing for the workers. If this assumption is satisfied, then our characterization is purely in terms of the physical environment.

We also show that there are allocations which can only be implemented by contract profiles which do not pin down single actions of the principals in the last stage of the game. An equilibrium contract typically specifies only a subset of the action space of the principal and not a single action. The reason is that there is a trade-off between committing to a small set of actions and having flexibility at the last stage of the game. On the one hand, more commitment can increase ex ante efficiency. On the other hand, more flexibility can deter certain deviations. Indeed, a deviation might be more attractive if the deviator knows exactly what actions his opponents will take at the last stage of the game. In this sense, equilibrium contracts are often *incomplete*. This observation might provide a new insight for why contracts are often incomplete in the real world.

There is another sense in which restricting attention to complete contracts is with the loss of generality. We show that there are allocations which can be supported as an equilibrium if contracts are required to be complete, but cannot be supported if contracts are allowed to be incomplete. This is because a principal might profitably deviate by offering an incomplete contract, but there might be no such a deviation in the form of a complete contract.

An Example.— To illustrate how our approach works, consider the following example. There are two firms (1 and 2) and a single consumer. Each firm can produce a good at no cost. The goods are close substitutes but not identical. The consumer has two equally likely types, A and B. If his type was A, he values firm 1's good at 8 and firm 2's good at 5. If his type was B, he values firm 2's good at 9 and firm 1's good at 6. His marginal value for the second good is zero. The action space of each firm is setting a price from the set $\{1, ..., 10\}$. Firms want to maximize profit and the agent wants to maximize his value for the good he purchases minus the price. If firms were to set prices simultaneously without being able to contract on contracts, the market price would be one. The joint profit of the firms would be maximized if the consumer buys the good from firm 1 at price 8 if his type is A and he buys from firm 2 at price 9 if his type is B. Below, we show that this allocation is an equilibrium outcome in our contracting game.

Let $[\varphi]$ denote the Godel code of the text φ and refer to $[\varphi]$ as the 'encoding' of φ. Consider the following two contract of Firm 1:

$$c_1^{n_2}([c_2],[m]) = \begin{cases} 8 & \text{if } [c_2] = n_2, \text{ and } [m] = [A] \\ 10 & \text{if } [c_2] = n_2, \text{ and } [m] \neq [A] \\ 1 & \text{otherwise.} \end{cases},$$

where c_2 denotes the contract of Firm 2 and m denotes the message of the consumer sent to Firm 1. This contract says that if the Godel code of Firm 2's contract is n_2 and the consumer reports type A, then the price of Firm 1 is 8. If the Godel code of Firm 2's contract is n_2 but the consumer does not report type A then the price is 10. Otherwise, the price is one. Similarly, define Firm 2's contract as follows:

$$c_2^{n_1}([c_1],[m]) = \begin{cases} 9 & \text{if } [c_1] = n_1, \text{ and } [m] = [B] \\ 10 & \text{if } [c_1] = n_1, \text{ and } [m] \neq [B] \\ 1 & \text{otherwise.} \end{cases}$$

Notice that if $[c_1^{n_2}] = n_1$ and $[c_2^{n_1}] = n_2$ then these contracts are cross-referential. If the firms offer these contracts then the consumer maximizes his payoff if he reports his true type to each firm and buys the product from Firm 1 if his type is A and from Firm 2 otherwise. Hence, these contracts implement the desired allocation. In addition, the firms have no incentive to offer a different contract because any deviation would result a price of one set by the other firm. In what follows, we construct such a pair of cross-referential contracts.

Before we proceed, we introduce two pieces of notations. First, the function $\langle x \rangle$ is the inverse operation to the Godel coding. That is, $< n >$ is the text whose Godel code is n. Second, if ϕ is a text, then $\phi^{(n_1,n_2)}$ is the same text as ϕ except that if ϕ contained the free variables x or y then the value of the free variable x is set to be n_1 and the value of the free variable y is set to be n_2. Now, consider the following two texts:

$$c_1^{x,y}([c_2],[m]) = \begin{cases} 8 & \text{if } [c_2] =< y^{(x,y)} >, \text{ and } [m] = [A] \\ 10 & \text{if } [c_2] =< y^{(x,y)} >, \text{ and } [m] \neq [A] \\ 0 & \text{otherwise.} \end{cases}$$

$$c_2^{x,y}([c_1],[m]) = \begin{cases} 9 & \text{if } [c_1] = \left[< x^{(x,y)} >\right], \text{ and } [m] = [B] \\ 10 & \text{if } [c_1] = \left[< x^{(x,y)} >\right], \text{ and } [m] \neq [B] \\ 0 & \text{otherwise.} \end{cases}$$

These texts are not contracts, because they contain free variables. However, if these free variables are evaluated at integers, they do become contracts. Let γ_1 and γ_2 denote the Godel codes of these two texts respectively. Then

$$c_1^{\gamma_1,\gamma_2}([c_2],[m]) = \begin{cases} 8 & \text{if } [c_2] = \left[< \gamma_2^{(\gamma_1,\gamma_2)} >\right], \text{ and } [m] = [A] \\ 10 & \text{if } [c_2] = \left[< \gamma_2^{(\gamma_1,\gamma_2)} >\right], \text{ and } [m] \neq [A] \\ 0 & \text{otherwise.} \end{cases}$$

and

$$c_2^{\gamma_1,\gamma_2}\left([c_1],[m]\right)=\begin{cases} 9 & \text{if } [c_1]=\left[<\gamma_1^{(\gamma_1,\gamma_2)}>\right], \text{ and } [m]=[B] \\ 10 & \text{if } [c_1]=\left[<\gamma_1^{(\gamma_1,\gamma_2)}>\right], \text{ and } [m]\neq[B] \\ 0 & \text{otherwise.}\end{cases}$$

The key observation is that $\left[<\gamma_1^{(\gamma_1,\gamma_2)}>\right]=\left[c_1^{\gamma_1,\gamma_2}\right]$ and $\left[<\gamma_2^{(\gamma_1,\gamma_2)}>\right]=\left[c_2^{\gamma_1,\gamma_2}\right]$. Therefore, the contract $c_1^{\gamma_1,\gamma_2}$ requires firm 1 to set a price of 8 if the contract of firm 2 is $c_2^{\gamma_1,\gamma_2}$ and the type of the consumer is A. It requires firm 1 to set a price of 10 if the contract of firm 2 is $c_2^{\gamma_1,\gamma_2}$ but the consumer's type is B. Finally, it requires setting a price of 1 whenever the contract of firm 2 is not $c_2^{\gamma_1,\gamma_2}$. The contract $c_2^{\gamma_1,\gamma_2}$ specifies similar instructions as a function of the contract of firm 1. Obviously, these contracts are cross-referential, and they implement the desired allocation.

Literature Review

As we discussed above, unlike this paper, the rest of the literature assumes that the contracts are neither observable nor contractible. Therefore, the principals can make their contracts contingent on the contracts of the other principals only through the reports of the agents. In order for the agents to communicate their contracts to the principals, their message spaces must be at least as large as the space of contracts. Since the contracts are mappings from the message spaces, it is not straightforward to construct such a message space. The goal of the literature is usually to establish that a desired message space exists and to understand how sensitive the set of equilibrium allocations is to the message spaces. One of the shortcomings of the literature is the lack of characterization of these allocations. Perhaps the main contribution of our paper to the literature is the full characterization of the equilibrium allocations.

Epstein and Peters (1999) show that in common agency models there exists a universal message space that is rich enough for the agents to communicate their private information as well as the contracts offered by the principals. They show a kind of revelation principle for these games. That is, any equilibrium in a communication game with a large enough message space can be implemented as an equilibrium in the game where the agents' message spaces are the universal message spaces. Since contracts are contractible in our model, the agents do not need to communicate the contracts they received.

Peters (2001) and Martimore and Stole (2002) show that a version of the Taxation Principle holds for common agency games. That is, any equilibrium in any communication game can be implemented as an equilibrium in a game where the principals offer menus of ordinary contracts. An ordinary contract is one which maps reports of types to outcomes. The agent then selects items from the menu of each principal.

Calzolari and Pavan (2006) and Yamashita (2007) develop revelation mechanisms with message spaces that are simpler than the universal message space of Epstein and Peters (1999). These message spaces are only rich enough to allow the agents to report a deviation of a principal and the allocations the agents can induce given the deviations. The significance of these results is that the simplified message spaces might make it possible to analyze the set of equilibrium outcomes.

The results mentioned above lead to characterizations of the implementable allocations only in special cases. A notable exception is Yamshita (2007) who proves a *Folk Theorem* in his environment if there are at least three agents. The author assumes that agents observe not only their own contracts but the contracts offered to the other agents too. Yamashita's Folk Theorem works as follows. The equilibrium mechanism of a principal asks the agents to report their types and to vote for a strongly incentive compatible allocation. The mechanism implements an allocation if it received the majority of the votes. Notice that if each agent votes for the same allocation, the vote of a single agent becomes irrelevant. Hence, for each strongly incentive compatible allocation, there is an equilibrium where each agent reports his type truthfully and votes for the allocation. If a principal deviates and offer a mechanism different from the one described above, the agents vote for an allocation which is worst for the deviator. Another feature of Yamshita's Folk Theorem is that the principals can be pushed down to their maxmin values instead of their minmax values. This is due to the assumption that each principal is forced to offer a contract which pins down a single action as a function of the messages of the agents. In particular, a principal is not allowed to take an action without participating in the contracting game. As we mentioned earlier, this assumption is with the loss of generality. We use the basic idea in Yamashita (2007) to prove that if there are at least three agents, the set of allocations implementable by contractible contracts is the same as the set of allocations implementable by ordinary contracts.

Our paper is also related to the literature on commitment devices. This literature considers situations where each player can arbitrarily restrict his action space as a function of the restrictions of the other players before playing a normal form game. Tennenholz (2006) models the commitment device space with a set of programs. A program receives the programs of the other players and outputs an action. Thennenholz (2006) proves a pure-strategy folk theorem for games with incomplete information. Kalai et. al. (2008) also consider commitment devices in two-player complete information games and prove a full folk theorem. The authors show how to implement correlated outcomes by independent randomizations by the players. Peters and Szentes (2008) analyze Bayesian games with commitment devices and prove a pure-strategy folk theorem. They use the same formalism as in this paper. In particular, the space of commitment devices is the set of definable functions.

2 The Model

2.1 The Physical Environment

Assume that each of n principals must choose from a finite set of feasible actions. The actions available to principal j are A_j and A denotes $\times_{j=1}^{n} A_j$. There are k agents. The finite type space of agent i is T^i, and T denotes $\times_{i=1}^{k} T^i$. The joint distribution of types is common knowledge. The payoff to principal j is given by $u_j : T \times A \to \mathbb{R}$. The payoff to agent i is $v_i : T \times A \to \mathbb{R}$. Principals and agents all maximize expected utility.

2.2 The Language and the Gödel Coding

We consider a formal language, which is sufficiently rich to allow its user to state propositions in arithmetic. Furthermore, the set of statements in this language is closed under the finite applications of the Boolean operations: $\urcorner$, $\vee$, and $\wedge$. This implies that one can express, for example, Fermat's Last Theorem:

$$\forall n, x, y, z \left\{ \left[(n \geq 3) \vee (x \neq 0) \vee (y \neq 0) \vee (z \neq 0) \right] \rightarrow (x^n + y^n \neq z^n) \right] \right\}.$$

In addition, one can also express statements in the language that involve any finite number of free variables. For example,"x is a prime number" is a statement in the language. The symbol x is a free variable in the statement. Another example for a predicate that has one free variable is "$x < 4$." One can substitute any integer into x and then the predicate is either true or false. This particular one is true if $x = 1, 2, 3$ and false otherwise.

Let $\mathfrak{L}$ be the set of all formulas of the formal language. Each of its element is a finite string of symbols. It is well known that one can construct a one-to-one function $\mathfrak{L} \rightarrow \mathbb{N}$. Let $[\varphi]$ be the value of this function at $\varphi \in \mathfrak{L}$, and call it the Gödel Code of the text φ.

Definition 1 *The function $f : \mathbb{N}^k \rightarrow 2^{\mathbb{N}}$ is said to be* definable *if there exists a first-order predicate ϕ in $k+1$ free variables such that $b \in f(a_1, ..., a_k)$ if and only if $\phi(a_1, ..., a_k, b)$ is true.*

To understand the definition better, consider the following correspondence: $f(n) = \{n, n+1\}$ for all $n \in \mathbb{N}$. In order to show that this correspondence is definable we have to construct the predicate required by the previous definition. Let

$$\phi(x, y) \equiv (y = x) \vee (y = x + 1).$$

Notice that for any pair of integers, a and b, $\phi(a, b)$ is true if and only if b is either a or $a + 1$. Therefore, the predicate ϕ indeed defines f.

2.3 The contracting game

The set of feasible contracts is the set of definable mappings from $\mathbb{N}^n \times \mathbb{N}^k \rightarrow 2^{\mathbb{N}}$. The first n arguments are the Godel codes of the contracts of all the principals. The next k arguments are the codes of the messages sent by the various agents. We denote the contract space of Principal j by C_j, and $C = \times_{j=1}^n C_j$. The timing of the game is as follows. Principals simultaneously submit contracts $(c_1, \ldots, c_n) \in C$. These contracts are publicly observable. Then, agents send messages to the principals. Let m_j^i denote the message sent by Agent i to Principal j. Finally, principals take actions simultaneously from the subsets of their action spaces determined by the contracts and messages. That is, Principal j can take action a_j only if

$$[a_j] \in c_j \left([c_1], ..., [c_n], \left[m_j^1\right], ..., \left[m_j^k\right] \right).$$

To make the notation a bit more transparent, we will abuse notation and use actions of the principals instead of their codes and write $c_j : \mathbb{N}^n \times \mathbb{N}^k \to 2^{A_j}$ while still thinking of c_j as a definable function.

We restrict attention to pure-strategy perfect Bayesian equilibria (PBE). That is, principals and the agents are required to play a weak perfect Bayesian equilibrium in every subgame generated by any contract profile $(c_1, ..., c_n)$.[5] The main result of this paper does not depend on the equilibrium concept as long as the players play *some* equilibrium in the subgames generated by the first-stage contracts. In particular, the set of sequential equilibria would be characterized by essentially the same constraints.

The restriction to pure strategies is purely for convenience. Allowing mixed strategies has no substantive consequences on our analysis but makes the notations more cumbersome. We shall discuss how to extend our results to mixed-strategy equilibria. We also point out that the existence of an equilibrium is only guaranteed if mixed strategies are allowed.

A deterministic allocation is a mapping from the type profile of the agents to the action profiles of the principals. Our strategy is to first analyze equilibria in games where contracts are observable but not contractible. These are the *communication games* analyzed in the common agency literature. We call these games *ordinary contracting games*. The analysis of these games leads to a full characterization of the contractible contracting games. However, these games are interesting for their own sakes. In the meet the competition example as well as in the example of the introduction, the ability to contract on contracts can lead to inefficient allocations. This is because the firms can collude through their contracts and behave as a monopoly. The policy implication of this observation is that prohibiting contracting on contracts might lead to more efficient allocations. In order to make such a claim, one has to characterize the equilibria of the ordinary contracting games.

3 Ordinary Contracting Games

The set of *ordinary contracts* is the set of definable mappings from $\mathbb{N}^k \to 2^{\mathbb{N}}$. The domain of these functions are the Godel codes of the messages sent by the agents. Let D_j denote the contract space of Principal j, and let $D = \times_{j=1}^n D_j$.The timing of the ordinary contracting game is as follows. Principals simultaneously select contracts $(d_1, \ldots, d_n) \in D$. These contracts are publicly observable. Then, agents send messages to the principals, $\left\{m^1, ..., m^k\right\} \in \mathbb{N}^{nk}$. Finally, principals take actions simultaneously, such that, Principal j can take action $a_j \in A_j$ if

$$[a_j] \in d_j \left(\left[m_j^1\right], ..., \left[m_j^k\right]\right).$$

[5] In order to guarentee that these subgames exist, one should describe the game such that the types of the agents are determined only after the contracts are offered by the principals. This way of modeling the game has no strategic implications but makes our terminology precise.

Again, for simplicity we use actions of the principals instead of their codes and write $d_j : \mathbb{N}^k \to 2^{A_j}$ while still thinking of d_j as a definable function. We restrict attention to pure-strategy PBE of this game.

We characterize the equilibria in these games by describing the best-response constraints of the principals and the agents. Notice that when an agent decides what messages to send to the principals, he knows his type and already observed the contract profile of the principals. Hence, the messages of the agents are functions of these two objects. Let $\beta^i : T^i \times D \to \mathcal{L}^n$ denote the strategy of Agent i, and let β^i_j denote the jth coordinate of β^i, that is, the message sent to Principal j by Agents i. Let β_j denote the messages received by Principal j, that is, $\left(\beta^1_j, ..., \beta^k_j\right)$. Principal j's action at the last stage of the game can depend on both the first-stage contract profile and the messages sent to him by the agents. Let $\alpha_j : \mathcal{L}^k \times D \to A_j$ denote the strategy of the principals at the last stage. Since Principal j's action must be consistent with his contract, $\alpha_j (m_j, d) \in d_j (m_j)$ must hold for all $m_j \in \mathcal{L}^k$, and for all $d = (d_j, d_{-j}) \in D$. As usual, α_{-j} denotes the action profile of principals other than Principal j, and β^{-i} denotes the message profile of agents other than Agent i.

The first constraint guarantees that each principal takes an action at the last stage which maximizes his payoff. For all j, $d \in D$:

$$\alpha_j (m_j, d) \in \arg \max_{a_j \in d_j(m)} E_t [u_j (t, a_j, \alpha_{-j}) : d, m_j, \beta, \alpha_{-j}] \tag{1}$$

for all $m_j \in \mathcal{L}^k$ and $d \in D$. The expectations are formed according to Bayes Rule if the message profile sent by the agents, m_j, is consistent with their equilibrium behavior. However, PBE imposes no restriction on the belief of Principal j if m_j is off the equilibrium path.[6]

The second constraint ensures that each agent maximizes his payoff by his message in every subgame generated by a contract profile. For all i, $t^i \in T^i$, and $d \in D$,

$$\beta^i \left(t^i, d\right) \in \arg \max_{m^i \in \mathcal{L}^n} E_{t^{-i}} \left[v_i \left(t, \alpha \left(\left(m^i, \beta^{-i}\right), d\right)\right) : d, t^i\right]. \tag{2}$$

The last constraint guarantees that no principal wants to deviate from his equilibrium contract in the first stage of the game. Let $(d^*_1, ..., d^*_n) = d^*$ denote the equilibrium contract profile. Then, for all j:

$$d^*_j \in \arg \max_{d_j \in D_j} E_t \left(u_j (t, \alpha (\beta)) : d_j, d^*_{-j}\right). \tag{3}$$

We claim the following

Proposition 1 *The strategy profile (d^*, β, α) constitutes a PBE in the Ordinary Contracting Game if and only if (1), (2), and (3) are satisfies.*

[6] A stronger equilibrium refinement concept imposes restrictions on the beliefs according to which the expectations are formed in (1), but has no other impact on our characterization result.

It turns out to be useful to define the set of those allocations that can be implemented in a subgame generated by *some* ordinary contract profile. To this end, let σ^d denote the set of those (α, β) pairs for which both (1) and (2) are satisfied. Then the set of allocations that can be implements in some subgame is defined as follows:

$$\mathcal{A} = \left\{ g : T \to A : \exists d \in D,\ \exists (\alpha, \beta) \in \sigma^d \text{ s.t. } g(t) = \alpha(\beta(t, d), d) \right\}.$$

Next, we characterize the set $\mathcal{A}$ in terms of the preferences of the agents.

Definition 2 *Let $g_j : T \to A_j$ for all $j = 1, ..., n$. Then the allocation $g = (g_1, ..., g_n)$ is* strongly incentive compatible *if for all $i \in \{1, ..., k\}$, $t^i \in T^i$, and $\left(t_1^i, ...t_n^i\right) \in \left(T^i\right)^n$:*

$$E_{t_{-i}}\left[v^i\left(t, \left(g\left(t^i, t^{-i}\right)\right)\right) : t^i\right] \geq E_{t_{-i}}\left[v^i\left(t, \left(g_1\left(t_1^i, t^{-i}\right), ..., g_n\left(t_1^i, t^{-i}\right)\right)\right) : t^i\right].$$

This definition is the streightforward extention of the standard notion of incentive compoatibility to a multi-principal setting. Indeed, this definition would coincide with the standard definition of incentive compatibility if one would require the inequality to hold only for those type vectors, $\left(t_1^i, ...t_n^i\right) \in \left(T^i\right)^n$, where $t_1^i = t_2^i = ... = t_n^i$. Such a constraint would require that no agent can benefit from mimicking another one of his type. In our multi-principal model, however, we have to take more complex deviations into account. This is because the messages of the agents are private, and therefore, an agent can report different types to different principals. Of course, the set of strongly incentive compatible allocations are also incentive compatible. The following example shows that the converse is not true.

Examle 1. Suppose that $n = 2$, $k = 1$, and $A_1 = A_2 = \{a_1, a_2\}$. The agent has two equally likely type, $T = \{1, 2\}$. The payoffs to the agent are described by the following matrix:

	a_1	a_2
a_1	0	1
a_2	1	0

.

The allocation g, where $g(i) = (a_i, a_i)$ for $i = 1, 2$, is obviosuly incentive compatible but not strongly incentive compatible.

Lemma 1 *The allocation $g : T \to A$ is strongly incentive compatible if and only if $g \in \mathcal{A}$.*

Proof. Suppose first that g is strongly incentive compatible. Fix an arbitrary element of T, say $t_d = \left(t_d^1, ..., t_d^k\right)$, and consider the following contract of Principal j:

$$d_j\left(\left[m_j^i\right]_{i=1}^k\right) = \left\{g_j\left(t^1, ..., t^k\right)\right\} \text{ where } t^i = m_j^i \text{ if } m_j^i \in T^i \text{ and } t_d^i \text{ otherwise.}$$

Notice that this contract pins down a single action for Principal j as a function of the message profile of the agents. Hence, the principals do not make any strategic choice in the subgame generated by $(d_1, ..., d_n)$. Since g is incentive compatible, truthtelling by the agents constitutes an equilibrium in the subgame. (That is, $m_i^j\left(t^i,d\right)=t^i$ for all i, t^i and j is an equilibrium.) This equilibrium obviously implements g.

Suppose now that $g\in\mathcal{A}$. This means that there exists a $d\in D$, and strategies of the principals and the agents, α and β, such that $(\alpha,\beta)\in\sigma^d$ and $g(t)=\alpha\left(\beta(t,d),d\right)$ for all t. In order to prove that the allocation g is strongly incentive compatible, we have to show that for all $i\in\{1,...,k\}$, $t^i\in T^i$ and $\left(t_1^i,...,t_n^i\right)\in\left(T^i\right)^n$:

$$E_{t_{-i}}\left[v^i\left(t,\left(g\left(t^i,t^{-i}\right)\right)\right):t^i\right]\geq E_{t_{-i}}\left[v^i\left(t,\left(g_1\left(t_1^i,t^{-i}\right),...,g_n\left(t_1^i,t^{-i}\right)\right)\right):t^i\right].$$

The left-hand side of this inequality is the expected equilibrium payoff of Agent i conditional on t^i in the subgame generated by d. The right-hand side is the expected payoff of Agent i conditional on t^i if he deviates and sends message $m_j^i\left(t_j^i,d\right)$ to Principal j instead of $m_j^i\left(t^i,d\right)$. Since $(\alpha,\beta)\in\sigma^d$ these deviations cannot be profitable and hence, the previous displayed inequality holds. ■

3.1 Examples for Ordinary Contracting Games

For simplicity, we identify the message of an agent with its Godel code in all the examples below. That is, instead of saying that an agent sends a message whose Godel code is q, we say that the agent sends the message q. (This does not cause confusion because the encoding is a bijection.)

Next, we show, by examples, that one cannot assume that the equilibrium contracts specify a single action for a principal as a function of the agents' messages. The contract d_j is said to be *complete* if $|d_j(q)|=1$ for all q, that is, d_j is a function from $\mathbb{N}^k$ to A_j. Restricting the contracts to be complete is with the loss of generality for two reasons. Example 2 shows that there are allocations which cannot be supported with complete contracts, but can be supported otherwise. Example 3 shows that there are allocations which can only be supported if contracts are required to be complete.

Example 2. Suppose that $n=2$ and $k=1$. Assume that the agent's type space is degenerate, $A_1=A_2=\{a,b\}$, and the payoffs of the principals are defined by the following matrix:

	a	b
a	2, 2, 0	0, 3, 3
b	1, 0, 0	1, 0, 1

,

where the first and second numbers in each cell describes the payoffs to Principal 1 and Principal 2, and the third number is the payoff to the agent.

Notice that the agent's payoff is zero whenever Principal 2 takes action a and positive otherwise. Therefore, whenever he can send a message which triggers action b by Principal 2, he will do so. In addition, given that Principal 2 takes action b, the agent prefers Principal 1 to take action a over action b. Consider the allocation (a,a). Principal 2 would like to deviate and take action b. Such a deviation can be punished by Principal 1 by taking action b. We show that the outcome (a,a) can be implemented as an equilibrium but cannot be implemented with complete contracts.

Define the equilibrium contracts of the principals as follows: $d_1(q) = A_1$ for all q, and $d_2(q) = a$ for all q. Since these contracts are constants in the messages of the agents, the strategy of the agent is irrelevant. Principal 1's strategy is the following. If he observes that Principal 2 offered a contract which allows taking action b for some reports of the agent, he takes action b, otherwise he takes action a. Obviously, none of the principals can increase his payoff by offering a different contract.

Next, we argue that (a, a) cannot be supported by complete contracts. Suppose that (d_1, d_2) supports (a, a) and d_1 is complete. Then, there exist an $q \in \mathbb{N}$ such that $d_1(q) = a$. Then Principal 2 can profitably deviate by offer a contract which specifies action b independently of the agent's report. This is because the agent report an $q \in \mathbb{N}$ to Principal 1 such that $d_1(q) = a$ and the outcome will be (a, b). This outcome maximizes the agent's payoff and provides Principal 2 with a payoff higher than the outcome (a, a) would.

Example 3. Suppose $n = 2$ and $k = 1$. Assume that the agent's type space is degenerate, $A_1 = A_2 = \{H, T\}$, and the payoffs of the principals are defined by the following matrix:

	H	T
H	1, -1, -1	-1, 1, 1
T	-1, 1, 1	1, -1, -1

,

where the first and second numbers in each cell describes the payoffs of Principal 1 and Principal 2, and the third number is the payoff to the agent. In this example, the two principals are playing the Matching Pennies Game, and the agent's payoff is identical to that of Principal 2.

We first show that if each principal is restricted to offer a complete contract then the payoff profile $(-1, 1, 1)$ can be supported as an equilibrium payoff profile. To see this consider the following contract of Principal 2: $d_2(1) = \{H\}$ and $d_2(q) = \{T\}$ if $q \neq 1$. Suppose that the complete contract of Principal 1 is d_1. Notice that $d_1(1)$ is either H or L. If $d_1(1) = \{H\}$ then the agent can send messages 1 and 2 to Principals 1 and 2 respectively which generates a payoff profile $(-1, 1, 1)$. Similarly, if $d_1(1) = L$, the agent can send the message 1 to both principals which again generates a payoff profile of $(-1, 1, 1)$. Therefore, no matter what the complete contract of Principal 1 is, the agent can always induce the payoff profile $(-1, 1, 1)$.

Suppose now that the principals are not restricted to offer complete contracts. Then there does not exist a pure strategy equilibrium in our game, because Principal 1 can always offer a contract d, such that $d(q) = \{H, L\}$. In addition, if we allow mixed strategies, the only equilibrium payoff profile was $(1/2, 1/2, 1/2)$.

Next we show that one cannot assume that the message space of an agent is his type space. To be more specific, the next example shows that the cardinality of the range of the equilibrium contracts must must be larger than the cardinality of the type space of the agent in order to implement certain allocations.

Example 3. Suppose that $n = 2$ and $k = 1$ and the type space of the agent is degenerate. The principals are playing the Prisoner's Dilemma. That is, $A_1 = A_2 = \{C, D\}$, and the payoffs are defined by the following matrix

	C	D
C	2, 2, 3	0, 3, 1
D	3, 0, 1	1, 1, 2

Again, the first two numbers are the payoffs to the principals and the third one is the payoff to the agent. Notice that the agent prefers the principals to cooperate to everything else, but prefers them to defect to (C, D) and (D, C). The agent has no private information in this example. Hence, if the action profile (C, C) could be implemented such that the message space of the agent is his type space then (C, C) would be supported as an equilibrium outcome by contracts which do not depend on the report of the agent. We show that this is impossible although (C, C) is implementable.

Suppose that d_1 and d_2 implement (C, C) and $d_i(q_1) = d_i(q_2)$ for all $q_1, q_2 \in N$ and $i \in \{1, 2\}$. If $d_1(q) = \{C\}$ for all q, Principal 2 can profitably deviate by offering a contract that specifies $\{D\}$. Hence, $d_1(q) = d_2(q) = \{C, D\}$ for all q, which implies that the Principals play the Prisoner's Dilemma in the last stage of the game, and therefore, $\{C, C\}$ cannot be implemented.

Now, we show that we can implement $\{C, C\}$ with the help of the agent. Consider the following contract

$$d_i(q) = \begin{cases} C & \text{if } q = 1, \\ D & \text{if } q \neq 1. \end{cases}$$

The strategy of the agent is defined such that he triggers (C, C) whenever he can. In particular, on the equilibrium path, the agent reports 1 to each principal. The agent has no incentive to deviate because his payoff if maximized. If one of the principal deviates, and offers a contract such that the agent cannot induce the action C, the agent reports 2 to the other principal and the outcome would be $\{D, D\}$.

4 Contractible Contracting Games

This section is devoted to the characterization of the equilibria in the contractible contracting game. We prove a Folk Theorem and show that an allocation is implementable if and only if it is strongly incentive compatible and the payoff of each principal is larger than his minmax value, to be defined later. To see that the allocation must be strongly incentive compatible, we first argue that any contract profile generates an ordinary contract profile. To this end, suppose that $(c_1^*, ..., c_n^*)$ is an equilibrium contract profile. For each j, define $d_j^* \in D_j$, such that $d_j^*(l_1, ..., l_k) = c_j^*([c_1^*], ..., [c_n^*], l_1, ..., l_k)$ for all $(l_1, ..., l_k) \in \mathbb{N}^k$. Notice that $d^* = (d_1^*, ..., d_n^*)$ is an ordinary contract profile and the subgame generated by c^* in the contractible contracting game is the same as the subgame generated by d^* in the ordinary contracting game. Since players are required to play an equilibrium in the subgame generated by the first-stage contract profile, we can conclude that any allocation that can be implemented as a PBE in the contractible contracting game must belong to $\mathcal{A}$. Therefore, by Lemma 1, the allocation must be strongly incentive compatible.

The difficult part of the theorem is to pin down the minmax values of the principals. The minmax value of Principal j is the lowest possible value what he can get in the ordinary contracting game if the goal of the other principals at the first stage of the game is to minimize his payoff. We shall prove that the minmax value of Principal j $(\in \{1,...,n\})$, $\underline{u_j}$, is defined as follows:

$$\underline{u_j} = \min_{d_{-j} \in D_{-j}} \max_{d_j \in D_j} \min_{(\alpha,\beta) \in \sigma(d_j,d_{-j})} E_t \left(u_j \left(t, \alpha \left(\beta \right) \right) : \left(d_j, d_{-j} \right) \right). \tag{4}$$

The meaning of this expression can be explained as follows. All the principals other than Principal j offer ordinary contracts at the first stage of the game in order to minimize the payoff of Principal j. Principal j also offers an ordinary contract which is a best response to the contracts of the others. These contracts generate a subgame in which there can be multiple equilibria. In this subgame, the principals and agents play an equilibrium which is the worst one for Principal j.

The fact that Principal j can only be punished by playing the worst equilibrium in the subgame is obvious because PBE requires the players to play an equilibrium in any subgame generated by a contract profile. The nontrivial part of our main result is the definition of $\underline{u_j}$. As we explained at the beginning of this section, the equilibrium contracts and a first-stage deviation of Principal j determines an ordinary contract profile. The formula in (4) essentially says that the ordinary contract profile of the principals other than Principal j does not depend on the deviation of Principal j, and hence, Principal j can best-responde to it. Since contracts are contractible, the ordinary contract profile of the principals other than Principal j can depend on the deviation of Principal j. Therefore, one might conjecture that the principals might be able to push Principal j's value below $\underline{u_j}$. For example, if Principal j would be restricted to offer ordinary contracts then the others could always offer contracts which are contingent on the ordinary contract of Principal j. Being able to offer these contingent contracts, is similar to being able to move after observing Principal j's contract, and hence, his lowest value would be

$$\max_{d_j \in D} \min_{d_{-j} \in D_{-j}} \min_{(\alpha,\beta) \in \sigma(d_j,d_{-j})} E_t \left(u_j \left(t, \alpha \left(\beta \right) \right) : \left(d_j, d_{-j} \right) \right).$$

Of course, Principal j is not restricted to offer ordinary contracts, and his contract can be contingent on the contracts offered by the other principals, which are contingent on his contract etc. In fact, because of this infinite regress problem, it is not even clear that the lowest value of Principal j is well-defined.

Nevertheless, we show that this value is well-defined and, interestingly, the most severe punishment inflicted on Principal j can be assumed to be invariant to his deviation. To be more specific, Proposition 2 shows that no matter what the contract profile of the principals is, there always exists an ordinary contract profile $d_{-j} \in D_{-j}$, such that for all $d_j \in D_j$, there is a way for Principal j to write a contract so that the generated ordinary contract profile is (d_j, d_{-j}). But then it is without the loss of generality to assume that the principals use the ordinary contract profile d_{-j} to punish Principal j.

We are ready to state our main result formally.

Theorem 1 *An allocation $g : T \to A$ is implementable as an equilibrium in the contractible contracting game if and only if (i) $g \in \mathcal{A}$, and (ii) for all $j \in \{1, ..., n\}$*

$$E_t u_j (t, g(t)) \geq \underline{u_j}.$$

We break the proof of the theorem into two parts. The if part is based on the same arguments as the ones used in the example of the introduction. We shall construct cross-referential contracts which support the desired allocation. Essentially, the contract of Principal j (for all j) specifies target codes, one for each of the other principals. If the Godel code of the contract of Principal q is the same as his target code for all q, then Principal j *cooperates*. If Principal q deviates, and the code of his contract is different from his target code, the contract of Principal j prescribes an ordinary contract which is used to minmax Principal q. The set of equilibrium contracts are cross-referential because the Godel code of Principal j's contract, which we have just described, is exactly the same as his target codes specified in the contracts of all the other principals.

Recall two pieces of notations from the introduction. First, if $l \in \mathbb{N}$ then $< l >$ denotes the text whose Gödel code is l. That is, $[< l >] = l$. Second, for any text φ, let $\varphi^{(l_1,...,l_n)}$ denote the text where if the letter x_q stands for a free variable in φ then x_q is substituted for l_q in φ for $q = 1, ..., n$. For example, if φ is "$x_1 < x_2$", $l_1 = 1$, and $l_2 = 2$ then $\varphi^{(l_1,l_2)}$ is $1 < 2$.[7]

Consider now the following text in n free variable: $< x_q >^{(x_1,...,x_n)}$, where $q \leq n$. Since the Godel coding is a bijection $< l_q >$ is a text for each $l_q \in \mathbb{N}$. Since $\varphi^{(l_1,...,l_n)}$ is defined for all φ and $(l_1, ..., l_n) \in \mathbb{N}^n$, $< l_q >^{(l_1,...,l_n)}$ is a text for all $(l_1, ..., l_n) \in \mathbb{N}^n$. It is a well-known result in Mathematical Logic that if $f(l_1, ..., l_n) = \left[< l_q >^{(l_1,...,l_n)}\right]$, then f is a definable function.

Proof of the "if" part of Therorem 1. Since the allocation g is in $\mathcal{A}$ there exists an ordinary contract profile $d^* = (d_1^*, ..., d_n^*)$, a strategy profile of the agents, $\beta^* = \left(\beta^{1*}, ..., \beta^{k*}\right)$, a third-stage strategy profile of the principals, $\alpha^* = (\alpha_1^*, ..., \alpha_n^*)$, such that $g(t) = \alpha^* (\beta^* (t, d^*), d^*)$ and both (1) and (2) are satisfied, that is, $(\alpha^*, \beta^*) \in \sigma^{d^*}$. In addition, let d_j^q denote the contract of Principal j which he uses to minmax Principal q. That is, the contract profile d_{-q}^q solves

$$\min_{d_{-q} \in D_{-q}} \max_{d_q \in D_q} \min_{(\alpha,\beta) \in \sigma^{(d_q, d_{-q})}} E_t \left(u_q (t, \alpha(\beta)) : (d_q, d_{-q})\right). \tag{5}$$

Consider the following text of Principal j, $c_j^{x_1,...,x_n}$, in n free variables:

$$c_j^{x_1,...,x_n} \left(([c_l])_{l=1}^n, \left(\left[m_j^i\right]\right)_{i=1}^k\right) =$$

[7] Of course, it is possible that the text φ does not contain some of the letters $\{x_1, ..., x_n\}$. In that case, there is no substitution for the missing letters in $\varphi^{(l_1,...,l_n)}$. For example, if φ is "$x_2 > 2$", then $\varphi^{(3,4)}$ is "$4 > 2$", because x_1 does not appear in φ.

$$\left\{ \begin{array}{ll} d_j^* \left(\left(\left[m_j^i \right] \right)_{i=1}^k \right) & \text{if } \left| \left\{ l : \left[< x_l >^{(x_1,...,x_m)} \right] \neq [c_l] \ \right\} \right| \neq 1, \\ d_j^q \left(\left(\left[m_j^i \right] \right)_{i=1}^k \right) & \text{if } \left\{ l : \left[< x_l >^{(x_1,...,x_n)} \right] \neq [c_l] \ \right\} = \{q\}, \end{array} \right. \tag{6}$$

for all $\left[m_j^i \right]_{i=1}^k \in \mathbb{N}^k$. This expression (6) is not a contract, but rather a contract with free variables. However, $c_j^{x_1,...,x_n}$ would become a contract if the free variables $x_1, ..., x_n$ are replaced by integers. Each of these contracts with free variables has a Godel code, so let $\gamma_j = \left[c_j^{x_1,...,x_n} \right]$. The functions $\left\{ c_i^{\gamma_1,...,\gamma_n} \right\}_i$ have no free variables, so they constitute a set of contracts. Notice that

$$c_j^{\gamma_1,...,\gamma_n} \left(([c_l])_{l=1}^n , \left(\left[m_j^i \right] \right)_{i=1}^k \right) \tag{7}$$

$$= \left\{ \begin{array}{ll} d_j^* \left(\left(\left[m_j^i \right] \right)_{i=1}^k \right) & \text{if } \left| \left\{ l : \left[< \gamma_l >^{(\gamma_1,...,\gamma_m)} \right] \neq [c_l] \ \right\} \right| \neq 1, \\ d_j^q \left(\left(\left[m_j^i \right] \right)_{i=1}^k \right) & \text{if } \left\{ l : \left[< \gamma_k >^{(\gamma_1,...,\gamma_m)} \right] \neq [c_l] \ \right\} = \{q\}. \end{array} \right. \tag{8}$$

The contract $c_j^{\gamma_1,...,\gamma_n}$ is definable because d_j^*, d_j^q and $f(l_1, ..., l_n) = \left[< l_q >^{(l_1,...,l_n)} \right]$ are all definable. Observe what happens when Principal q offers contract $c_q^{\gamma_1,...,\gamma_n}$ for all $q = 1, ..., n$. Principal j needs to check whether the Godel code of $< \gamma_l >^{(\gamma_1,...,\gamma_m)}$ is equal to the Godel code of $c_l^{\gamma_1,...,\gamma_n}$. The integer γ_l is the Godel code of the contract with free variable $c_l^{x_1,...,x_n}$. Principal j's contract says to take this contract with free variable, fix the free variables at $\gamma_1, ..., \gamma_n$ (which gives the contract $c_l^{\gamma_1,...,\gamma_n}$), then evaluate its Godel code. This is what is to be compared with the Godel code of the contract offered by Principal l. Of course, if Principal l offers $c_l^{\gamma_1,...,\gamma_n}$ these are the same. In fact, if Principal l offers $c_l^{\gamma_1,...,\gamma_n}$ for all $l \in \{1, ..., n\}$, then Principal j ends up with the ordinary contract d_j^* according to the first line of (7). Therefore, if Principal j offers contract $c_j^{\gamma_1,...,\gamma_n}$ for all j then the resulting subgame is generated by the ordinary contract profile d^*. Define the strategies of the agents and the principals as $\beta^* (t, d^*)$ and $\alpha^* (d^*)$. These strategies obviously support the allocation g. It remains to specify the strategies of the players off the equilibrium path and show that no player can profitably deviate.

Next we define the second-stage strategies of the agents and the third-stage strategies of the principals off the equilibrium path. (It is enough to define these strategies in subgames which are resulted from a deviation of a single principal.) Suppose that Principal q offers a contract c_q instead of $c_q^{\gamma_1,...,\gamma_n}$. Let d_q denote $c_q \left([c_q] , \left(\left[c_j^{\gamma_1,...,\gamma_n} \right] \right)_{j \neq q} \right)$. As a result of this deviation, according to the second line of (7), Principal j will end up with the ordinary contract d_j^q for all $j \neq q$. Therefore, the subgame resulting from the deviation of Principal q is generated by the ordinary contract profile $d = \left(d_q, d_{-q}^q \right)$. Define the strategies of the agents and the principals, $\alpha(d)$ and $\beta(d)$, so that the expected payoff if Principal q is minimized. That is, $(\alpha(d), \beta(d))$ solves

$$\min_{(\alpha,\beta) \in \sigma^d} E_t \left(u_q \left(t, \alpha(\beta) \right) : d \right). \tag{9}$$

Finally, we argue that neither the principals nor the agents have incentives to deviate from the equilibrium strategies. First, if Principal j offers contract $c_j^{\gamma_1,...,\gamma_n}$ for all j then no player can profitably deviate in the subgame is generated by the ordinary contract profile d^* because $(\alpha^*, \beta^*) \in \sigma^{d^*}$. In fact, we have defined the strategies of the players, $\alpha(d)$ and $\beta(d)$, in any relevant subgame generated by an ordinary contract profile, d, such that $(\alpha, \beta) \in \sigma^d$. Therefore, we only have to show that no principal can profitably deviate at the first stage of the game. Recall that if Principal q offers the contract c_q instead of $c_q^{\gamma_1,...,\gamma_n}$, then his payoff is (9). Hence, the maximum payoff he can achieve by deviating from his equilibrium contract is

$$\max_{d_q \in D_q} \min_{(\alpha,\beta)\in\sigma^{\left(d_q, d_{-q}^q\right)}} E_t \left(u_q \left(t, \alpha\left(\beta\right)\right) : \left(d_q, d_{-q}^q\right)\right).$$

By (5), the previous expression can be rewritten as

$$\min_{d_{-q}\in D_{-q}} \max_{d_q \in D_q} \min_{(\alpha,\beta)\in\sigma^{(d_q,d_{-q})}} E_t \left(u_q \left(t, \alpha\left(\beta\right)\right) : \left(d_q, d_{-q}\right)\right) = \underline{u_q}.$$

This implies that Principal q can achieve at most $\underline{u_q}$ by deviating at the first-stage. Therefore, by (ii) of the hypothesis of the theorem, no deviation is profitable. ■

Next, we turn our attention to the more difficult "only if" part of the proof. Let G_d denote the subgame generated by the ordinary contract profile $d \in D$.

Definition 3 *The subgames G_d and $G_{d'}$ $(d, d' \in D)$ are said to be equivalent, $G_d \sim G_{d'}$, if the set of equilibrium outcomes are the same in the two subgames.*[8]

The next proposition states that for all $c_{-j} \in C_{-j}$ there exists a $d_{-j} \in D_{-j}$ such that for all $d_j \in D_j$, Principal j can write a contract so that the subgame generated by the contract profile is equivalent to $G_{(d_j,d_{-j})}$. That is, no matter what the equilibrium contracts are, there always exists an ordinary contract profile d_{-j}, such that Principal j can induce a subgame $G_{(d_j,d_{-j})}$ for all d_j by an appropriate deviation. This implies that it is without loss of generality to assume that the contractual punishment for any deviation by Principal j is simply d_{-j}. That is, the punishment does not depend on the deviation itself, only on the identity of the deviator.

To state this result formally, for all $c = (c_1, ..., c_n) \in C$ let $d(c) \in D$ denote the ordinary contract profile generated by c. That is, $d_j(c) = c_j([c_1], ..., [c_n])$ for all $j \in \{1, ..., n\}$.

Proposition 2 *Let $c = (c_1^*, ..., c_n^*) \in C$. Then, for all j there exists a $d_{-j} \in D_{-j}$, such that for all $d_j \in D_j$ there exists a $c_j \in C_j$ such that $G_{(d_j,d_{-j})} \sim G_{d\left(c_j, c_{-j}^*\right)}$.*

Proof. See the Appendix. ■

This proposition is key to the "only if" part of the theorem. Since the proof of the proposition is lengthy and technical it is relegated to the Appendix. Here, we sketch the proof for the case where there are two principals and there are no agents. Since there are no agents, and therefore the restrictions on the action spaces cannot depend on the messages, a contract of a principal is just a definable mapping from the codes of the contracts to the subsets of the codes of the action

[8] Whether or not two subgames are equivalent depends on the particular equilibrium concept. However, it will become clear from the way this difinition is used that our results do not depend on the refinement concept.

space of the principal. Similarly, and ordinary contract is a subset of the codes of the action space of a principal. For all $d_1 \in D_1$, define

$$S(d_1) = \{d_2 : \exists c_1 \; c_1([c_1],[c_2^*]) = d_1, \; c_2^*([c_1],[c_2^*]) = d_2\}.$$

That is, $S(d_1)$ is the set of those d_2s for which Principal 1 is able to offer a contract such that the generated subgame is $G_{(d_1,d_2)}$. The statement of the proposition is equivalent to $\cap_{d_2 \in D_2} S(d_2) \neq \{\emptyset\}$. Suppose by contradiction that $\cap_{d_2 \in D_2} S(d_2) = \{\emptyset\}$. This implies that for all $d_2 \in D_2$ there exists a d_1 such that $d_2 \notin S(d_1)$. Therefore, one can construct a function, $f : D_2 \to D_1$, such that $d_2 \notin S(f(d_2))$. Since D_1 and D_2 are finite sets, the funcion f is definable.[9] Consider now the following contract in one free variable:

$$c_1^x([c_2]) = f\left(c_2^*\left(\left[< x >^{(x)}\right]\right)\right).$$

Let γ denote the Godel code of this contract. Then $c_1^\gamma([c_2]) = f(c_2^*([c_1^\gamma]))$. Notice that, by the definition of the function f, $c_2^*([c_1^\gamma]) \notin S(f(c_2^*([c_1^\gamma])))$. Substituting the previous equality into $f(c_2^*([c_1^\gamma]))$ we get

$$c_2^*([c_1^\gamma]) \notin S(c_1^\gamma([c])). \tag{10}$$

On the other hand, by the definition of S, $c_2([c_1]) \in S(c_1([c_2]))$ for all c_1,c_2 . Therefore,

$$c_2^*([c_1^\gamma]) \in S(c_1^\gamma([c_2^*])). \tag{11}$$

Of course, (10) and (11) cannot be true simultaneously, and hence, $\cap_{d_2 \in D_2} S(d_2) \neq \{\emptyset\}$.

We point out that the difficulty of generalizing this argument for the case when there are agents is that the ordinary contract space of Principal j, D_j, is not finite. Therefore, the function f is not necessarily definable. The proof in the Appendix takes advantage of the fact that although these spaces are infinite, the range of any ordinary contract is finite.

Proof of the "only if" part of Theorem 1. We have already established in the text before the statement of the theorem that $g \in \mathcal{A}$. We only have to show that the payoff of Principal j in every equilibrium is at least $\underline{u_j}$ for all $j \in \{1, ..., n\}$. Suppose that $(c_1^*, ..., c_n^*) \in C$ is an equilibrium contract profile. According to Proposition 2 there exists a $d'_{-j} \in D_{-j}$ such that Principal j can generate a subgame which is equivalent to $G_{\left(d_j, d'_{-j}\right)}$ for all $d_j \in D_j$. Let β^* and α^* denote the second-stage equilibrium strategies of the agents and the third-stage equilibrium strategies of the principals, respectively. Then Principal j's equilibrium payoff is weakly larger than

$$\begin{aligned}\max_{d_j \in D_j} E_t\left(u_j(t, \alpha^*(\beta^*)) : \left(d_j, d'_{-j}\right)\right) &\geq \max_{d_j \in D_j} \min_{(\alpha,\beta) \in \sigma^{(d_j, d_{-j})}} E_t\left(u_j(t, \alpha(\beta)) : \left(d_j, d'_{-j}\right)\right) \\ &\geq \min_{d_{-j} \in D_{-i}} \max_{d_j \in D_j} \min_{(\alpha,\beta) \in \sigma^{(d_j, d_{-j})}} E_t\left(u_j(t, \alpha(\beta)) : (d_j, d_{-j})\right) = \underline{u_j}.\end{aligned}$$

■

[9] The sets D_1 and D_2 are finite because there are no agents. Therefore an ordinary contract is a restriction on the actions space. There are only finitely many such restrictions because the action space of each principal is finite.

Whether an allocation is strognly incentive compatible only depends on the preferences of the agents. Hence part (i) of the statement of Theorem 1 is a property of an allocation which depends only on the physical environment. However, the minmax values of the principals are defined in terms of equilibria in subgames of the ordinary contracting game. It is desirable to characterize even these minmax values in terms of the physical environment. Next, we show that one additional assumption leads to such a characterization.

Assumption 1. For all j there exist $\overline{a}_j \in A_j$, $a^j_{-j} \in A_{-j}$, and $U_j : T \to \mathbb{R}$ such that
(i) $u_j\left(t, \overline{a}_j, a_{-j}\right) \geq U_j\left(t\right)$ for all $a_{-j} \in A_{-j}$, and
(ii) $u_j\left(t, a_j, a^j_{-j}\right) \leq U_j\left(t\right)$ for all $a_j \in A_j$.

This assumption is satisfied in many important economic applications. The action $\overline{a}_j$ can often be thought as a default action of Principal j which allows him not to participate in the interaction with the agents. If the principals are sellers and the agents are buyers then $\overline{a}_j$ means that Principal j does not sell his products. If the principals are employers and the agents are workers then this action corresponds to the choice of not employing any woker. The action profile a^j_{-j} can be interpreted as an action profile of the principals (other than Principal j) which excludes Principal j from participation. In the buyer-seller example, this can be accomplished by setting prices so low that Principal j cannot make a positive profit by selling his products. Similarly, in the employer-worker example, the principals can set wages higher than the productivity of the workers.

Theorem 2 *Suppose that Assumption 1 is satisfied. Then the allocation $g : T \to A$ is implementable as an equilibrium in the contractible contracting game if and only if (i) $g \in \mathcal{A}$, and (ii) $E_t u_j\left(t, g\left(t\right)\right) \geq E_t U_j\left(t\right)$ for all $j \in \{1, ..., n\}$.*

Proof. By Theorem 1, we only have to show that $E_t U_j\left(t\right) = \underline{u}_j$ for all j. Consider first the following ordinary contract of Principal q ($q \neq j$):

$$\widetilde{d}_q\left(l_1, ..., l_k\right) = \left\{a^j_q\right\} \text{ for all } \left(l_1, ..., l_k\right) \in \mathbb{N}.$$

Suppose that Principal q offers d_q for all $q\left(\neq j\right)$. Then Principal $q\left(\neq j\right)$ ends up taking action a^j_q no matter what the messages of the agents and the contract of Principal j. Therefore, by part (ii) of Assumption 1, the expected payoff of Principal j is at most $U_j\left(t\right)$ in every subgame $G_{\left(d_j, \widetilde{d}_{-j}\right)}$. Hence, $E_t U_j\left(t\right) \geq \underline{u}_j$.

Now, consider the following contract of Principal j:

$$\widetilde{d}_j\left(l_1, ..., l_k\right) = \left\{\overline{a}_j\right\} \text{ for all } \left(l_1, ..., l_k\right) \in \mathbb{N}.$$

Suppose that Principal j offers d_j. Then Principal j ends up taking action a^j_j no matter what the messages of the agents and the contracts of the other Principals are. Therefore, by part (i) of Assumption 1, the expected payoff of Principal j is at least $U_j\left(t\right)$ in every subgame $G_{\left(\widetilde{d}_j, d_{-j}\right)}$. Hence, $E_t U_j\left(t\right) \leq \underline{u}_j$. ■

Next, we show that if information is complete, our Theorem 1 also leads to a characterization of the equilibria in terms of the physical environment without any reference to ordinary contracting games.

Theorem 3 *Suppose that* $|T_i| = 1$ *for all* i. *Then the allocation* $(a_1^*, ..., a_n^*) = a^* \in A$ *is implementable as subgame perfect Nash equilibrium if and only if*

$$u_j(a^*) \geq \min_{a_{-i}} \max_{a_i} u_j(a_i, a_{-i}) = u_j^*. \tag{12}$$

A notable feature of this corollary is that whether an allocation is implementable does not depend on the number and on the preferences of the agents.

Proof. Since information is complete, every allocation is strongly incentive compatible. By Theorem 1, we only have to show that $\underline{u_j} = u_j^*$. Notice that Principal j can offer the contract d_j, such that $d_j(q_1, ..., q_k) = \{A_j\}$ for all $q_1, ..., q_k \in \mathbb{N}$. This means that no matter what the messages of the agents, Principal j can take any of his action in the subgame generated by the contracts. Therefore, Principal j can best-respond to the action profile of the other principals and can achieve a value of at least u_j^*. This shows that $\underline{u_j} \leq u_j^*$. In order to prove that $\underline{u_j} \geq u_j^*$, let $a_{-j}^j = \left(a_q^j\right)_{q \neq j} \in A_{-j}$ be a solution to $\min_{a_{-j}} \max_{a_j} u_j(a_j, a_{-j})$. Define Principal q's contract, d_q, as $d_q(l_1, ..., l_k) = a_q^j$ for all $l_1, ..., l_k \in \mathbb{N}$. That is, no matter what the messages of the agents are, the principals other than j will take action a_{-j}^j. Of course, Principal j can achieve at most u_j^*, hence, $\underline{u_j} \geq u_j^*$. ■

5 Applications and Examples

5.1 Welfare and Policy Implications

Next, we identify environments where the competition among the principals leads to a Pareto efficient allocations if contracts are not contractible. However, if contracts are contractible, any efficiency gain generated by the competition can disappear due to the collusion among the principals. Therefore, a policy implication of our results is that contracting on contracts should be prohibited in these environments.

Suppose that $k = 1$ and $A_i = X \times P$ where X is a finite set and P is finite subset of $\mathbb{R}$. If Principal j takes action $a_j = (x_j, p_j)$ then the payoff to the agent is

$$v(t, a) = \max\left\{0, \max_{i \in \{1,...,n\}} V(t, x_i) - p_i\right\}.$$

The interpretation of this expression is that the agent has the option to opt out and receive his reservation value, normalized to be zero. Otherwise, the agent's payoff is determined by the action taken by a principal which is best for him. This assumption is made so that the single-principal model is comperable with the multi-principal one. This implies that the agent can only enter in an exclusive relationship with a principal. In order to resolve ties, we assume that if $V(t, x_j) - p_j = V(t, x_q) - p_q$ and $j < q$ then the agent strictly prefers (x_j, p_j) to (x_q, p_q).

The payoff to Principal j is defined as follows:

$$u_j(t,a) = \begin{cases} u(x_j) + p_j & \text{if } j = \min \arg\max_{q \in \{1,\ldots,n\}} [V(t,x_q) - p_q] \text{ and } V(t,x_j) - p_j \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This means that a principal can only achieve a strictly positive payoff if his action maximizes the payoff of the agent. In addition, his name must be the smallest one among those who maximize the agent's payoff.

This model is the discrete version of the standard principal-agent model with adverse selection where the principal's payoff does not depend on types. This model has several interpretations. First, one can think of the agent as a buyer and the principals as sellers. The type of the agent is his valuation, x_j is the quality (or quantity) of the product of Principal j, and p_j is the price. Second, the agent is a potential employee and the principals are employers. The pair, (x_j, p_j), is a labor contract, where p_j is the wage and x_j specifies other characteristics of the job such as the number of vacation days, health insurance benefits etc. The type of a worker describes his taste for the various characteristics of the job. Of course, there are many other interpretations too.

Before we proceed we make a few technical assumptions.

Assumption 2. (i) For all $t \in T$, there exists $\mathbf{p}_c(t) \in P$ such that

$$(\mathbf{x}_c(t), \mathbf{p}_c(t)) = \arg \max_{\substack{(x,p) \in X \times P \\ u(x)+p \geq 0 \\ V(t,x)-p \geq 0}} V(t,x) - p,$$

and (ii) for all $x \in X : u(x) + \min P < 0 < u(x) + \max P$.

Part (i) says that there are transfers which implement the Pareto efficient allocation without violating the participation constraint of the players and the agent's payoff is maximized subject to the principals' participation constraints. Part (ii) says that the transfer can be small (large) enough so that the payoff of a principal is negative (positive) no matter what action he takes. We point out that we need these assumptions only because we consider a discrete model, and both of these assumptions are satisfied in the standard continuous version of our environment. Finally, notice that $(\mathbf{x}_m, \mathbf{p}_m)$ as well as $\mathbf{x}_c(t)$ are generically unique.

Suppose first that $n = 1$, that is, the principal has monopoly power. The characterization of the principal's payoff maximizing contract is based on the Revelation Principle. Let $\mathbf{x}$, $\mathbf{p} : T \to (X, P)$

$$\begin{aligned} &\max_{\mathbf{x},\, \mathbf{p}} E_t [u(\mathbf{x}(t)) + \mathbf{p}(t)] \\ \text{s.t. } V(t, \mathbf{x}(t)) - \mathbf{p}(t) &\geq V(t, \mathbf{x}(t')) - \mathbf{p}(t') \text{ for all } t, t' \in T. \\ \text{and } V(t, \mathbf{x}(t)) - \mathbf{p}(t) &\geq 0. \end{aligned}$$

The first constraint is the incentive compatibility constraint and the second one is the participation constraint of the agent. By part (ii) of Assumption 2, the value of this problem is positive. Let $(\mathbf{x}_m, \mathbf{p}_m)$ denote the solution to this problem.

The Pareto Optimal outcome is the solution to the following problem:

$$\max_{\mathbf{x}} Eu\left(\mathbf{x}\left(t\right)\right) + V\left(t, \mathbf{x}\left(t\right)\right)$$

Let $\mathbf{x}_c$ denote the solution.

Proposition 3 *Suppose that Assumption 2 is satisfied. If $n > 1$, each equilibrium implements $\mathbf{x}_c$ in the ordinary contracting game. In the contractible contracting game, there exists an equilibrium which implements $\mathbf{x}_m$.*

Proof. Suppose by contradiction that contracts are not contractible and there exists an equilibrium and $t \in T$, such that $\mathbf{x}_c\left(t\right)$ is not implemented if the agent's type is t. Since the agent prefers to contract with the principal with the *smallest* name, all but Principal 1 receives a payoff of zero even conditional on the type of the agent. We show that Principal 2 can increase his payoff by deviation at the contracting stage. Consider the following contract: $d\left(l\right) = \left(\mathbf{x}_c\left(t\right), \mathbf{p}_c\left(t\right)\right)$ for all l. By part (i) of Assumption 2, the agent with type t, and perhaps with other types too, will interact with Principal 2, and hence, Principal 2 can achieve a positive payoff. In addition, this payoff is strictly positive generically.

Suppose now that contracts are contractible. In order to verify that $\left(\mathbf{x}_m, \mathbf{p}_m\right)$ is an equilibrium outcome, we verify parts (i) and (ii) of the statement of Theorem 1. To see that $\left(\mathbf{x}_m, \mathbf{p}_m\right) \in \mathcal{A}$, consider the following ordinary contract:

$$d_m\left(l\right) = \begin{cases} \left(\mathbf{x}_m\left(t\right), \mathbf{p}_m\left(t\right)\right) & \text{if } < l >= t \in T, \\ \left\{\left(\mathbf{x}_m\left(t\right), \max P\right) : t \in T\right\} & \text{if } < l > \notin T. \end{cases}$$

Consider the subgame where each principal offers the contract d_m. Since $\left(\mathbf{x}_m\left(t\right), \mathbf{p}_m\left(t\right)\right)$ was incentive compatible, the agent with type t reporting $[t]$ is an equilibrium strategy. Obviously, this strategy implements $\left(\mathbf{x}_m\left(t\right), \mathbf{p}_m\left(t\right)\right)$ in the subgame, and hence, $\left(\mathbf{x}_m, \mathbf{p}_m\right) \in \mathcal{A}$. Notice that in this subgame each principal receives a non-negative payoff. Hence, in order to verify part (ii) of Theorem 1, it is enough to show that the minmax value of each principal is at most zero. To this end, consider the following contract:

$$\underline{d}\left(l\right) = \begin{cases} \left(x, \min P\right) & \text{if } < l >= x \in X, \\ X & \text{if } < l > \notin X. \end{cases}$$

Suppose that all principals but Principal j offers this contract. Then the agent can generate any element from X and pay the smallest transfer. Therefore, by part (ii) of Assumption 2, the highest payoff Principal j can achieve is zero. ■

5.2 Comparison Between Ordinary and Contractible Contracts

The next proposition states that the ability to contract on contracts expands the set of implementable allocations.

Proposition 4 *The set of allocations in the contractible contracting game is larger than in the ordinary contracting game.*

Proof. Suppose that (d^*, β^*, α^*) implements an allocation in the ordinary contracting game. We construct equilibrium strategies in the contractible contracting game which implements the same allocation. Define the contract for Principal j, $c_j^* \in C_j$, as follows: $c_j^*\left(([c_q])_{q=1}^n, \left([m_j^i]\right)_{i=1}^k\right) = d_j^*\left(\left([m_j^i]\right)_{i=1}^k\right)$ for all $c \in C$ and $m \in \mathcal{L}^k$. In a subgame generated by the contract profile c, define the second-stage strategies of the agents as $\alpha^*(d(c), .)$, and the third stage strategies of the principals as $\beta^*(d(c), .)$.

We have to show that the players have no incentives to deviate. First, recall that the subgame generated by c in the contractible contracting game is the same as the subgame generated by $d(c)$ in the ordinary contracting game. Since β^* and α^* were equilibrium strategies in the ordinary contracting game, all players play a Weak Perfect Bayesian Equilibrium in every subgame. We only have to show that principals have no incentive to deviate at the contracting stage. Suppose that Principal j offers the contract c_j instead of c_j^*. This deviation results a subgame generated by $\left(c_j\left([c_j], \left[c_{-j}^*\right]\right), d_{-j}^*\right)$. Notice that Principal j could generate the same subgame in the ordinary contracting game by offering $c_j\left([c_j], \left[c_{-j}^*\right]\right)$. Since this deviation was not profitable in the ordinary contracting game, offering c_j is not profitable in the contractible contracting game. ■

The next example shows that the set of allocations implementable by contractible contracting game can be strictly larger than the set of allocations implementable by ordinary contracts.

Example 5. Suppose that $n = 2$ and $k = 1$. Let $A_1 = \{a_1, a_2\}$ and $A_2 = \{b_1, b_2\}$. The type space of the agent is degenerate. Payoffs are as follows:

	b_1	b_2
a_1	$2, 1, 0$	$1, 3, 1$
a_2	$0, 0, 2$	$0, 0, 3$

Again, the first and second numbers in each cell are the payoffs to the principals and the third one is the payoff to the agent. We show that the outcome (a_1, b_1) cannot be implemented as an equilibrium of the ordinary contracting game, but can be implemented by contractible contracts.

Suppose that the equilibrium contract profile (d_1^*, d_2^*) implements (a_1, b_1). Notice that, given a_1, Principal 2 prefers action b_2 to action b_1. We show that Principal can profitably deviate by offering the contract d_2' such that $d_2'(q) = \{b_2\}$ for all q. If there is a q such that $d_1^*(q) = \{a_2\}$, the agent will send such a message, and therefore (d_1^*, d_2^*) cannot implement (a_1, b_1). We can conclude that $a_1 \in d_1^*(q)$ for all q. Since a_1 strictly dominates a_2, Principal 1 always takes action a_1. Hence, if Principal 2 offers d_2' the outcome will be $\{a_1, b_2\}$ which is strictly preferred by Principal 2 to (a_1, b_1).

Next, we show that the outcome (a_1, b_1) can be implemented by contractible contracts. In order to do so we have to verify that both (i) and (ii) of the statement of Theorem 1 hold. First, the allocation $(a_1, b_1) \in \mathcal{A}$, because it is an equilibrium in the subgame generated by (d_1, d_2) where

$d_1(q) = \{a_1\}$ and $d_2(q) = \{b_1\}$ for all q. Second, we argue that $\underline{u_j} \leq u_j(a_1, b_1)$ for $j = 1, 2$. The action profile (a_1, b_1) maximizes Principal 1's payoff, and hence, $\underline{u_1} \leq u_1(a_1, b_1)$. Now, define Principal 1's contract, d_1^2, such that as $d_1^2(q) = \{a_2\}$ for all q. This contract generates a payoff of zero to Principal 2, no matter what the rest of the strategies are. Since zero is the smallest possible payoff to Principal 2 we conclude that $\underline{u_2} = 0 < u_2(a_1, b_1) = 1$.

The next proposition identifies some environments where the allocations of the contractible contracting game and that of the ordinary contracting game are the same.

Proposition 5 *Suppose that $k \geq 3$. Then the set of allocations in the contractible contracting game is the same as in the ordinary contracting game.*

Notice that Proposition 5 and Theorem 1 imply a kind of Folk Theorem for ordinary contracting games. This result is similar to the main theorem in Yamashita (2007). The main difference is that principals can be pushed down to their maxmin values in Yamashita (2007) instead of to their minmax value as in this paper. This is due to Yamashita's restriction to complete contracts. That is, Yamashita forces the principals to participate in the contracting game and specify a single action as a function of the agents' messages. This restriction makes it easier for the other principals to punish. We point out that the proof of this proposition is an adaptation of Yamashita's idea to our setting.

Proof. By Proposition 4, we only have to show that any equilibrium outcome in the contractible contracting game can also be implemented by ordinary contracts. Suppose that $(c_1^*, ..., c_n^*) \in C$ is an equilibrium contract profile and (β^*, α^*) are the corresponding equilibrium strategies of the agents and the principals at the second and third stages of the game. Recall that $d(c^*)$ denotes the ordinary contract profile generated by c^*, that is, $d_j(c^*) = c_j^*([c_1^*], ..., [c_n^*])$. According to the proof of the "if only" part of Theorem 1, it is without loss of generality to assume that if Principal q deviates at the contracting stage and offers c_q, the resulting subgame is generated by the contract profile $\left(d_q\left(c_q, c_{-q}^*\right), d_q^{-q}\right)$. In addition, the strategies of the agents and the principals in the subgames depend only on these ordinary contracts, but not on the actual deviation.

Next, we construct equilibrium strategies in the ordinary contracting game which implement the same allocation as (c^*, β^*, α^*). The idea of the equilibrium construction is the following. Each agent sends a pair of messages to each principal. The first message is used to report a deviating principal, and the second one corresponds to the message of the agent in the contractible contracting game. If the majority of the agents report to Principal j that Principal q deviated then Principal j punishes Principal q by d_q^j. Each agent reports deviations truthfully because missreporting a deviation by a single agent has no impact. If agents do not report any deviation then the second messages have the same consequences as in the subgame generated by $d(c^*)$. Formally, let $\left(l_j^i, m_j^i\right) \left(\in \mathcal{L}^2 = \mathcal{L}\right)$ denote the pair of messages of Agent i to Principal j. Define the ordinary contract of Principal j as follows:

$$\overline{d}_j\left(\left[\left(l_j^i, m_j^i\right)\right]_{i=1}^k\right) = \begin{cases} d_j^q\left(\left[m_j^i\right]_{i=1}^k\right) & \text{if } \left|\left\{i : l_i^j = q\right\}\right| > k/2, \\ d_j\left(c^*\right)\left(\left[m_j^i\right]_{i=1}^k\right) & \text{otherwise.} \end{cases} \tag{13}$$

The intuition behind this contract is the following. If the majority of the agents report that Principal q deviated then the messages of the agents are evaluated according to d_j^q, which is the contract used by Principal j to punish Principal q in the contractible contracting game. If no deviation is reported by the majority of the agents then the messages of the agents are evaluated according to $d_j(c^*)$. Define the agents' strategies as follows. If Principal j offers $\overline{d}_j$ for all j then Agent i sends the message $\left(0, \beta_j^{*i}\left(t^i, c^*\right)\right)$ to Principal j. If Principal j offers $\overline{d}_j$ for all $j \neq q$, and Principal q offers d_q, then Agent i send the message $\left(q, \beta_j^{*i}\left(\left(t^i, d_q, d_{-q}^q\right)\right)\right)$ to Principal j, and $\left(\beta_q^{*i}\left(t^i, \left(d_q, d_{-q}^q\right)\right)\right)$ to Principal q. The third-stage strategies of the principals are defined as follows. If Principal q offers $\overline{d}_q$ for all q and $\left|\left\{i : l_i^j = q\right\}\right| < k/2$ for all q then Principal j takes action $\alpha_j\left(\left[m_j^i\right]_{i=1}^k, c^*\right)$. (According to the second line of (13), this action is consistent with the contracts.) If Principal q offers the contract $d_q \neq \overline{d}_q$ but the other principals do not deviate and $\left|\left\{i : l_i^j = q\right\}\right| > k/2$ then Principal j takes action $\alpha_j\left(\left[m_j^i\right]_{i=1}^k, \left(c_q, c_{-q}^*\right)\right)$, where $d_q\left(c_q, c_{-q}^*\right) = d_q$. (According to the first line of (13), this action is consistent with the contracts.) The strategies in the rest of the subgames can arbitrarily defined. Notice that these strategies indeed implement the same allocation as (c^*, β^*, α^*).

It remained to show that players cannot increase their payoffs by deviating. First, each agent reports truthfully a deviation by each principal. This is because, given the strategy of the other agents, any deviation from this strategy would have no effect on the payoffs. Second, we defined the rest of the strategies in the relevant subgames so that they correspond to the equilibrium strategies in the contractible contracting game. Therefore, if there was a profitable deviation in a subgame, then there would be a profitable deviation from (β^*, α^*). Finally, principals cannot achieve a higher payoff by offering a different contract, because any such a deviation would be punished the same way as a deviation in the contractible contracting game. ■

5.3 Incompleteness of Contracts

The following example shows that equilibrium contracts cannot be assumed to be complete even if contracts are contractible.

Example 6. Suppose that $n = 2$ and $k = 1$. The action space of Principal 1 is $\{x, y\}$, and the action space of Principal 2 is $\{a_1, a_2, s\}$. The type space of the agent is $\{1, 2\}$ and each type is equally likely. The payoff of Principal 1 is constant zero. The following tables represent the payoffs to Principal 2 and to the agent, respectively:

$t = 1$	a_1	a_2	s
x	$1, 1.1$	$-3, 0.1$	$0, -1$
y	$0, 1$	$-3, 0$	$1, -1$

,

$t = 2$	a_1	a_2	s
x	$-3, 0$	$1, 1.1$	$0, -1$
y	$-3, 0.1$	$0, 1$	$1, -1$

First, we show that the constant allocation (x, s) can be implemented with incomplete contracts. Then we show that the same allocation cannot be implemented with complete contracts.

In order to show that the outcome (x, s) is an equilibrium outcome, we have to verify the two conditions of Theorem 1. The constant action profile (x, s) belongs to $\mathcal{A}$ because it is the unique outcome in the subgame generated by (d_1, d_2), where $d_1(n) = \{x\}$ and $d_2(n) = \{s\}$ for all n. It remains to show that the minmax value of Principal 2, defined by (4), is weakly smaller than zero. In order to do so, consider the following contract of Principal 1: $d_1(n) = \{x, y\}$ for all $n \in \mathbb{N}$. We show that for all $d_2 \in D_2$, there is an equilibrium in the subgame $G_{(d_1,d_2)}$ such that the payoff of Principal 2 is at most zero. We have to analyze four different cases depending on the range of d_2.

Case 1. There exist n_1 and n_2 such that $d_2(n_1) = \{a_1\}$ and $d_2(n_2) = \{a_2\}$. Define the agent's strategy as follows: $m_1^2(1) = n_1$, $m_1^2(2) = n_2$, and $m_1^1 \equiv 1$. Principal 1's strategy is to take action y. These strategies constitute an equilibrium in the subgame $G_{(d_1,d_2)}$ and result a payoff of 0 to Principal 2.

Case 2. There exists an n_1 such that $d_2(n_1) = \{a_1\}$ but there does not exist n_2 such that $d_2(n_2) = \{a_2\}$. Define the agent's strategy such that $m_1^2 \equiv n_1$ and $m_1^1 \equiv 1$. Principal 2's strategy is defined as follows. If $s \in d_2\left(m_1^2\right)$ then he takes action s, otherwise he takes action a_1. Principal 1 always takes action x. These strategies constitute an equilibrium in the subgame $G_{(d_1,d_2)}$ and result an expected payoff of minus two to Principal 2.

Case 3. There exists an n_2 such that $d_2(n_2) = \{a_2\}$ but there does not exist n_1 such that $d_2(n_1) = \{a_1\}$. Define the agent's strategy such that $m_1^2 \equiv n_2$ and $m_1^1 \equiv 1$. Principal 2's strategy is defined as follows. If $s \in d_2\left(m_1^2\right)$ then he takes s, otherwise he takes action a_2. Principal 1 always takes action x. Again, these strategies constitute an equilibrium in the subgame $G_{(d_1,d_2)}$ and result an expected payoff of minus two to Principal 2.

Case 4. Suppose that there does not exist n_i such that $d_2(n_i) = \{a_i\}$ for $i = 1, 2$. Define the agent's strategy such that then $m_1^2 \equiv n$ if there exists an n such that $d_2(n) = \{a_1, a_2\}$. Otherwise, $m_1^2 \equiv 1$. In addition, $m_1^1 \equiv 1$. Principal 2's strategy is defined as follows. If $s \in d_2\left(m_1^2\right)$ then he takes s, otherwise he takes action a_2. Principal 1 always takes action x. These strategies constitute an equilibrium in the subgame $G_{(d_1,d_2)}$ and the payoff of Principal 2 is at most zero.

Now we show that (x, s) cannot be implemented with complete contracts. Suppose by contradiction that there is an equilibrium implementing (x, s), and Principal 1's contract is complete. Let d_1^2 denote the ordinary contract used by Principal 1 to punish deviations of Principal 2. (The existence of such an ordinary contract is guaranteed by Proposition 2.) The contract d_1^2 is complete because Principal 1's equilibrium contract is complete. We have to consider two cases. Case 1: There exists an n such that $d_1^2(n) = \{x\}$. Then, consider the following ordinary contract of Principal $2: d_2(1) = \{a_1\}$ and $d_2(n) = \{a_2\}$ if $n \neq 1$. According to Proposition 2, Principal 2 can induce a subgame equivalent to $G_{\left(d_1^2,d_2\right)}$. Then the agent can generate the action profile (x, a_1) if $t = 1$ and the action profile (x, a_2) if $t = 2$. (He can do so, for example, by sending message n to Principal 1 and is n and $m_1^2(1) = 1$ and $m_1^2(2) = 2$ to Principal 2. Notice that these action profiles the unique maximizers of the agent's payoff, hence he will generate this outcome in $G_{\left(d_1^2,d_2\right)}$. But this outcome provides Principal 2 with a payoff of one which is strictly larger then his payoff from (x,s), and hence, the deviation generating $G_{\left(d_1^2,d_2\right)}$ is profitable. Case 2: Suppose that $d_1(n)$ for all n. Then Principal 2 can deviate and generate $G_{\left(d_1^2,d_2\right)}$ where $d_2(n)=s$ for all n. The outcome of this subgame is (y,s), which generates a payoff of one to Principal 2. Hence, Principal 2 can again profitably deviate.

6 Appendix: The Proof of Proposition 2

For all j, $d_i \in D_i$, and $m=(m_j,m_{-j})$ define

$$H_j^{d_i}(m)=\left\{d_i\left(m_j',m_{-j}\right):m_j'\in\mathbb{N}\right\}.$$

Notice that $H_j^{d_i}(m)\subset 2^{A_i}$. Now, consider $\mathcal{H}_i : D_i \to 2^{A_i}\times\prod_{j=1}^k 2^{2^{A_i}}$ defined as

$$\mathcal{H}_i(d_i)=\left\{\left(d_i(m),H_1^{d_i}(m),...,H_k^{d_i}(m)\right):m\in\mathbb{N}^k\right\}.$$

Lemma 2 *Suppose that d, $d'\in D$ and $\mathcal{H}_i(d_i)=\mathcal{H}(d_i')$ for all $i=1,...,n$. Then $G_d\sim G_{d'}$.*

Let us assume by contradiction that

$$\cap_{d_i\in D_i}S(d_i)=\{\emptyset\}.$$

Then there exists a function, $\mathcal{F}:\mathcal{H}_{-i}(D_{-i})\to\mathcal{H}_i(D_i)$, such that,

$$d_{-i}\notin S(d_i) \tag{14}$$

if $\mathcal{F}(\mathcal{H}_{-i}(d_{-i}))=\mathcal{H}_i(d_i)$. Now, define a function $g:\mathcal{H}_i(D_i)\to D_i$, such that $\mathcal{H}_i(g(\mathcal{H}_i(d_i)))=\mathcal{H}_i(D_i)$. Furthermore, define the function $f:D_{-i}\to D_i$, such that $f(d_{-i})=g(\mathcal{F}(\mathcal{H}_{-i}(d_{-i})))$. Notice that the domains of both $\mathcal{F}$ and g are finite, and hence, f is a definable function. In addition,

$$d_{-i}\notin S(f(d_{-i})), \tag{15}$$

by (14) and the definitions of $\mathcal{F}$ and g. Finally, we are ready to prove the proposition. Define the following contract for Principal i in one free variable:

$$c_i^x([c_i],[c_{-i}],m)=f\left(c_{-i}\left(\left[<x>^{(x)}\right],[c_{-i}],m\right)\right),$$

for all $m\in\mathbb{N}^k$. Let γ denote the Godel code of this contract. Then

$$c_i^\gamma([c_i],[c_{-i}],m)=f\left(c_{-i}\left([c_i^\gamma],[c_{-i}],m\right)\right). \tag{16}$$

First, notice that

$$c_{-i}\left([c_i^\gamma],[c_{-i}]\right)\in S\left(c_i^\gamma\left([c_i^\gamma],c_{-i}\right)\right) \tag{17}$$

by the definition of the set S. On the other hand, by (15),

$$c_{-i}\left(\left[c_i^{\gamma}\right],\left[c_{-i}\right]\right) \notin S\left(f\left(c_{-i}\left(\left[c_i^{\gamma}\right],\left[c_{-i}\right],m\right)\right)\right).$$

This can be rewritten by (16) as

$$c_{-i}\left(\left[c_i^{\gamma}\right],\left[c_{-i}\right]\right) \notin S\left(c_i^{\gamma}\left(\left[c_i\right],\left[c_{-i}\right]\right)\right).$$

But this contradicts to (17), and hence, $\cap_{d_i \in D_i} S\left(d_i\right) \neq \{\emptyset\}$.

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