Recursive Vector Expected Utility

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Abstract

This paper proposes and axiomatizes a recursive version of the *vector expected utility* (VEU) decision model (Siniscalchi, 2009). Recursive VEU preferences are dynamically consistent and "consequentialist." Dynamic consistency implies standard Bayesian updating of the baseline (reference) prior in the VEU representation, but imposes no constraint on the adjustment functions and one-step-ahead adjustment factors. This delivers both tractability and flexibility.

Recursive VEU preferences are also consistent with a dynamic, i.e. intertemporal extension of atemporal VEU preferences. Dynamic consistency is characterized by a time-separability property of adjustments—the VEU counterpart of Epstein and Schneider (2003)'s rectangularity for multiple priors.

A simple exchangeability axiom ensures that the baseline prior admits a representation à la de Finetti, as an integral of i.i.d. product measures with respect to a unique probability μ. Jointly with dynamic consistency, the same axiom also implies that μ is updated via Bayes' Rule to provide an analogous representation of baseline posteriors.

Finally, an application to a dynamic economy à la Lucas (1978) is sketched.

Keywords: Ambiguity, reference prior, dynamic consistency, exchangeability.

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1 Introduction

Siniscalchi (2009) (henceforth S09) proposes and axiomatizes the "vector expected utility" decision model—VEU for short. According to this model, the individual evaluates uncertain prospects, or acts, by a process suggestive of *anchoring and adjustment* (Tversky and Kahneman, 1974). The "anchor" is the expected utility of the prospect under consideration, computed with respect to a *baseline probability*; the "adjustment" depends upon its *exposure to distinct sources of ambiguity*, as well as its *variation* away from the anchor at states that the individual deems ambiguous. Formally, an act f, mapping each state $\omega \in \Omega$ to a consequence $x \in X$, is evaluated via the functional

$$V(f) = \mathrm{E}_p[u \circ f] + A\left(\left(\mathrm{E}_p[\zeta_i \cdot u \circ f]\right)_{0 \le i < n}\right). \tag{1}$$

In Eq. (1), $u : X \to \mathbb{R}$ is a von Neumann-Morgenstern utility function; p is a *baseline probability* on Ω, and E_p is the corresponding expectation operator; $n \le \infty$ and, for $0 \le i < n$, ζ_i is a random variable, or *adjustment factor*, that satisfies $\mathrm{E}_p[\zeta_i] = 0$; and the function $A : \mathbb{R}^n \to \mathbb{R}$ satisfies $A(0) = 0$ and $A(-\phi) = A(\phi)$ for every vector $\phi \in \mathbb{R}^n$. As is demonstrated in S09, VEU specifications of preferences can be analytically tractable, yet retain sufficient flexibility to accommodate a wide variety of preference patterns and ambiguity attitudes. For instance, a simple VEU specification can rationalize the well-known "reflection example" proposed by Machina (2009); by way of contrast, popular models of choice, including Choquet expected utility and all decision models satisfying the "Uncertainty Aversion" axiom of Schmeidler (1989) are inconsistent with the preferences in Machina's example (Baillon et al., forthcoming).

The objective of the present paper is to develop and axiomatize *recursive* VEU preferences. Following Epstein and Schneider (2003) (henceforth ES), the basic objects of choice are contingent consumption plans, or simply "plans," adapted to a given sequence of progressively finer partitions—that is, an event tree. The individual is endowed with preferences conditional upon every time and state (subject to the natural measurability restrictions). In the proposed recursive formulation, preferences conditional upon (t, ω) are represented by a functional $U_t(f, \omega)$ that satisfies the recursive relation

$$U_t(f, \omega) = u(f_t(\omega)) + \beta \mathrm{E}_p[U_{t+1}(f, \cdot) | \mathscr{F}_t(\omega)] + A_{t,\omega}\left(\beta \mathrm{E}_p[\zeta^{t,\omega} \cdot U_{t+1}(f, \cdot) | \mathscr{F}_t(\omega)]\right). \tag{2}$$

In Eq. (2), f denotes a plan, which is a collection of suitably measurable acts f_t representing state-contingent consumption at time t, and $\mathscr{F}_t(\omega)$ denotes the cell of the time-t partition containing ω. The adjustment factors $\zeta^{t,\omega}$ and adjustment functions $A_{t,\omega}$ are permitted to vary with time and state; the details are discussed in Sec. 2.3.

Two key features of the recursive VEU representation are worth emphasizing. First, at each time t and state ω, the baseline probability employed in Eq. (2) is simply the *Bayesian update of the time-0 baseline prior p*. This reinforces the central role of the baseline prior in the VEU model. Furthermore, it suggests that its "anchoring and adjustment" interpretation extends to dynamic settings: the DM acts as if she first specified the law of motion for the underlying uncertainty, and then adjusted it in view of ambiguity. Finally, as in the static setting, the fact that all expectations are taken with respect to a single probability measure can enable tractable specifications, again even in dynamic settings.

The second noteworthy feature is the fact that the adjustment factors $\zeta^{t,\omega}$ are $\mathscr{F}_{t+1}$-measurable. The interpretation is that, in each time period, *ambiguity concerns one-step-ahead probabilities*, rather than the entire continuation process. This is in line with the intuition that the DM distorts the law of motion to account for ambiguity, but does so in a manner consistent with recursion.

Recursive VEU preferences also admit a direct, "dynamic VEU" representation that explicitly involves contingent consumption plans:

$$\tilde{U}_t(f,\omega) = \mathrm{E}_p\left[\sum_{\tau\geq t}\beta^{\tau-t}\,u\circ f_\tau \middle| \mathscr{F}_t(\omega)\right] + \tilde{A}_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\geq t+1}\zeta^{t,\omega}_\tau\cdot\beta^{\tau-t}\,u\circ f_\tau\middle|\mathscr{F}_t(\omega)\right]\right). \tag{3}$$

The adjustment functions and factors in Eq. (3) are required to satisfy a *time-separability* condition, reminiscent of ES's notion of *rectangularity* (their Def. 3.1), and even more closely Maccheroni et al. (2006b)'s Eq. (11). This dynamic representation suggests that recursive VEU preferences are a *bona fide* intertemporal version of atemporal VEU preferences.

While, as noted above, the adjustment function is permitted to vary with time and state, this is of course not required; indeed, S09 (see in particular Prop. 3 and 4) suggests ways to ensure axiomatically that the same adjustment function is used in every period. On the other hand, it is interesting to note that the recursive VEU specification can model an individual who may react to information by becoming more or less ambiguity-averse, but do so in a fully time-consistent way.

Finally, a notion of *exchangeability* is introduced in a setting in which uncertainty is captured by the realizations of a sequence of random variables $X_0, X_1, \ldots$ taking value in some finite set $\mathscr{X}$. The proposed notion delivers a representation of the baseline prior p à la De Finetti: for every event E,

$$p(E) = \int_{\Delta(\mathscr{X})} \ell^\infty(E) d\mu(\ell),$$

where $\Delta(\mathscr{X})$ is the set of probabilites on $\mathscr{X}$, with typical element ℓ, and ℓ^∞ represents the i.i.d. product measure corresonding to $\ell \in \Delta(\mathscr{X})$. For recursive VEU preferences, the baseline posteriors $p(\cdot|\mathscr{F}_t(\omega))$ admit a similar representation, but the integrating measure is obtained from μ via Bayes' Rule. This provides a behavioral foundation for parametric learning in the VEU model.

As in ES, the key axiom adopted in this paper is *dynamic consistency*. This requires that, if the DM deems a plan f at least as good as another plan f' at time $t+1$, regardless of the realization of the time-t uncertainty, then she should also rank f above f' at time t. ES axiomatize recursive maxmin expected-utility (MEU) preferences (Gilboa and Schmeidler, 1989); they show that dynamic consistency ensures that the DM's set of initial beliefs will be updated prior-by-prior via Bayes' rule. An analogous property holds here: as noted above, the DM's baseline prior is updated via Bayes' Rule. Similar properties do not hold uniformly for all recursive models of choice under ambiguity; for instance, in the recursive smooth model of Klibanoff et al. (2009), Bayesian updating of the second-order prior is not a consequence of dynamic consistency alone.

It is also worth noting that the proposed notion of exchangeability is compatible with full dynamic consistency. Epstein and Seo (2010) axiomatize an exchangeable version of the atemporal multiple-priors model; however, they employ a weak (i.e. partial) dynamic consistency condition. By way of contrast, their exchangeability requirement is stronger than the notion adopted here.

As an example, this paper considers a version of the Lucas (1978) economy wherein agents have recursive VEU preferences. A characterization of equilibrium asset prices is provided, along with preliminary observations on the features of the stochastic discount factor.

2 Notation and Definitions

2.1 Basics

The following notation is standard. Consider a set Ω (the state space) and a sigma-algebra Σ of subsets of Ω (events). It will be useful to assume that the sigma-algebra Σ is *countably generated*: that is, there is a countable collection $\mathscr{S} = (S_i)_{i\geq 0}$ such that Σ is the smallest sigma-algebra containing $\mathscr{S}$. All finite and countably infinite sets, as well as all Borel subsets of Euclidean n-space, and more generally all standard Borel spaces (Kechris, 1995) satisfy this assumption.

Information is described via an event tree. Formally, fix a sequence $(\mathscr{F}_t)_{t\geq 0}$ of sigma-algebra generated by progressively finer, finite partitions of Ω; $\mathscr{F}_0$ is assumed to be trivial. For every $\omega \in \Omega$ and $t \geq 0$, denote by $\mathscr{F}_t(\omega)$ the cell of the partition generating $\mathscr{F}_t$ that contains ω. It is convenient to refer to a pair (t, ω) as a *node*, which evokes the underlying event tree. Also assume that

$$\forall (E_k)_{k\geq 0} \subset \bigcup_{t\geq 0} \mathscr{F}_t \quad \text{s.t.} \quad \forall k,\ E_k \supset E_{k+1}: \quad \bigcap_{k\geq 0} E_k \neq \emptyset. \tag{4}$$

That is, any decreasing sequence of conditioning events has a non-empty intersection. This assumption holds if, for instance, Ω is the set of realizations of a sequence $(X_t)_{t\geq 0}$ of random variables, such that each X_t takes values in some finite set $\mathscr{X}_t$, and $\mathscr{F}_t = \sigma(X_0, \ldots, X_{t-1})$ (in particular, $\mathscr{F}_0 = \{\Omega\}$ by convention). In this case, every $E_t \in \mathscr{F}_t$ is of the form $E_t = \{(x_0, \ldots, x_{t-1})\} \times \prod_{\tau\geq t} \mathscr{X}_\tau$; a sequence (E_k) as in Eq. (4) must then be of the form $\{(x_0, \ldots, x_{t_k-1})\} \times \prod_{\tau\geq t_k} \mathscr{X}_\tau$ for some non-decreasing sequence $(t_k)_{k\geq 0}$, and if $t_k \to \infty$, then $\bigcap_{k\geq 0} E_k$ consists of the point $(x_t)_{t\geq 0} \in \Omega$.

Denote by $B_0(\Sigma)$ the set of Σ-measurable real functions with finite range, and by $B(\Sigma)$ its sup-norm closure. The set of countably additive probability measures on Σ is denoted by $ca_1(\Sigma)$. For any probability measure $\pi \in ca_1(\Sigma)$ and function $a \in B(\Sigma)$, let $\mathrm{E}_\pi[a] = \int_\Omega a\, d\pi$, the standard Lebesgue integral of a with respect to π. Finally, $a \circ b : \mathscr{X} \to \mathscr{Z}$ denotes the composition of the functions $b : \mathscr{X} \to \mathscr{Y}$ and $a : \mathscr{Y} \to \mathscr{Z}$.

Additional notation is useful to streamline the definition and analysis of the VEU representation. Given $m \in \mathbb{Z}_+ \cup \{\infty\}$ and a finite or countably infinite collection $z = (z_i)_{0\leq i<m}$ of elements of $B(\Sigma)$, let $\mathrm{E}_\pi[z \cdot a] = (\mathrm{E}_\pi[z_i \cdot a])_{0\leq i<m}$ if $m > 0$, and $\mathrm{E}_\pi[z \cdot a] = 0$ if $m = 0$. For any collection

$F \subset B(\Sigma)$, let $\mathscr{E}(F;\pi,z) = \{\mathrm{E}_\pi[z \cdot a] \in \mathbb{R}^m : a \in F\}$. Finally, let 0_m denote the zero vector in $\mathbb{R}^m$.

Turn now to the decision setting. Consider a convex set X of consequences (outcomes, prizes). As in Anscombe and Aumann (1963), X could be the set of finite-support lotteries over some underlying collection of deterministic prizes (e.g. consumption), endowed with the usual mixture operation. Alternatively, the set X might be endowed with a subjective mixture operation, as in Casadesus-Masanell et al. (2000) or Ghirardato et al. (2003). It is convenient to assume that a preference $\succcurlyeq^X$ on X is given at the outset; the orderings considered below will be assumed to extend $\succcurlyeq^X$ in a suitable sense.

The objects of choice in an atemporal setting are *acts*, i.e. Σ-measurable functions from Ω to X; the set of all simple (i.e. finite-range) acts is denoted F^s. As e.g. in Schmeidler (1989), denote by F^b the set of acts f for which there exist $x, x' \in X$ such that $x \succcurlyeq^X f(\omega) \succcurlyeq^X x'$ for all $\omega \in \Omega$.

Turn now to the objects of choice in a dynamic setting. For a given filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, a *time-$t$ act* is an $\mathscr{F}_t$-measurable act; the set of time-t acts is denoted by F_t. A *plan* is a sequence $f = (f_t)_{t\geq 0}$, where, for every $t \geq 0$, f_t is a time-t act. Note that, since each $\mathscr{F}_t$ is finite, $F_t \subset F^s$ for all $t \geq 0$. We restrict attentions to *bounded plans*, i.e. plans for which there exist $x, x' \in X$ such that $x \succcurlyeq^X f_t(\omega) \succcurlyeq^X x'$. The set of all bounded plans is denoted F^p.

Given a sequence $(x_t)_{t\geq 0}$ of consequences that is bounded for $\succcurlyeq^X$ (that is, for all $t \geq 0$, $x_t \in X$ and there exist $x, x' \in X$ such that $x \succcurlyeq^X x_t \succcurlyeq^X x'$ for all $t \geq 0$), abuse notation and denote the plan that delivers x_t in each state ω by $(x_t)_{t\geq 0}$. The set of plans corresponding to non-contingent, bounded consequence streams is denoted F^{cs}. In particular, if $x_t = x$ for some $x \in X$ and all $t \geq 0$, denote the corresponding act simply by x. The set of such constant plans is denoted F^c. It is sometimes convenient to denote $(x_t)_{t\geq 0}$ by $(x_1, \ldots, x_t, \ldots)$ or similar notation.

Finally, given a function $u : X \to \mathbb{R}$ and a set F of acts, let $u \circ F = \{u \circ f \in B(\Sigma) : f \in F\}$.

2.2 VEU representation: atemporal setting

Begin by reviewing the definition of VEU preferences in an atemporal setting. It is convenient to state it for an arbitrary collection of atemporal acts. It will be assumed throughout that the preference relation of interest extends $\succcurlyeq^X$ (i.e. agrees with $\succcurlyeq^X$ on constant acts).

Definition 1 *Let F be a non-empty subset of F^b. A tuple (u, p, n, ζ, A) is a* ***VEU representation***

of a preference relation $\succcurlyeq$ *on* F *if*

1. $u : X \to \mathbb{R}$ *is non-constant and affine,* $p \in ca_1(\Sigma)$*,* $n \in \mathbb{Z}_+ \cup \{\infty\}$ *and* $\zeta = (\zeta_i)_{0\le i<n}$*;*

2. *for every* $0 \le i < n$*,* $\zeta_i \in B(\Sigma)$ *and* $\mathrm{E}_p[\zeta_i] = 0$*.*

3. $A : \mathscr{E}(u \circ F; p, \zeta) \to \mathbb{R}$ *satisfies* $A(0_n) = 0$ *and* $A(\varphi) = A(-\varphi)$ *for all* $\varphi \in \mathscr{E}(u \circ F; p, \zeta)$*;*

4. *for all* $a, b \in u \circ F$*,* $a(\omega) \ge b(\omega)$ *for all* $\omega \in \Omega$ *implies* $\mathrm{E}_p[a] + A(\mathrm{E}_p[\zeta \cdot a]) \ge \mathrm{E}_p[b] + A(\mathrm{E}_p[\zeta \cdot b])$*;*

5. *for every pair of acts* $f, g \in F$*,*

$$f \succcurlyeq g \quad \Leftrightarrow \quad \mathrm{E}_p[u \circ f] + A\left(\mathrm{E}_p[\zeta \cdot u \circ f]\right) \ge \mathrm{E}_p[u \circ g] + A\left(\mathrm{E}_p[\zeta \cdot u \circ g]\right). \tag{5}$$

In keeping with standard terminological conventions, a *VEU preference* is a binary relation that admits a VEU representation. The statement that (u, p, n, ζ, A) is "a VEU representation on F" will be employed when the focus is on the properties of the tuple (u, p, n, ζ, A) itself, rather than the preference that it defines or represents via Eq. (5).

Conditions 1 – 5 are discussed in S09. The monotonicity requirement in Condition 4 also has a differential characterization: see Ghirardato and Siniscalchi (2009) for details.

Example 1 (The three-color-urn Ellsberg paradox) A ball will be drawn from an urn containing 30 amber balls and 60 blue and green balls; the relative proportion of blue vs. green balls is unspecified. Let the state space be $\Omega = \{\alpha, \beta, \gamma\}$ in obvious notation, and let the prize space be $X = \{0, 1\}$. The DM is asked to rank the acts $f_\alpha = (1, 0, 0)$ vs. $f_\beta = (0, 1, 0)$, i.e. (representing acts as vectors) a bet on amber vs. a bet on blue; then, she is asked to rank $f_{\alpha\gamma} = (1, 0, 1)$ vs. $f_{\beta\gamma} = (0, 1, 1)$, i.e. a bet on amber or green vs. a bet on blue or green.

The modal preferences are $f_\alpha \succ f_\beta$ and $f_{\alpha\gamma} \prec f_{\beta\gamma}$, which contradict the existence of probabilistic beliefs. To accommodate these preferences within the VEU framework, choose a uniform baseline prior p, define a single ($n = 1$) adjustment factor $\zeta_0 = (0, 1, -1)$, and let the adjustment function be $A(\varphi) = -|\varphi|$. Notice how the specification of ζ_0 captures the fact that ambiguity about β and γ "cancels out."

Detailed calculations are provided in S09. For a smooth VEU specification that is also consistent with these preferences, let e.g. $A(\varphi) = -\varphi \cdot \tanh \varphi$. □

S09 also provides the following notion of "parsimonious" VEU representation. Adapting a definition due to Ghirardato et al. (2004), say that an act $f \in F$ is **crisp** if, for every $x \in X$ that satisfies $f \sim x$, and for every $g \in F^s$ and $\lambda \in (0,1]$,

$$\lambda g + (1-\lambda)x \sim \lambda g + (1-\lambda)f. \tag{6}$$

Definition 2 *Let F denote either F^s or F^b. A VEU representation (u,p,n,ζ,A) of a preference relation $\succcurlyeq$ on F is **sharp** if $(\zeta_i)_{0\le i<n}$ is orthonormal and, for any crisp act $f \in F$, $\mathrm{E}_p[\zeta \cdot u \circ f] = 0_n$.*

Theorem 1 in S09 shows that, if a preference admits a VEU representation, then it also admits a sharp VEU representation. The latter conveys more information about the underlying preferences: see §4.2 in S09. However, non-sharp specifications may sometimes be more convenient for analytical purposes. The same is true in the current setting.

2.3 Recursive VEU preferences

Turn now to the dynamic decision environment. The main object of interest is now an "adapted" collection of preference relations:

Definition 3 *A **preference system** is a collection $(\succcurlyeq_{t,\omega})_{t\ge 0,\omega\in\Omega}$ such that, for every $t \ge 0$:*

1. *for every $\omega \in \Omega$, $\succcurlyeq_{t,\omega}$ is a binary relation on F^p;*
2. *$x \succcurlyeq_{t,\omega} x'$ iff $x \succcurlyeq^X x'$ for all $x, x' \in X$; and*
3. *$\omega' \in \mathscr{F}_t(\omega)$ implies $\succcurlyeq_{t,\omega} = \succcurlyeq_{t,\omega'}$.*

To simplify notation, I will write $(\succcurlyeq_{t,\omega})$ in lieu of $(\succcurlyeq_{t,\omega})_{t\ge 0,\omega\in\Omega}$, and similarly (f_t) in lieu of $(f_t)_{t\ge 0}$.

A representation of the preference system $(\succcurlyeq_{t,\omega})$ can be provided in two ways. The first, and most convenient for applications, is recursive:

Definition 4 *A tuple $\left(u,\beta,p,(n_{t,\omega},\zeta^{t,\omega},A_{t,\omega})_{t,\omega}\right)$ is a **recursive VEU representation** of a preference system $(\succcurlyeq_{t,\omega})$ if:*

1. *for every node (t,ω), $\omega' \in \mathscr{F}_t(\omega)$ implies $(n_{t,\omega},\zeta^{t,\omega},A_{t,\omega}) = (n_{t,\omega'},\zeta^{t,\omega},A_{t,\omega'})$; furthermore, $\left(\frac{\beta}{1-\beta}u, p(\cdot|\mathscr{F}_t(\omega)), n_{t,\omega}, \zeta^{t,\omega}, A_{t,\omega}\right)$ is a VEU representation on F_{t+1}, and $\zeta^{t,\omega}$ is $\mathscr{F}_{t+1}$-measurable.*
2. *for every $f \in F^p$, the adapted process $\left(U_t(f,\cdot)\right)_{t\ge 0}$ recursively defined by*

$$U_t(f,\omega) = u(f_t(\omega)) + \beta \mathrm{E}_p[U_{t+1}(f,\cdot)|\mathscr{F}_t(\omega)] + A_{t,\omega}\left(\beta \mathrm{E}_p[\zeta^{t,\omega} \cdot U_{t+1}(f,\cdot)|\mathscr{F}_t(\omega)]\right), \tag{7}$$

is bounded: $\sup_{t,\omega}|U_t(f,\omega)| < \infty$*;*

3. for every node (t,ω) *and plans* $f, g \in F^p$, $f \succcurlyeq_{t,\omega} g$ *iff* $U_t(f,\omega) \geq U_t(g,\omega)$*.*

A recursive VEU representation is ***sharp*** *if every VEU representation in 1 is sharp as per Def. 2.*

Example 2 Consider a coin of ambiguous bias—the individual ignores which of the two sides is more likely to come up. The coin will be tossed twice; represent the state space in obvious notation as $\Omega = \{HH, HT, TH, TT\}$. The event tree is defined by $\mathscr{F}_0 = \{\Omega\}$; $\mathscr{F}_1 = \{\{HH, HT\}, \{TH, TT\}\}$. In other words, at date 0 the individual has no information; at date 1 she learns the outcome of the first coin toss; and at date 2 all uncertainty is resolved. We consider a plan f corresponding to a bet on H *on the first toss and* T *on the second.*

To characterize VEU preferences, assume that the baseline prior p is uniform, that utility is linear, and that, at every t and ω, the adjustment function is $A_{t,\omega}(\varphi) = -\theta|\varphi|$, for some suitable $\theta > 0$. The adjustment factors are defined in Table 1, which also explicitly describes the components (acts) f_0, f_1, f_2 of the plan f.

ω	HH	HT	TH	TT
$\zeta^{1,HH}, \zeta^{1,HT}$	1	-1	0	0
$\zeta^{1,TH}, \zeta^{1,TT}$	0	0	1	-1
$\zeta^{0,\omega}$	1	1	-1	-1
f_2	0	1	0	1
f_1	1	1	0	0
f_0	0	0	0	0

Table 1: Adjustment factors in the coin-toss example.

I now calculate the utility indices assigned to f at various times t and states ω. First, at the terminal date $t = 2$, $U_2(f,\omega) = f_2(\omega)$, due to the assumption of linear utility. Moving back one period, we have

$$U_1(f, HH) = U_1(f, HT) = 1 + \beta\left[\frac{1}{2}0 + \frac{1}{2}1\right] - \theta\left|\beta\left[\frac{1}{2}(0)(1) + \frac{1}{2}(1)(-1)\right]\right| = 1 + \frac{1}{2}\beta(1-\theta);$$

similarly $U_1(f, TH) = U_1(f, TT) = 0 + \frac{1}{2}\beta(1-\theta)$.

Finally, at time 0,

$$U_0(f,\omega)=0+\beta\left\{\frac{1}{2}\left[1+\frac{1}{2}\beta(1-\theta)\right]+\frac{1}{2}\left[0+\frac{1}{2}\beta(1-\theta)\right]\right\}$$
$$-\beta\theta\left|\frac{1}{2}\left[1+\frac{1}{2}\beta(1-\theta)\right](1)+\frac{1}{2}\left[0+\frac{1}{2}\beta(1-\theta)\right](-1)\right|=\frac{1}{2}(1-\theta)(\beta+\beta^2).$$

□

As noted in the Introduction, recursive VEU preferences are fully specified by "one-step-ahead" VEU representations, as per part 1 of Def. 4. The utility process $(U_t(f,\cdot))$ is defined recursively via Eq. (7). Proposition 2 in the Appendix employs standard contraction-mapping techniques to show that Eq. (7) always admits a unique *bounded* solution, so the boundedness requirement in part 2 of Def. 4 is not restrictive.[1]

The one-step ahead representations employ a rescaled utility function $\frac{\beta}{1-\beta}u$. To see why this is necessary, observe that, from Eq. (7), $U_t(f,\omega)$ is the sum of the utility delivered at time t in state w, $u(f_t(\omega))$, and the VEU evaluation of $U_{t+1}(f,\cdot)$, discounted by β. If f delivers prizes bounded below by x and above by x', for suitable $x,x'\in X$, then it stands to reason that $U_{t+1}(f,\omega')\in[\frac{u(x)}{1-\beta},\frac{u(x')}{1-\beta}]$ for all $\omega'\in\Omega$; indeed, Proposition 2 shows that this is the case. Thus, to ensure that $U_t(f,\omega)$ is well-defined for all $f\in F^p$, the one-step-ahead VEU representation employed in Eq. (7) must be defined for functions taking values in $\frac{\beta}{1-\beta}u(X)$, as required in part 1 of Def. 4.

It is worth emphasizing that the monotonicity requirement in Condition 4 is *only* imposed on the (induced) preferences over acts in F_{t+1}—not over plans. Verifying this condition is no harder than checking for monotonicity in the atemporal setting; in particular, the sufficient conditions in Appendix A of S09 apply, or one can employ the characterization provided by Ghirardato and Siniscalchi (2009, Prop. 23). As shown in the proof of necessity in Theorem 1, Condition 4 is enough (given the overall structure of recursive VEU preferences) to deliver monotonicity with respect to arbitrary plans.

[1]In principle, there could be unbounded solutions to Eq. (7); such solutions would define preferences that do not necessarily satisfy the axioms in the following section.

2.4 Infinite-horizon representation and time-separable adjustments

A preference system can alternatively be represented by functionals defined over entire (continuation) utility processes. This representation will in general not be especially useful in applications; however, it is a direct extension of the atemporal VEU representation in Def. 1. Theorem 1 shows that the recursive and dynamic VEU representations (with time-separable adjustments) describe the *same* behavior; hence, the following definition effectively serves as a bridge between the atemporal theory in S09 and the recursive theory that is the focus of this paper.

Definition 5 *A tuple* $\left(u,\beta,p,(\tilde{n}_{t,\omega},\tilde{\zeta}^{t,\omega},\tilde{A}_{t,\omega})_{t,\omega}\right)$ *is a* ***dynamic VEU representation*** *of a preference system* $(\succcurlyeq_{t,\omega})$ *if:*

1. $u: X\to\mathbb{R}$ *is non-constant affine,* $\beta\in(0,1)$*,* $p\in ca_1(\Sigma)$*,* $\tilde{n}_{t,\omega}\in\mathbb{Z}_+\cup\{\infty\}$ *and* $\tilde{\zeta}^{t,\omega}=(\zeta^{t,\omega}_{i\tau})_{0\le i<\tilde{n}_{t,\omega},\tau>t}$*, with* $\zeta^{t,\omega}_{i\tau}$ $\mathscr{F}_\tau$*-measurable for all* i *and* τ*; if* $\omega'\in\mathscr{F}_t(\omega)$*, then* $(n_{t,\omega},\tilde{\zeta}^{t,\omega},\tilde{A}_{t,\omega})=(n_{t,\omega'},\tilde{\zeta}^{t,\omega'},\tilde{A}_{t,\omega'})$*; and* $p(\mathscr{F}_t(\omega))>0$ *for all* (t,ω)*;*

2. for all (t,ω) *and* $0\le i<\tilde{n}_{t,\omega}$*,* $\sup_{\tau>t,\omega'\in\Omega}|\zeta^{t,\omega}_{i\tau}(\omega')|<\infty$*; for all* $\tau>t$*,* $\mathrm{E}_p[\zeta^{t,\omega}_{i\tau}|\mathscr{F}_t(\omega)]=0$*;*

3. for all (t,ω)*,* $\tilde{A}_{t,\omega}:\left\{\mathrm{E}_p\left[\sum_{\tau\ge t+1}\beta^{\tau-t}\zeta^{t,\omega}_{\tau}\cdot u\circ f_\tau|\mathscr{F}_t(\omega)\right]: f\in F^p\right\}\to\mathbb{R}$ *satisfies* $\tilde{A}_{t,\omega}(0_{\tilde{n}_{t,\omega}})=0$*,* $\tilde{A}_{t,\omega}(\varphi)=\tilde{A}_{t,\omega}(-\varphi)$*, and* $\tilde{A}_{t,\omega}=\tilde{A}_{t,\omega}$ *for all* $\omega'\in\mathscr{F}_t(\omega)$*;*

4. for every (t,ω)*, the functional* $\tilde{U}_{t,\omega}:F^p\to\mathbb{R}$ *defined by*

$$\tilde{U}_t(f,\omega)=\mathrm{E}_p\left[\sum_{\tau\ge t}\beta^{\tau-t}u\circ f_\tau\Big|\mathscr{F}_t(\omega)\right]+\tilde{A}_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\ge t+1}\beta^{\tau-t}\zeta^{t,\omega}_{\tau}\cdot u\circ f_\tau\Big|\mathscr{F}_t(\omega)\right]\right). \tag{8}$$

is monotonic: if $f,g\in F^p$ *are such that* $u(f_\tau(\omega'))\ge u(g_\tau(\omega'))$ *for all* $\omega'\in\mathscr{F}_t(\omega)$ *and* $\tau\ge t$*, then* $\tilde{U}_{t,\omega}(f)\ge\tilde{U}_{t,\omega}(g)$*.*

5. for every node (t,ω)*, the preference* $\succcurlyeq_{t,\omega}$ *is represented by* $U_t(f,\omega)$

A dynamic VEU representation has ***time-separable adjustments*** *if*

$$\begin{aligned}\tilde{A}_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\ge t+1}\beta^{\tau-t}\zeta^{t,\omega}_{\tau}\cdot u\circ f_\tau\Big|\mathscr{F}_t(\omega)\right]\right)&=\tilde{A}_{t,\omega}\left(\beta\mathrm{E}_p\left[\zeta^{t,\omega}_{t+1}\cdot U_{t+1}(f,\cdot)\Big|\mathscr{F}_t(\omega)\right]\right)+\\&+\beta\mathrm{E}_p\left[\tilde{A}_{t+1,\cdot}\left(\mathrm{E}_p\left[\sum_{\tau\ge t+2}\beta^{\tau-t-1}\zeta^{t+1,\cdot}_{\tau}\cdot u\circ f_\tau\Big|\mathscr{F}_{t+1}(\cdot)\right]\right)\Big|\mathscr{F}_t(\omega)\right].\end{aligned} \tag{9}$$

As in Def. 4, the adjustment functions $\tilde{A}_{t,\omega}$ and factors $\tilde{\zeta}^{t,\omega}$ are state-dependent and constant on every cell $\mathscr{F}_t(\omega)$; the time-$\tau>t$ components $\zeta^{t,\omega}_{i\tau}$ are $\mathscr{F}_\tau$–measurable.

Conditions 1–5 mirror their counterparts in Def. 1. Time-separability of adjustments corresponds to ES's notion of *rectangularity*: cf. their Def. 3.1, and also Maccheroni et al. (2006b, Eq. (11)). The intuition is as follows. Fixing a plan $f=(f_\tau)\in F^p$, and a node (t,ω), the DM can compute adjustments to the entire utility stream $(u\circ f_\tau)_{\tau\ge t}$: this is what the l.h.s. of Eq. (19) does. Alternatively, the DM can compute an adjustment to the *continuation values* $U_{t+1}(f,\cdot)$, which reflects ambiguity about the resolution of time-$(t+1)$ uncertainty; but this still does not capture ambiguity about the resolution of subsequent uncertainty. To remedy this, the DM considers the possible ways in which time-$(t+1)$ uncertainty can resolve, i.e. imagine her situation at a node of the form $(t+1,\omega')$ for some $\omega'\in\mathscr{F}_t(\omega)$, and compute the adjustment to the subsequent utility stream $(u\circ f_\tau)_{\tau\ge t+1}$; then, to evaluate such future adjustments from the perspective of the current node (t,ω), the individual takes the appropriate conditional expectation. Eq. (19) states that these two procedures are equivalent.

Thus, time-separability of adjustments, like rectangularity, captures the assumption that ambiguity about the immediate future can somehow be isolated from ambiguity about the distant future. This is essential (that is, necessary and sufficient) for dynamic VEU preferences to admit a recursive VEU representation as well—that is, for dynamic consistency to hold.

3 Axiomatic Characterization of VEU preferences

The axiomatics are similar in spirit to Epstein and Schneider (2003) (ES henceforth), but some modifications are necessary due to the nature of VEU preferences. Specifically, at each node (t,ω), preferences are assumed to satisfy a slight strengthening of the VEU axioms that, essentially, ensures that *baseline* preferences over consequence streams admit a time-separable representation. Then, one adds axioms that deliver geometric discounting for consequence stream, consequentialism, and dynamic consistency.

Begin by adapting the VEU axioms in S09. Mixtures of plans are interpreted state-wise and time-wise: that is, if $f=(f_t)_{t\ge0}$ and $g=(g_t)_{t\ge0}$, then $\alpha f+(1-\alpha)g=\big(\alpha f_t+(1-\alpha)g_t\big)_{t\ge0}$, where $[\alpha f_t+(1-\alpha)g_t](\omega)=\alpha f_t(\omega)+(1-\alpha)g_t(\omega)$ for all t,ω.

Axiom 1 (Weak Order) *For each (t,ω), $\succcurlyeq_{t,\omega}$ is transitive and complete.*

Axiom 2 (Monotonicity) *For each* (t,ω) *and all plans* $f=(f_t)_{t\geq 0}, g=(g_t)_{t\geq 0}\in F^p$*, if* $f_\tau(\omega')\succsim_{t,\omega} g_\tau(\omega')$ *for all* $\tau\geq 0$ *and* $\omega'\in\Omega$*, then* $f\succsim_{t,\omega} g$*.*

Axiom 3 (Continuity) *For each* (t,ω) *and all plans* $f,g,h\in F^p$*, the sets* $\{\alpha\in[0,1]:\alpha f+(1-\alpha)g\succsim_{t,\omega} h\}$ *and* $\{\alpha\in[0,1]:h\succsim_{t,\omega}\alpha f+(1-\alpha)g\}$ *are closed.*

Axiom 4 (Non-Degeneracy) *For each* (t,ω)*, not for all* $f,g\in F^p$*,* $f\succsim_{t,\omega} g$*.*

Axiom 5 (Weak Certainty Independence) *For all plans* $f,g\in F^p$*,* $x,y\in X$*, and* $\alpha\in(0,1)$*:* $\alpha f+(1-\alpha)x\succsim_{t,\omega}\alpha g+(1-\alpha)y$ *implies* $\alpha f+(1-\alpha)x\succsim_{t,\omega}\alpha g+(1-\alpha)y$*.*

The following axiom is a version of monotone continuity, adapted to the present dynamic setting. It is required because, while each partition $\mathscr{F}_t$ is finite, the time horizon is infinite, and so there is a countable infinity of events that are relevant to the individual's decisions. The following notation streamlines the statement of the axiom: for $f\in F^p$ and $x\in X$, let $[f=x]=\{(t,\omega):f_t(\omega)=x\}$.

Axiom 6 (Monotone Continuity) *Fix a node* (t,ω) *and consequences* $x,y,z\in X$ *such that* $x\succ y\succ z$*. Consider sequences* $(f^k)_{k\geq 0},(g^k)_{k\geq 0}\subset F^p$ *such that, for every* $k\geq 0$ *and* (t',ω')*, either* $f^k_{t'}(\omega')=z$ *and* $g^k_{t'}(\omega')=x$ *or* $f^k_{t'}(\omega')=x$ *and* $g^k_{t'}(\omega')=z$*. If* $[f^k=z]\supset[f^{k+1}=z]$ *and* $\bigcap_{k\geq 0}[f^k=z]=\emptyset$*, then there is* $k\geq 0$ *such that* $f^k\succ_{t,\omega} y\succ_{t,\omega} g^k$*.*

I now strengthen the axiomatics relative to the atemporal setting. First, note that, unlike ES, the Monotonicity and Weak Certainty Independence are exactly as in the atemporal setting: they are merely adapted to the present environment. However, in keeping with the spirit of the VEU decision model, *baseline* preferences are required to satisfy an additional time-separability assumption. To formalize it and the central axiom for VEU preferences, Complementary Independence, the following definitions are required.

Definition 6 *Two plans* $f=(f_t)_{t\geq 0},\bar f=(\bar f_t)_{t\geq 0}\in F^p$ *are* ***complementary*** *if and only if, for any two states* $\omega,\omega'\in\Omega$*, and all times* $t\geq 0$*,*

$$\frac{1}{2}f_t(\omega)+\frac{1}{2}\bar f_t(\omega)\sim^X\frac{1}{2}f_t(\omega')+\frac{1}{2}\bar f_t(\omega').$$

If two plans $f,\bar f\in F^p$ *are complementary, then* $(f,\bar f)$ *is referred to as a* ***complementary pair****.*

Def. 6 adapts the corresponding notion in S09, but is weaker in the present setting. In particular, the $\frac{1}{2}:\frac{1}{2}$ mixtures of time-t acts in two complementary plans must be constant within each time period, but may vary across time periods. The intuition is that there is no need to "smooth *discounted* utilities across time": ambiguity only concerns states and events.

Next, adopt the notion of *crisp acts* in S09, who in turn borrows/adapts it from Ghirardato et al. (2004). A crisp act "behaves like a constant" in mixtures:

Definition 7 *A plan $f \in F^p$ is **crisp** if and only if, for any node (t,ω), prize $x \in X$, plan $g \in F^p$ and $\lambda \in (0,1)$,*

$$f \sim_{t,\omega} x \quad \Rightarrow \quad \lambda f + (1-\lambda) g \sim_{t,\omega} \lambda x + (1-\lambda) g.$$

The two main time-separability assumptions can now be stated. First, adopt the Monotonicity axiom of ES, but restrict it to comparisons of complementary plans:

Axiom 7 (Complementary Stream Monotonicity) *For each (t,ω) and all complementary plans $f=(f_t)_{t\geq 0}, \bar{f}=(\bar{f}_t)_{t\geq 0} \in F^p$, $\big(f_t(\omega)\big)_{t\geq 0} \succcurlyeq_{t,\omega} (\bar{f}_t(\omega))_{t\geq 0}$ for all $\omega \in \Omega$ implies $f \succcurlyeq_{t,\omega} \bar{f}$.*

Second, ensure that there is no hedging of *discounted utilities* across time:

Axiom 8 (Crisp Streams) *Every consequence stream $(x_t) \in F^{cs}$ is crisp.*

Finally, the key axiom for recursive VEU preferences, Complementary Independence, is stronger than in S09, because the notion of complementarity is weaker.

Axiom 9 (Complementary Independence) *For each (t,ω), any two complementary pairs $(f,\bar{f})$ and $(g,\bar{g})$ in F^p, and all $\alpha \in [0,1]$: $f \succcurlyeq_{t,\omega} \bar{f}$ and $g \succcurlyeq_{t,\omega} \bar{g}$ imply $\alpha f + (1-\alpha) g \succcurlyeq_{t,\omega} \alpha \bar{f} + (1-\alpha)\bar{g}$.*

A final assumption is needed (unmodified from S09):

Axiom 10 (Complementary Translation Invariance) *For each (t,ω), all complementary pairs $(f,\bar{f})$ in F^p, and all $x,\bar{x} \in X$ with $f \sim_{t,\omega} x$ and $\bar{f} \sim_{t,\omega} \bar{x}$: $\frac{1}{2} f + \frac{1}{2}\bar{x} \sim_{t,\omega} \frac{1}{2}\bar{f} + \frac{1}{2} x$.*

Next, turn to the axioms pertaining to dynamic choice. I maintain the terminology in ES.

Axiom 11 (Conditional Preference—CP) *For all nodes (t,ω) and $f, f' \in F^p$: if $f_\tau(\omega') = f'_\tau(\omega')$ for all $\tau \geq t$ and $\omega' \in \mathscr{F}_t(\omega)$, then $f \sim_{t,\omega} f'$.*

Axiom 12 (Risk Preference—RP) *For every* $(x_t) \in F^{cs}$, *and* $x, x', y, y' \in X$: *if* $(x_0, \ldots, x_{\tau-2}, x, x', x_{\tau+1}, x_{\tau+2} \ldots) \succsim_{t,\omega}$ $(x_0, \ldots, x_{\tau-2}, y, y', x_{\tau+1}, x_{\tau+2} \ldots)$ *for some* (t, ω) *and* $\tau \geq t$, *then this is true for every* (t, ω) *and* $\tau \geq t$.

Axiom 13 (Impatience—IMP) *For all nodes* (t, ω), $x \in X$, *and* $f, f^*, f^{**} \in F^p$: *if* $f^* \prec_{t,\omega} f \prec_{t,\omega} f^{**}$ *and* $f^n = (f_0, \ldots, f_n, x, x, \ldots)$, *then* $f^* \prec_{t,\omega} f^n \prec_{t,\omega} f^{**}$ *for all large* n.

As usual, for $\tau > t$, say that $A \in \mathscr{F}_t$ is $\succsim_{t,\omega}$–null if $(f_{t'}(\omega'))_{t' \geq 0} = (f'_{t'}(\omega'))_{t' \geq 0}$ for all $\omega' \notin A$ implies that $f \sim_{t,\omega} f'$.

The key axiom is the standard consistency requirement, applied to decision trees based upon the filtration $\mathscr{F}$.

Axiom 14 (Dynamic Consistency—DC) *For every node* (t, ω) *and* $f, f' \in F^p$: *if* $f_\tau = f'_\tau$ *for all* $\tau \leq t$, *and* $f \succsim_{t+1,\omega'} f'$ *for all* ω', *then* $f \succsim_{t,\omega}$; *and the latter ranking is strict if the former is strict at every* ω' *in a* $\succsim_{t,\omega}$*–non-null event.*

Finally, ensure that all conditioning events "matter":

Axiom 15 (Full Support–FS) *Every* $A \in \bigcup_{t \geq 0} \mathscr{F}_t$ *is* $\succsim_0$*–non-null.*

Note that, since there is a single time-0 preference, we can write $\succsim_0$ in lieu of $\succsim_{0,\omega}$.

The main characterization result can now be stated.

Theorem 1 *Fix a preference system* $(\succsim_{t,\omega})$ *on* F^p. *The following statements are equivalent:*

(1) Axioms 1–15 hold

(2) $(\succsim_{t,\omega})$ *admits a (sharp) recursive VEU representation* $\left(u, \beta, p, (n_{t,\omega}, \zeta^{t,\omega}, A_{t,\omega})_{t \geq 0, \omega \in \Omega}\right)$.

(3) $(\succsim_{t,\omega})$ *admits a dynamic VEU representation* $\left(u, \beta, p, (\tilde{n}_{t,\omega}, \tilde{\zeta}^{t,\omega}, \tilde{A}_{t,\omega})_{t \geq 0, \omega \in \Omega}\right)$ *with time-separable adjustments.*

In (2), p *and* β *are unique, and if* $\left(\bar{u}, \beta, p, (\bar{n}_{t,\omega}, \bar{\zeta}^{t,\omega}, \bar{A}_{t,\omega})_{t \geq 0, \omega \in \Omega}\right)$ *is another recursive VEU representation of* $(\succsim_{t,\omega})$, *then* $p' = p$, $\bar{u} = \alpha u + \gamma$ *for some* $\alpha, \gamma \in \mathbb{R}$ *with* $\alpha > 0$, *and for every* (t, ω) *there is a linear surjection* $T_{t,\omega} : \mathscr{E}(\frac{\beta}{1-\beta} \bar{u} \circ \mathscr{F}_{t+1}; p(\cdot | \mathscr{F}_t(\omega)), \bar{\zeta}^{t,\omega}) \to \mathscr{E}(\frac{\beta}{1-\beta} u \circ \mathscr{F}_{t+1}; p(\cdot | \mathscr{F}_t(\omega)), \zeta)$

such that

$$\forall \bar{a} \in \frac{\beta}{1-\beta} u \circ \mathscr{F}_{t+1}, \quad T_{t,\omega}\left(\mathrm{E}_p[\bar{\zeta}^{t,\omega} \cdot \bar{a}]\right) = \frac{1}{\alpha}\mathrm{E}_p[\zeta^{t,\omega} \cdot \bar{a}] \quad \textit{and} \quad \bar{A}_{t,\omega}\left(\mathrm{E}_p[\bar{\zeta}^{t,\omega} \cdot \bar{a}]\right) = \alpha A_{t,\omega}\left(T_{t,\omega}\left(\mathrm{E}_p[\bar{\zeta}^{t,\omega} \cdot \bar{a}]\right)\right). \tag{10}$$

Finally, if the representation in (2) is sharp, then $n_{t,\omega} \leq |\mathscr{F}_{t+1}| - 1$.

For elaboration on the uniqueness statement, please refer to S09.

4 Exchangeability

In order to analyze exchangeability for recursive VEU preferences, assume that the underlying uncertainty concerns the realization of a stochastic process $(X_t)_{t\geq 0}$, where each random variable X_t takes values in the finite set $\mathscr{X}$. This is a special case of the environment considered in Sec. 2, with $\mathscr{F}_t = \sigma(X_0, \ldots, X_{t-1})$ for all $t \geq 0$; it is also, essentially, the environment considered by Epstein and Seo (2010). Recall that, for every $E_t \in \mathscr{F}_t$,

$$E_t = \{(x_0, \ldots, x_{t-1})\} \times \mathscr{X}^\infty. \tag{11}$$

We also let $\Sigma = \sigma(\bigcup_{t\geq 0} \mathscr{F}_t)$, the product sigma-algebra on $\mathscr{X}^\infty$.

Now fix $t > 0$ and $(x_0, \ldots, x_{t-1}) \in \mathscr{X}^t$. For $y, z \in X$, the notation $(y, (x_0, \ldots, x_{t-1}); z)$ indicates the plan $f \in F^p$ such that $f_\tau = z$ for $\tau \neq t$, $f_t(\omega) = y$ for $\omega \in \{(x_0, \ldots, x_{t-1})\} \times \mathscr{X}^\infty$, and $f_t(\omega) = y$ for $\omega \notin \{(x_0, \ldots, x_{t-1})\} \times \mathscr{X}^\infty$. For $y > z$, every such plan is a "bet" on the event "$X_0 = x_0, \ldots, X_{t-1} = x_t$" that "pays" y at time t if it is successful, and z otherwise; the bet also pays z at all other times. Finally, for $t > 0$, let $\mathscr{P}^t$ denote the class of permutations of $\{0, \ldots, t-1\}$.

The main axiom can now be stated. First, although this paper focuses on recursive VEU preferences, the characterization of exchangeability applies as long as $\succcurlyeq_0$ is a dynamic VEU preference, i.e. the time-0 component of a dynamic VEU representation as per Def. 5. Thus, the focus will be on a dynamic VEU preference $\succcurlyeq_0$, and the axiom will concern $\succcurlyeq_0$ alone, rather than an entire preference system.

For a (time-0) EU preference $\succcurlyeq_0^*$, the intuition behind exchangeability is simple: a bet on some time-t realization $(x_0, \ldots, x_{t-1})$ should be just as good as a bet on any permutation $(x_{\pi(0)}, \ldots, x_{\pi(t-1)})$, for $\pi \in \mathscr{P}^t$. This basic intuition underlies the exchangeabiity axiom(s) considered, for instance, in Epstein and Seo (2010). The proof of Theorem 2 shows that, indeed, such condition is equivalent to the assumption that the probability measure p^* representing the EU preference $\succcurlyeq_0^*$ is indeed exchangeable, and hence admits a de Finetti-style representation as an average of product probability measures.

While $\succcurlyeq_0$ is not an EU preference in general, the objective of this section is to indentify a condition that ensures that the *baseline prior* p in the VEU representation of $\succcurlyeq_0$ is exchangeable. The restriction just described would be much too strong; however, a suitable assumption can be formulated leveraging the notion of complementarity, which is central to VEU preferences.

The basic intuition is as follows. Recall that complementary plans (or, in S09, acts) are ranked solely via their baseline EU evaluation, as adjustment terms cancel out. Now suppose that a bet f "on" a sequence $(x_0,\dots,x_{t-1})$ is (weakly) preferred to a complementary plan $\bar{f}$, which must then be, essentially, a bet "against" the same sequence, possibly involving different prizes. Then, if $g,\bar{g}$ are complementary, g represents a bet "on" a permutation $(x_{\pi(0)},\dots,x_{\pi(t-1)})$ of the original sequence, and $\bar{g}$ represents a bet "against" that permutation, the exchangeability intuition suggests that $g \succcurlyeq_0 \bar{g}$—provided, of course, the prizes involved in the bets f,g and, respectively, $\bar{f},\bar{g}$ are the same. This is precisely what the following axiom requires.

Axiom 16 (Complementary Exchangeability) *For all $t>0$, $(x_0,\dots,x_{t-1})\in\mathscr{X}^t$, $\pi\in\mathscr{P}^t$, and $y,z\in X$ with $y\succ z$: if $(f,\bar{f})$ and $(g,\bar{g})$ are complementary pairs with $f=(y,(x_0,\dots,x_{t-1});z)$, $g=(y,(x_{\pi(0)},\dots,x_{\pi(t-1)});z)$, and $\frac{1}{2}f_\tau+\frac{1}{2}\bar{f}_\tau=\frac{1}{2}g_\tau+\frac{1}{2}\bar{g}_\tau$ for all $\tau\geq 0$, then $f\succcurlyeq_0\bar{f}$ implies $g\succcurlyeq_0\bar{g}$.*

The main result of this section can now be stated. Denote $ba_1(2^{\mathscr{X}})$ simply by $\Delta(\mathscr{X})$; for $\ell\in\Delta(\mathscr{X})$, let ℓ^∞ be the corresponding i.i.d. product measure on $\mathscr{X}^\infty$, endowed with the product sigma-algebra.

Theorem 2 *Let $\succcurlyeq_0$ be a dynamic VEU preference, with baseline prior p. Then Axiom 16 holds if and only if there exists a (unique) $\mu\in\Delta(\mathscr{X})$ such that $p=\int_{\Delta(\mathscr{X})}\ell^\infty\,d\mu(\ell)$.*

The result holds a fortiori for recursive VEU preferences. Thus, Complementary Exchangeability is compatible with full dynamic consistency; by comparison, as noted in the Introduction, Epstein and Seo (2010) are careful to point out that their notion of exchangeability can only be accommodated in a setting where a weak form of dynamic consistency holds.

Recall that Bayesian updating of the baseline prior is a key feature of recursive VEU preferences, and one that is ensured by Axiom DC. Theorem 2 then has an immediate,[2] but important Corollary:

Corollary 1 *Let* $(\succcurlyeq_{t,\omega})$ *be a recursive VEU preference system, with* $p \in ca_1(\Sigma)$ *as baseline prior; also let* μ *be as in Theorem 2. Then, for every* $t > 0$, $(x_0, \ldots, x_{t-1}) \in \mathscr{X}^t$, *and rectangle* $A = A_0 \times A_2 \times \ldots \in \Sigma$,

$$p(A|X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) = \int_{\Delta(\mathscr{X})} \prod_{\tau \geq t} \ell(A_\tau) d\mu_{(x_0,\ldots,x_{t-1})}(\ell),$$

where

$$d\mu_{(x_0,\ldots,x_{t-1})}(\ell) = \frac{\prod_{\tau=0}^{t-1} \ell(x_\tau) d\mu(\ell)}{\int_{\Delta(\mathscr{X})} \prod_{\tau=0}^{t-1} \bar{\ell}(x_\tau) d\mu(\bar{\ell})}.$$

In other words, once Dynamic Consistency and Complementary Exchangeability are combined, one concludes that (1) all conditional preferences are exchangeable, and (2) the beliefs $\mu_{(x_0,\ldots,x_{t-1})}$ are obtained from μ via Bayes' Rule. This provides a straightforward behavioral foundations for specifications of recursive VEU preferences that feature "learning about parameters."

5 Applications: Lucas (1978)

I now sketch an application of recursive VEU preferences to the celebrated dynamic stochastic general-equilibrium model developed by Lucas (1978). The objective of the discussion is to highlight the similarities with the usual EU analysis of this well-known model, and the tractability and flexibility of recursive VEU preferences.

The economy features a single, stand-in consumer, characterized by recursive VEU preferences. The stochastic environment is Markovian, with infinite horizon; states of nature can be represented as $\omega = (x_0, x_1, \ldots)$, where $x_t \in \mathscr{X} \equiv \mathbb{R}^J_+$, for some $J \geq 1$ (the interpretation will be provided momentarily). The filtration $\mathscr{F}_0, \mathscr{F}_1, \ldots$ reveals the information that, at time t and state $\omega = (x_\tau)_{\tau \geq 0}$, only the initial realizations $(x_0, x_1, \ldots, x_{t-1})$ is known; this filtration is defined in the usual way.

[2] Just note that $p(A|X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) = \frac{\int_{\Delta(\mathscr{X})} \prod_{\tau=0}^{t-1} \ell(x_\tau) \prod_{\tau \geq t} \ell(A_\tau) d\mu(\ell)}{\int_{\Delta(\mathscr{X})} \prod_{\tau=0}^{t-1} \bar{\ell}(x_\tau) d\mu(\bar{\ell})}$.

To interpret, imagine that there are J firms, denoted $j = 1,\dots,J$, producing a single consumption good. Firm j's ouput at time t is a random variable $\tilde{x}_t^j$, with realization x_t^j. The vector of random output and realized output for all firms are denoted $\tilde{x}_t$ and x_t respectively. Thus, each state encodes the sequence of output realizations for each firm in the economy.

To simplify notation, it is convenient to write expectations in the form $\mathrm{E}_x[a] = \int a(\tilde{x})\,d\pi(\tilde{x}|x)$, where I denote the representative agent's prior by π and reserve the lowercase letter p for prices. Similarly, let $I_x(a(\tilde{x})) = \mathrm{E}_x[a] + A(\mathrm{E}_x[a\cdot\zeta])$: this indicates that we assume *stationary* adjustment functions and one-period-ahead adjustment factors $\zeta = (\zeta_i)_{0\le i<n}$. Also assume for simplicity that A is concave and suitably differentiable, with i-th partial derivative $\partial A/\partial\varphi_i$.

Note that, in this Markovian formulation, adjustment factors are necessarily "about" the current-period realization of output $\tilde{x}$ (omitting time indices for simplicity), which will become known only in the next period. In other words, one can write $\zeta = \zeta(\tilde{x})$. Also observe that the analysis would not change much if adjustment factors and functions were allowed to vary with the current state x.

In each period, the agent is endowed with a fraction of shares in each firm, and is entitled to a corresponding share of their output (dividend). This constitutes the agent's sole source of income. The agent then decides how much dividend to consume, and how much to adjust his ownership shares.

Thus, in this economy, an **equilibrium** is a price function $p:\mathscr{X}\to\mathbb{R}_+$ and a value function $v:\mathbb{R}^{J+1}\to\mathbb{R}$ such that:

$$v(z,x) = \max_{z'\in[0,1]^J,\,c\ge 0} u(c) + I_x(\beta v(z',\tilde{x}))$$

$$\text{s. to}\quad c + p(x)\cdot z' \le [x + p(x)]\cdot z$$

and furthermore, for each $x\in\mathscr{X}$, $v(1_J,x)$ is attained by

$$c = \sum_j x_j \quad\text{and}\quad z' = 1_S.$$

The latter is the market-clearing condition for both consumption and asset markets. This notion follows closely Lucas's definition, except that the VEU functional I_x is employed in lieu of the standard expectation operator.

The following preliminary results can be established:

Proposition 1 *(1) If u or $\mathscr{X}$ are bounded, then for every price function $p(\cdot)$ there exists a unique value function $v(\cdot,\cdot)$, concave in z.*

(2) If $p(\cdot)$ and $v(\cdot,\cdot)$ constitute an equilibrium, then for all $j=1,\dots,J$ and $x\in\mathscr{X}$,

$$p_j(x)=\mathrm{E}_x\left[M_x(\tilde{x})\cdot[\tilde{x}_j+p_j(\tilde{x})]\right],$$

where M_x is the stochastic discount factor:

$$M_x(\tilde{x})=\beta\frac{u'\left(\sum_k\tilde{x}_k\right)}{u'\left(\sum_k x_k\right)}\left\{1+\sum_{0\le i<n}\frac{\partial A(\mathrm{E}_x[\beta v(1_J,\tilde{x})\cdot\zeta])}{\partial\varphi_i}\zeta_i\right\}.$$

This result follows by adapting Propositions 1 and 2 in Lucas (1978); the proof is omitted. A general proof of existence of equilibrium is *in progress*; for specific parameterizations of the adjustment factors and function, existence can be established directly, and of course it is not an issue if the horizon is assumed to be finite.

It is important to emphasize that, in (3), the usual pricing formula via a pricing kernel or stochastic discount factor (SDF) is obtained: informally, "price = E[*SDF* $\cdot$ *payoff*]." It must be emphasized that the SDF emerges easily from calculations, just as in the EU case; analogous pricing exercises employing ambiguity-sensitive preferences often require more complex manipulations (e.g. Ju and Miao, 2009).

Furthermore, the SDF for VEU preferences has an interesting multiplicative form: it equals the standard, EU SDF times an "ambiguity adjustment."

A Proof of Theorem 1

Sufficiency: (1) ⇒ (3) ⇒ (2). I proceed in a roughly similar fashion as ES's proof of Theorem 1. However, differences in the properties of the underlying static preferences necessitate several departures. The proof of sufficiency is divided up into a sequence of steps.

Atemporal VEU representation. Consider the state space $\hat{\Omega}\equiv\mathscr{T}\times\Omega$, where $\mathscr{T}=\{t:t\ge 0\}$, endowed with the sigma-algebra $\hat{\Sigma}$ generated by $\hat{\mathscr{F}}\equiv\bigcup_{t\ge0}\{\{t\}\times E\,:\,E\in\mathscr{F}_t\}$. Note that, since $\hat{\mathscr{F}}$ is countably infinite, $\hat{\Sigma}$ is countably generated.

It is possible, and useful, to provide further details on the sigma-algebra $\hat{\Sigma}$. I claim that $\hat{\Sigma}$ consists of all countable unions of elements of $\hat{\mathscr{F}}$ (including by convention the empty union, which is equal to $\emptyset$). Denote by $\tilde{\Sigma}$ the collection of sets just described; clearly, $\tilde{\Sigma} \subset \hat{\Sigma}$, so it is enough to show that $\tilde{\Sigma}$ is a sigma-algebra. Note first that $\hat{\mathscr{F}}$ is countably infinite, because each $\mathscr{F}_t$ is finite. Therefore, $\hat{\Omega} \in \tilde{\Sigma}$. It is clear that $\tilde{\Sigma}$ is closed under countable unions, so it remains to be shown that it is closed under complements. Fix $\hat{E} \in \tilde{\Sigma}$; then there is a (possibly empty) collection $\hat{\mathscr{E}} \subset \hat{\mathscr{F}}$ such that $\hat{E} = \bigcup \hat{\mathscr{E}}$. But then the (possibly empty) collection $\hat{\mathscr{F}} \setminus \hat{\mathscr{E}}$ is countable and its union is $\hat{\Omega} \setminus \hat{E}$. This proves the claim.

Every plan $f = (f_t) \in F^p$ maps to a bounded, $\hat{\Sigma}$–measurable function $\hat{f} : \hat{\Omega} \to X$ (i.e. a bounded act on $\hat{\Omega}$), by letting $\hat{f}(t,\omega) = f_t(\omega)$ for all (t,ω). Conversely, given a bounded $\hat{\Sigma}$–measurable $\hat{f} : \hat{\Omega} \to X$, let $f_t(\omega) = \hat{f}(t,\omega)$ for all (t,ω); then $f_t^{-1}(x) = \{\omega : f_t(\omega) = x\} = \{\omega : \hat{f}(t,\omega) = x\} = \{\omega : \{t\} \times E \subset \hat{f}^{-1}(x)\}$, which is a union of (finitely many) elements of $\mathscr{F}_t$.

The (sole) preference relation $\succcurlyeq_0 \equiv \succcurlyeq_{0,\omega}$ then induces a preference ordering $\hat{\succcurlyeq}$ over the set $\hat{F}^b$ of bounded acts on $\hat{\Omega}$. Axioms 1–5 and 10 translate directly into the corresponding axioms of S09; moreover, for $x, y \in X$, with the usual abuse of notation, $x \hat{\succcurlyeq} y$ iff $x \succcurlyeq_0 y$ iff $x \succcurlyeq^X y$. Axiom 9 instead yields a stronger form of the Complementary Independence axiom, because the notion of complementarity in Def. 6 in is weaker than the corresponding notion in S09. As in ES, the intuition is that ambiguity pertains to smoothing across states, not across time.

So, it remains to be shown that $\hat{\succcurlyeq}$ satisfies Monotone Continuity. Fix consequences $x \hat{\succ} y \hat{\succ} z$ and consider a sequence $(\hat{A}^k)_{k\geq 0} \subset \hat{\Sigma}$ such that $\hat{A}^k \supset \hat{A}^{k+1}$ and $\bigcap_{k\geq 0} \hat{A}^k = \emptyset$. For every k, define $f^k, g^k \in F^p$ so that $f_t^k(\omega) = z$ and $g_t^k(\omega) = x$ for $(t,\omega) \in \hat{A}^k$, and $f_t^k(\omega) = x$ and $g_t^k(\omega) = z$ for $(t,\omega) \notin \hat{A}^k$. Thus, f^k and g^k correspond to the acts $z\,\hat{A}^k\,x$ and $x\,\hat{A}^k\,z$ on $\hat{\Omega}$. Then the sequences (f^k) and (g^k) satisfy the assumptions of Axiom 6, so there is k such that $f^k \succ_0 y \succ_0 g^k$, hence $z\,\hat{A}^k\,x \hat{\succ} y \hat{\succ} x\,\hat{A}^k\,z$, as required by Monotone Continuity.

Hence, Theorem 1 in S09 delivers $(u, \hat{p}, \hat{\zeta}, \hat{A})$ such that $V_0(\hat{f}) = \mathrm{E}_{\hat{p}}[u \circ \hat{f}] + \hat{A}(\mathrm{E}_{\hat{p}}[\hat{\zeta} \cdot u \circ \hat{f}])$ represents $\hat{\succcurlyeq}$, and hence also $\succcurlyeq_0$. Assume throughout that this representation is sharp. It is convenient to map $\hat{\zeta} = (\hat{\zeta}_i)_{0\leq i<n}$ to a collection of plans $(\zeta_i)_{0\leq i<n}$, where $\zeta_{it}(\omega) = \hat{\zeta}_i(t,\omega)$ for all (t,ω). If $f \in F^p$ is the plan corresponding to $\hat{f} \in B(\hat{\Sigma}, u(X))$, it is useful to denote the projection map from $\mathscr{T} \times \Omega$ to Ω by π_Ω, and write

$$\mathrm{E}_{\hat{p}}[u \circ \hat{f}] = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[u \circ \hat{f} 1_{\{t\}\times\Omega}] = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[u \circ f_t \circ \pi_\Omega 1_{\{t\}\times\Omega}] \equiv \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[u \circ f_t; t]$$

and, similarly,

$$\mathrm{E}_{\hat{p}}[\hat{\zeta}_i u \circ \hat{f}] = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\hat{\zeta}_i u \circ \hat{f} 1_{\{t\}\times\Omega}] = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[(\zeta_{it} \circ \pi_\Omega)(u \circ f_t \circ \pi_\Omega) 1_{\{t\}\times\Omega}] = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_{it} u \circ f_t; t]$$

and thus $V_0(f) = \sum_{t\geq 0} \mathrm{E}_{\hat{p}}[u \circ f_t; t] + A\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t \cdot u \circ f_t; t]\right)$.

Factorizing $\hat{p}$; Conditional expectation of adjustment factors. The next step is to employ the stronger axioms adopted here (i.e. Axiom 7, Complementary Stream Monotonicity; Axiom 8, Crisp Streams; and the stronger form of Complementary Independence) to specialize the representation. First, consider a consequence stream $(x_t) \in F^{cs}$: we have, using the fact that states do not influence outcomes,

$$V_0\big((x_t)\big) = \sum_{t\geq 0} \hat{p}(\{t\} \times \Omega) u(x_t) + \hat{A}\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t] u(x_t)\right).$$

Now let $(\bar{x}_t) \in F^{cs}$ be such that $\frac{1}{2}u(x_t) + \frac{1}{2}u(\bar{x}_t) = \gamma$ for all t; since (x_t) is bounded, such sequence can always be constructed, as shown in S09. Notice that

$$\hat{A}\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t] u(\bar{x}_t)\right) = \hat{A}\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t][2\gamma - u(x_t)]\right) = \hat{A}\left(-\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t\right) = \hat{A}\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t] u(x_t)\right),$$

where the second equality follows from the fact that $\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t] = \mathrm{E}_{\hat{p}}[\hat{\zeta}] = 0$ and the third from symmetry of $\hat{A}$. Furthermore, let $x, \bar{x} \in X$ be such that $(x_t) \sim_0 x$ and $(\bar{x}_t) \sim_0 \bar{x}$ (these consequences exist by standard arguments). Since any two consequence streams are complementary in the sense of Def. 6, Axiom 9 implies that then $\frac{1}{2}(x_t) + \frac{1}{2}(\bar{x}_t) \sim_0 \frac{1}{2}x + \frac{1}{2}\bar{x}$. Then the equality of adjustment terms derived above, and the fact that $\frac{1}{2}u(x_t) + \frac{1}{2}u(\bar{x}_t) = \gamma$ for all t, imply that $\hat{A}\left(\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t; t] u(x_t)\right) = 0$. Thus,

$$V_0\big((x_t)\big) = \sum_{t\geq 0} u(x_t) \hat{p}(\{t\} \times \Omega). \tag{12}$$

Furthermore, by Axiom 8, (x_t) is crisp, and since the VEU representation chosen above is sharp, $\sum_{t\geq 0} \mathrm{E}_{\hat{p}}[\zeta_t \cdot u(x_t); t] = 0$. Finally, for every $t \geq 0$, fix $x, y \in X$ with $x \succ^X y$ and construct a stream (x_τ) such that $x_t = x$ and $x_\tau = y$ for $\tau \neq t$; then $0 = \sum_{\tau\geq 0} \mathrm{E}_{\hat{p}}[\zeta_\tau \cdot u(x_\tau); \tau] = \sum_{\tau\geq 0} \mathrm{E}_{\hat{p}}[\zeta_\tau \cdot u(y); \tau] + \mathrm{E}_{\hat{p}}[\zeta_t \cdot (u(x) - u(y)); t] = \mathrm{E}_{\hat{p}}[\zeta_t; t][u(x) - u(y)]$. Thus,

$$\zeta_0 = 0, \qquad \forall t > 0, \quad \mathrm{E}_{\hat{p}}[\zeta_t; t] = 0 \tag{13}$$

where the first equality follows from the fact that ζ_0 is measurable wrto $\mathscr{F}_0 = \{\Omega\}$.

I now claim that, for every $t \geq 0$, $\hat{p}(\{t\} \times \Omega) > \hat{p}(\{t+1\} \times \Omega) > 0$. To see this, Suppose first that $\hat{p}(\{t\} \times \Omega) = 0$ for some $t \geq 0$. By Monotonicity and Non-Degeneracy (Axioms 2 and 4), there exist $x, y \in X$ with $x \succ y$. Fix $(z_t) \in F^{cs}$: by Eq. (12) and the stated assumption, $(z_0, \ldots, z_{t-1}, x, z_{t+1}, \ldots) \sim (z_0, \ldots, z_{t-1}, y, z_{t+1}, \ldots)$; but then, Axiom 12, also $(z_0, \ldots, z_{\tau-1}, x, z_{\tau+1}, \ldots) \sim (z_0, \ldots, z_{\tau-1}, y, z_{\tau+1}, \ldots)$ for any $\tau \geq 0$. By Eq. (12), this implies that $\hat{p}(\{\tau\} \times \Omega) = 0$ for all $\tau \geq 0$: contradiction.

Similarly, suppose that $\hat{p}(\{t\} \times \Omega) \leq \hat{p}(\{t+1\} \times \Omega)$ for some $t \geq 0$. Then, with $x, y, (z_t)$ as above, Eq. (12) implies $(z_0, \ldots, z_{t-1}, x, y, z_{t+2}, \ldots) \preccurlyeq (z_0, \ldots, z_{t-1}, y, x, z_{t+2}, \ldots)$ and Axiom 12 then implies $(z_0, \ldots, z_{\tau-1}, x, y, z_{\tau+2}, \ldots) \preccurlyeq (z_0, \ldots, z_{\tau-1}, y, x, z_{\tau+2}, \ldots)$ for all $\tau \geq 0$, so $\hat{p}(\{\tau\} \times \Omega) \leq \hat{p}(\{\tau+1\} \times \Omega)$ for all $\tau \geq 0$. In particular, $0 < \hat{p}(\{0\} \times \Omega) \leq \hat{p}(\{\tau\} \times \Omega)$ for all $\tau \geq 0$, which contradicts the fact that $\sum_{\tau \geq 0} \hat{p}(\{\tau\} \times \Omega) = 1$.

Now let $\lambda_t = \hat{p}(\{t\} \times \Omega)$ for all t; also, write $\mathrm{E}_{\hat{p}}[\cdot | \{t\} \times \Omega] \equiv \mathrm{E}_{\hat{p}}[\cdot | t]$. Then $\lambda_t > \lambda_{t+1} > 0$ for all $t \geq 0$, and for any $f \in F^p$,

$$V_0(f) = \sum_{t \geq 0} \lambda_t \mathrm{E}_{\hat{p}}[u \circ f_t \circ \pi_\Omega | t] + \hat{A}\left(\sum_{t \geq 0} \lambda_t \mathrm{E}_{\hat{p}}[\zeta_t \cdot u \circ f_t \circ \pi_\Omega | t] \right).$$

From $\hat{p}$ on $\hat{\Omega}$ to p on Ω. The next step is to show that the conditional expectations with respect to $\hat{p}$ can be replaced with unconditional ones with respect to a measure p on Ω. To simplify the notation, let $\hat{p}(E|t) \equiv \hat{p}(E \times \mathscr{T} | \Omega \times \{t\})$. A preliminary result is required; just like atemporal VEU preferences represent the DM's ranking of complementary *acts* via their baseline EU evaluation, recursive VEU preferences represent the ranking of complementary *pairs* via their baseline *discounted* EU evaluation. This requires a proof, because the notion of complementary plans is weaker than is assumed in S09. The proof relies upon Eq. (13), and thus on Axiom 8 and the strengthened Axiom 9.

Claim: if $(f, \bar{f})$ are complementary plans, then

$$f \succcurlyeq_0 \bar{f} \quad \Leftrightarrow \quad \sum_{t \geq 0} \lambda_t \mathrm{E}_{\hat{p}}[u \circ f_t \circ \pi_\Omega | t] \geq \sum_{t \geq 0} \lambda_t \mathrm{E}_{\hat{p}}[u \circ \bar{f}_t \circ \pi_\Omega | t]. \tag{14}$$

To see this, suppose that $(\gamma_t)_{t \geq 0} \subset \mathbb{R}$ is such that, for every $t \geq 0$, $u \circ f_t + u \circ \bar{f}_t = \gamma_t$. Then, for every $t \geq 0$, $\lambda_t \mathrm{E}_{\hat{p}}[\zeta_t \cdot u \circ \bar{f}_t \circ \pi_\Omega | t] = \mathrm{E}_\pi[\zeta_t \cdot u \circ \bar{f}_t; t] = \mathrm{E}_\pi[\zeta_t \cdot \gamma_t; t] - \mathrm{E}_\pi[\zeta_t \cdot u \circ f_t; t] = -\mathrm{E}_\pi[\zeta_t \cdot u \circ f_t; t] =$

$-\lambda_t \mathrm{E}_\pi[\zeta_t \cdot u \circ f_t \circ \pi_\Omega | t]$, where the third equality follows from Eq. (13). Since $\hat{A}$ is symmetric around 0, it follows that $A\left(\sum_{t\geq 0} \lambda_t \mathrm{E}_{\hat{p}}[\zeta_t \cdot u \circ f_t \circ \pi_\Omega | t]\right) = \hat{A}\left(\sum_{t\geq 0} \lambda_t \mathrm{E}_{\hat{p}}[\zeta_t \cdot u \circ \bar{f}_t \circ \pi_\Omega | t]\right)$, and so $V_0(f) \geq V_0(\bar{f})$ reduces to Eq. (14).

Return to the main step. I claim that, for all $t \geq 0$ and $E \in \mathscr{F}_t$, if $\tau \geq t$ then $\hat{p}(E|\tau) = \hat{p}(E|t)$. To see this, note first that it is wlog to assume that $0 \in \operatorname{int} u(X)$. Now fix $t, E \in \mathscr{F}_t$ and find $x, y \in X$ such that $u(x), u(y) > 0$ and $u(x)\lambda_t = u(y)\lambda_{t+1}$. To see that such prizes exist, recall that $\lambda_t > \lambda_{t+1} > 0$, and pick y such that $u(y) > 0$. Then $u(y)\lambda_t > u(y)\lambda_{t+1} > 0$, so there is x with $u(x) \in (0, u(y))$ such that $u(x)\lambda_t = u(y)\lambda_{t+1}$. Finally, define $z = \frac{\lambda_t}{\lambda_t + \lambda_{t+1}} x + \frac{\lambda_{t+1}}{\lambda_t + \lambda_{t+1}} y$. Note that

$$\sum_{\tau=0}^{t-1} \lambda_\tau u(z) + \lambda_t u(x) + \lambda_{t+1} u(y) + \sum_{\tau > t+1} \lambda_\tau u(z) = \sum_{\tau=0}^{t-1} \lambda_\tau u(z) + (\lambda_t + \lambda_{t+1}) u(z) + \sum_{\tau > t+1} \lambda_\tau u(z) = u(z).$$

Since $0 \in \operatorname{int} u(X)$, it is then possible to construct plans $f, \bar{f}$ such that, for some $\alpha > 0$,

$$\forall \tau, \omega, \qquad u \circ f_\tau(\omega) = \begin{cases} \alpha[u(x) - u(z)] & \tau = t,\ \omega \in E \\ \alpha[u(y) - u(z)] & \tau = t+1,\ \omega \in E \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and } u \circ \bar{f}_\tau(\omega) = -u \circ f_\tau(\omega).$$

Clearly, $f, \bar{f}$ are complementary. Furthermore, the above calculation implies that $\left(f_\tau(\omega)\right)_{\tau\geq 0} \sim \left(\bar{f}_\tau(\omega)\right)_{\tau \geq 0}$ for all ω, because $\sum_{\tau\geq 0} \lambda_\tau u \circ f_\tau(\omega) = 0 = -\sum_{\tau \geq 0} \lambda_\tau u \circ f_\tau(\omega) = \sum_{\tau\geq 0} \lambda_\tau u \circ \bar{f}_\tau(\omega)$. Axiom 7 then implies that $f \sim_0 \bar{f}$. Therefore,

$$\lambda_t \alpha[u(x) - u(z)]\hat{p}(E|t) + \lambda_{t+1}\alpha[u(y) - u(z)]\hat{p}(E|t+1) =$$
$$= \lambda_t \alpha(-1)[u(x) - u(z)]\hat{p}(E|t) - \lambda_{t+1}\alpha(-1)[u(y) - u(z)]\hat{p}(E|t+1)$$

which simplifies to

$$\lambda_t [u(x) - u(z)]\hat{p}(E|t) + \lambda_{t+1}[u(y) - u(z)]\hat{p}(E|t+1) = 0.$$

Move $u(z)$ to the rhs and divide both sides by $\lambda_t \hat{p}(E|t) + \lambda_{t+1}\hat{p}(E|t+1)$:

$$\frac{\lambda_t \hat{p}(E|t)}{\lambda_t \hat{p}(E|t) + \lambda_{t+1}\hat{p}(E|t+1)} u(x) + \frac{\lambda_{t+1}\hat{p}(E|t+1)}{\lambda_t \hat{p}(E|t) + \lambda_{t+1}\hat{p}(E|t+1)} u(y) = u(z) = \frac{\lambda_t}{\lambda_t + \lambda_{t+1}} u(x) + \frac{\lambda_{t+1}}{\lambda_t + \lambda_{t+1}} u(y).$$

Since $x \not\sim y$ because $\lambda_t > \lambda_{t+1}$, this requires

$$\frac{\lambda_t \hat{p}(E|t)}{\lambda_t \hat{p}(E|t) + \lambda_{t+1}\hat{p}(E|t+1)} = \frac{\lambda_t}{\lambda_t + \lambda_{t+1}} \quad \Leftrightarrow \quad (\lambda_t + \lambda_{t+1})\hat{p}(E|t) = \lambda_t \hat{p}(E|t) + \lambda_{t+1}\hat{p}(E|t+1)$$

which implies that $\hat{p}(E|t)=\hat{p}(E|t+1)$, as required.

Now consider $\mathscr{S}=\{\emptyset\}\cup\bigcup_{t\geq 0}\mathscr{F}_t$. Then $\mathscr{S}$ is a semiring;[3] Define $p:\mathscr{S}\to[0,1]$ by $p(E)=\hat{p}(E|t)$ for $E\in\mathscr{F}_t$, and $p(\emptyset)=0$. It remains to be shown that p thus defined is countably additive on $\mathscr{S}$, as then the Caratheodory extension theorem will imply that p is countably additive on $\sigma(\mathscr{S})$. Thus, suppose $(E_k)_{k\geq 0}$ is a collection of pairwise disjoint elements of $\mathscr{S}$ such that $\bigcup_k E_k=E\in\mathscr{S}$ (assume $E\neq\emptyset$, or there is nothing to show). Let $\bar{E}_k=E\setminus\bigcup_{\ell=0}^k E_k$; if all such sets are non-empty, $(\bar{E}_k)$ satisfies the conditions of Eq. (4), and therefore $\bigcap_k\bar{E}_k\neq\emptyset$, contradiction. Thus, $\bar{E}_k=\emptyset$ for some k, and hence $E=\bigcup_{\ell=0}^k E_\ell$, $E_k=\emptyset$ for $\ell>k$. Let t_ℓ be such that $E_\ell\in\mathscr{F}_{t_\ell}$ for each $\ell=0,\ldots,k$, and define $t=\max_{\ell=0,\ldots,k}t_\ell$. Then $\hat{p}(E_\ell|t)=\hat{p}(E_\ell|t_\ell)=p(E_\ell)$, and furthermore, if $E\in\mathscr{F}_{\bar{t}}$, then also $\bar{t}\leq t$ and so $\hat{p}(E|t)=\hat{p}(E|\bar{t})=p(E)$. Hence $p(E)=\hat{p}(E|t)=\sum_{\ell=0}^k p(E_\ell|t)=\sum_{\ell=0}^k p(E_\ell)=\sum_{k\geq 0}p(E_k)$, as required. This completes the proof of the claim.

Finally, for every plan $f\in F^p$, $\mathrm{E}_{\hat{p}}[u\circ f_t\circ\pi_\Omega|t]=\sum_{E\in\mathscr{F}_t}\hat{p}(E|t)u\circ f_t(\omega_E)=\sum_{E\in\mathscr{F}_t}p(E)u\circ f_t(\omega_E)=\mathrm{E}_p[u\circ f_t]$, where ω_E is a generic element of E. Similarly, $\mathrm{E}_{\hat{p}}[\zeta_{it}\cdot u\circ f_t\circ\pi_\Omega|t]=\mathrm{E}_p[\zeta_t\cdot u\circ f_t]$. Thus, we can write

$$V_0(f)=\sum_{t\geq 0}\lambda_t\mathrm{E}_p[u\circ f_t]+\hat{A}\left(\sum_{t\geq 1}\lambda_t\mathrm{E}_p[\zeta_t\cdot u\circ f_t]\right)=\mathrm{E}_p\left[\sum_{t\geq 0}\lambda_t u\circ f_t\right]+\hat{A}\left(\mathrm{E}_p\left[\sum_{t\geq 1}\zeta_t\cdot\lambda_t u\circ f_t\right]\right)$$

where $\mathscr{E}_p[\zeta_{it}]=0$ for all i,t due to Eq. (13). Since $\zeta_{i0}=0$, the $t=0$ term in the adjustment can be omitted.

The argument can now be replicated for each (t,ω), leading to representations

$$V_t(f,\omega)=\mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq 0}\lambda_\tau^{t,\omega}u^{t,\omega}\circ f_t\right]+\hat{A}_{t,\omega}\left(\mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq t+1}\lambda_\tau^{t,\omega}\zeta_\tau^{t,\omega}\cdot u^{t,\omega}\circ f_t\right]\right).$$

The final part of the argument used in the proof of ES's Lemma A.1 applies here, too; see also Appendix B therein and note that, while I do not assume ES's Axiom BW, I do restrict attention to plans (and constant streams) that are bounded above and below. Conclude that, under Axioms RP, CP and IMP, for some Bernoulli utility u and discount factor $\beta\in(0,1)$,

$$U_t(f,\omega)=\mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq t}\beta^{\tau-t}u\circ f_\tau\right]+A_{t,\omega}\left(\mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq t+1}\beta^{\tau-t}\zeta_\tau^{t,\omega}\cdot u\circ f_\tau\right]\right),\qquad(15)$$

[3] By definition $\emptyset\in\mathscr{S}$. If $A,B\in\mathscr{S}$, then $A\cap B$ is either empty or it coincides with one of A or B. Finally, if $A,B\in\mathscr{S}$, then $A\setminus B$ is either empty or, if $A\in\mathscr{F}_t$ and $B\in\mathscr{F}_{t'}$ with $t'>t$, it is a union of elements of $\mathscr{F}_{t'}\subset\mathscr{S}$.

where the adjustment function has been adjusted to account for the normalization of the discount factors; this is a dynamic VEU representation (the properties of $A_{t,\omega}$ and $\zeta^{t,\omega}$ in Def. (5) correspond directly to properties of adjustment functions/factors in the atemporal VEU representation). Furthermore, $\mathrm{E}_{p^{t,\omega}}[\zeta^{t,\omega}_{i\tau}]=0$ for all i,τ, and for all complementary pairs $f,\bar{f}$,

$$f \succcurlyeq_{t,\omega} \bar{f} \quad \Leftrightarrow \quad \mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq t}\beta^{\tau-t}u\circ f_\tau\right] \geq \mathrm{E}_{p^{t,\omega}}\left[\sum_{\tau\geq t}\beta^{\tau-t}u\circ \bar{f}_\tau\right]. \tag{16}$$

Axiom DC and first recursive formulation. Now consider the implications of Axiom DC. Fix a node (t,ω) and a plan f; then, define a plan g as follows. Let $g_\tau = g_t$ for $\tau\leq t$; then, for each ω', choose $g_{t+1}(\omega')\in X$ such that $g_{t+1}(\omega')\sim_{t+1,\omega'} f$ [note that the lhs represents a constant stream equal to $g_{t+1}(\omega')$ in every state and period]; finally, for $\tau>t+1$, let $g_\tau = g_{t+1}$.

Hence, by construction (and using the representation derived above), $g\sim_{t+1,\omega'} f$ for all ω', and Axiom DC then implies that $f\sim_{t,\omega} g$. Furthermore, by Eq. (15) and the fact that $g_\tau(\omega')= g_{t+1}(\omega)$ for $\tau>t+1$ and $\omega'\in\mathscr{F}_t(\omega)$, $\sum_{\tau\geq t+1}\beta^{\tau-t-1}u\circ g_\tau(\omega') = U_{t+1}(f,\omega')$ for all $\tau\geq t+1$ and all $\omega'\in\mathscr{F}_t(\omega)$. Hence,

$$\begin{aligned} U_t(f,\omega) =& U_t(g,\omega) = \mathrm{E}_{p^{t,\omega}}\left[u\circ f_t + \beta U_{t+1}(f,\cdot)\right] + A_{t,\omega}\left(\mathrm{E}_{p^{t,\omega}}\left[\zeta^{t,\omega}_{t+1}\cdot\beta U_{t+1}(f,\cdot)\right]\right) = \\ =& u\circ f_t(\omega) + \beta\mathrm{E}_{p^{t,\omega}}\left[U_{t+1}(f,\cdot)\right] + A_{t,\omega}\left(\beta\mathrm{E}_{p^{t,\omega}}\left[\zeta^{t,\omega}_{t+1}\cdot U_{t+1}(f,\cdot)\right]\right).\end{aligned}$$

As in Def. 4, henceforth denote the size of the vector $\zeta^{t,\omega}$ by $n_{t,\omega}$. This is *almost* the recursive representation in Eq. (7), except for the appearence of the so far arbitrary measures $p^{t,\omega}$.

Bayesian updating and full support. I now show that $p^{t,\omega}=p^0(\cdot|\mathscr{F}_t(\omega))$. As above, it is wlog to assume that $0\in\operatorname{int} u(X)$; let $z\in X$ be such that $u(z)=0$. Also choose $\epsilon>0$ so that $[-\epsilon,2\epsilon]\subset u(X)$, and let $x\in X$ be such that $u(x)=\epsilon$. Now fix a node (t,ϵ) and $E\in\mathscr{F}_{t+1}(\omega)$.

Define a plan f by letting $f_\tau=z$ for $\tau\leq t+1$ or $\tau>t+2$, $f_{t+2}(\omega')=x$ for $\omega'\in E$, and $f_{t+2}(\omega')=z$ otherwise. Finally, let $\bar{f}$ be a plan complementary with f and such that $f\sim_{t+1,\omega}\bar{f}$. To see that such a plan exists, let $y\in X$ be such that $u(y)=2\epsilon$, and $w\in X$ be such that $u(w)= -\epsilon$; then the plan g' that yields x at time $t+2$ for $\omega'\in E$, and y elsewhere, is complementary with f and at least as good as f by Monotonicity; similarly, the plan g'' that yields w at time $t+2$ for $\omega'\in E$ and z elsewhere is also complementary with f and at most as good as f. Mixtures of g' and g'' are complementary to f, and one such mixture $\bar{f}$ will be conditionally indifferent to it given $(t+1,\omega)$.

By Eq. (16) we then have

$$\beta p^{t+1,\omega}(E)\epsilon = \beta\left(\gamma - p^{t+1,\omega}(E)\epsilon\right) \quad\Leftrightarrow\quad p^{t+1,\omega}(E) = \frac{1}{2}\gamma,$$

where $u(f_\tau(\omega')) + u(\bar{f}_\tau(\omega')) = \gamma$ for all (τ,ω'). Let $\bar{z} \in X$ be such that $u(\bar{z}) = \frac{1}{2}\gamma$.

Now consider complementary plans $(g,\bar{g})$ such that $g_\tau = \bar{g}_\tau = \bar{z}$ for $\tau \le t+1$ or $\tau > t+2$, $g_{t+2}(\omega') = f_{t+2}(\omega')$ and $\bar{g}_{t+2}(\omega') = \bar{f}_{t+2}(\omega')$ for $\omega' \in \mathscr{F}_{t+1}(\omega)$, and $g_{t+2}(\omega') = \bar{g}_{t+2}(\omega') = \bar{z}$ elsewhere. Then by Axiom CP we again have $g \sim_{t+1,\omega} \bar{g}$. Furthermore, by Axiom DC, since g and $\bar{g}$ agree outside $\mathscr{F}_{t+1}(\omega)$, $g \sim_{t,\omega} \bar{g}$. Again by Eq. (16),

$$\frac{1}{2}\gamma(1+\beta) + \beta\left(p^{t,\omega}(E)\epsilon + p^{t,\omega}(\mathscr{F}_{t+1}(\omega)\setminus E)\cdot 0 + [1-p^{t,\omega}(\mathscr{F}_{t+1}(\omega))]\frac{1}{2}\gamma\right) + \frac{\beta^2}{1-\beta}\frac{1}{2}\gamma =$$
$$= \frac{1}{2}\gamma(1+\beta) + \beta\left(p^{t,\omega}(E)(\gamma-\epsilon) + p^{t,\omega}(\mathscr{F}_{t+1}(\omega)\setminus E)\cdot\gamma + [1-p^{t,\omega}(\mathscr{F}_{t+1}(\omega))]\frac{1}{2}\gamma\right) + \frac{\beta^2}{1-\beta}\frac{1}{2}\gamma.$$

Canceling common terms yields

$$p^{t,\omega}(E)\epsilon = p^{t,\omega}(E)(\gamma-\epsilon) + p^{t,\omega}(\mathscr{F}_{t+1}(\omega)\setminus E)\cdot\gamma \quad\Leftrightarrow\quad p^{t,\omega}(E) = p^{t,\omega}(\mathscr{F}_{t+1}(\omega))\frac{1}{2}\gamma,$$

i.e., substituting for $\frac{1}{2}\gamma$,

$$p^{t,w}(E) = p^{t,\omega}(\mathscr{F}_{t+1}(\omega))\cdot p^{t+1,\omega}(E).$$

The result will follow once it is shown that $p^0(\mathscr{F}_t(\omega)) > 0$ for all (t,ω). But this follows from Axiom FS. Suppose that $p^0(\mathscr{F}_t(\omega)) = 0$. If the plans f, f' satisfy $f_\tau(\omega') = f'_\tau(\omega')$ for all τ and $\omega' \notin \mathscr{F}_t(\omega)$, it follows that $\mathrm{E}_{p^0}[u\circ f_t] = \mathrm{E}_{p^0}[u\circ f_t 1_{\Omega\setminus\mathscr{F}_t(\omega)}] = \mathrm{E}_{p^0}[u\circ f'_t 1_{\Omega\setminus\mathscr{F}_t(\omega)}] = \mathrm{E}_{p^0}[u\circ f'_t]$, and similarly $\mathrm{E}_{p^0}[\zeta_{t+1}\cdot u\circ f_t] = \mathrm{E}_{p^0}[\zeta_{t+1}\cdot u\circ f'_t]$, so $f \sim_0 f'$: that is, $\mathscr{F}_t(\omega)$ is $\succcurlyeq_0$-null, contradiction.

Summing up, henceforth we let $p \equiv p^0$ and obtain the recursive representation

$$U_t(f,\omega) = u\circ f_t(\omega) + \beta\mathrm{E}_p\left[U_{t+1}(f,\cdot)\middle|\mathscr{F}_t(\omega)\right] + A_{t,\omega}\left(\beta\mathrm{E}_p\left[\zeta^{t,\omega}_{t+1}\cdot U_{t+1}(f,\cdot)\middle|\mathscr{F}_t(\omega)\right]\right); \tag{17}$$

since $U_t(f,\omega)$ is bounded by monotonicity and the assumption that $f \in F^p$, the above equation shows that **(2) holds**, taking $\zeta^{t,\omega} = \zeta^{t,\omega}_{t+1}$.

Furthermore, we obtain the non-recursive representation

$$U_t(f,\omega) = \mathrm{E}_p\left[\sum_{\tau\ge t}\beta^{\tau-t}u\circ f_\tau\middle|\mathscr{F}_t(\omega)\right] + A_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\ge t+1}\beta^{\tau-t}\zeta^{t,\omega}_\tau\cdot u\circ f_\tau\middle|\mathscr{F}_t(\omega)\right]\right). \tag{18}$$

Time-separability. Now use Eq. (18) to substitute for $U_{t+1}(f,\omega')$ in Eq. (17):

$$U_t(f,\omega) = u\circ f_t(\omega) + \beta \mathrm{E}_p\left[\mathrm{E}_p\left[\sum_{\tau\geq t+1}\beta^{\tau-t-1}u\circ f_\tau \middle| \mathscr{F}_{t+1}(\cdot)\right]\middle|\mathscr{F}_t(\omega)\right]+$$
$$+\beta\mathrm{E}_p\left[A_{t+1,\cdot}\left(\mathrm{E}_p\left[\sum_{\tau\geq t+2}\beta^{\tau-t-1}\zeta_\tau^{t+1,\cdot}\cdot u\circ f_\tau\middle|\mathscr{F}_{t+1}(\cdot)\right]\right)\middle|\mathscr{F}_t(\omega)\right]+$$
$$+A_{t,\omega}\left(\beta\mathrm{E}_p\left[\zeta_{t+1}^{t,\omega}\cdot U_{t+1}(f,\cdot)\middle|\mathscr{F}_t(\omega)\right]\right).$$

Comparing the above expression with Eq. (18) yields

$$A_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\geq t+1}\beta^{\tau-t}\zeta_\tau^{t,\omega}\cdot u\circ f_\tau\middle|\mathscr{F}_t(\omega)\right]\right) = A_{t,\omega}\left(\beta\mathrm{E}_p\left[\zeta_{t+1}^{t,\omega}\cdot U_{t+1}(f,\cdot)\middle|\mathscr{F}_t(\omega)\right],\omega\right)+ \tag{19}$$
$$+\beta\mathrm{E}_p\left[A_{t+1,\cdot}\left(\mathrm{E}_p\left[\sum_{\tau\geq t+2}\beta^{\tau-t-1}\zeta_\tau^{t+1,\cdot}\cdot u\circ f_\tau\middle|\mathscr{F}_{t+1}(\cdot)\right]\right)\middle|\mathscr{F}_t(\omega)\right].$$

Therefore, **(3) holds as well**. In particular, note that $\zeta^{t,\omega}$ in Def. 4 is $\tilde{\zeta}_{t+1}^{t,\omega}$ in Def. 5, and $A_{t,\omega}$ in Def. 4 coincides with the restriction of $\tilde{A}_{t,\omega}$ in Def. 5 to $\mathscr{E}(\frac{\beta}{1-\beta}u\circ F_{t+1},p,\zeta^{t,\omega})$.

Finally, iterating Eq. (19) yields

$$A_{t,\omega}\left(\mathrm{E}_p\left[\sum_{\tau\geq t+1}\beta^{\tau-t}\zeta_\tau^{t,\omega}\cdot u\circ f_\tau\middle|\mathscr{F}_t(\omega)\right]\right) = \sum_{\tau\geq t}\beta^{\tau-t}\mathrm{E}_p\left[A_{\tau,\cdot}\left(\beta\mathrm{E}_p\left[\zeta_{\tau+1}^{\tau,\cdot}\cdot U_{\tau+1}(f,\cdot)\middle|\mathscr{F}_\tau(\cdot)\right]\right)\middle|\mathscr{F}_t(\omega)\right]. \tag{20}$$

Necessity of the axioms. It is clear that **(3)** $\Rightarrow$ **(2)**, defining $A_{t,\omega},\zeta^{t,\omega}$ as indicated after Eq. (19) above, so assume (2) and show that it implies (1).

Suppose that the preference system $(\succcurlyeq_{t,\omega})$ has a recursive VEU representation as per Def. 4. A VEU-like representation over F^p will be constructed first; then, the latter will be employed to verify that the axioms hold. [Note that, alternatively, one could construct a time-separable, dynamic VEU representation from the intermediate one obtained here, and hence show that (2) implies (3); then, showing that (3) implies (1) would be straightforward.]

Preliminaries. Since the baseline p is fixed in this argument, write $\mathrm{E}_t[a]$ in lieu of $\mathrm{E}_p[a|\mathscr{F}_t]$ and $\mathrm{E}_{t,\omega}[a]$ in lieu of $\mathrm{E}_p[a|\mathscr{F}_t(\omega)]$. Similarly, write $I_{t,\omega}(a) = \mathrm{E}_{t,\omega}[a] + A_{t,\omega}(\mathrm{E}_{t,\omega}\zeta_{t+1}^t\cdot a])$.

Since $I_{t,\omega}$ is a niveloid, it has a (minimal) niveloidal extension from $(1-\beta)^{-1}\beta u\circ F_{t+1} = B_0(\sigma(\mathscr{F}_{t+1}),(1-\beta)^{-1}\beta u(X))$ to all of $B_0(\sigma(\mathscr{F}_{t+1}))$ (Maccheroni et al., 2006a, p. 1476); denote this extension by $\hat{I}_{t,\omega}$. Also recall that a niveloid satisfies a Lipschitz condition with Lipschitz constant 1. Finally, for every $a \in B_0(\sigma(\mathscr{F}_{t+1}))$, denote by $\hat{I}_t(a)$ the $\mathscr{F}_t$-measurable map $\omega \mapsto I_{t,\omega}(a)$.

Existence of utility over F^p. Define the set $\mathscr{U}$ of sequences $\upsilon = (u_t)_{t\geq 0}$ of real-valued functions on Ω such that each u_t is $\mathscr{F}_t$–measurable, and $\|\upsilon\| \equiv \sup_{t,\omega} |u_t(\omega)| < \infty$. There is a natural mapping between $\mathscr{U}$ and the set of bounded, $\hat{\Sigma}$-measurable functions on $\hat{\Omega} = \mathscr{T} \times \Omega$, where $\hat{\Omega}$ and $\hat{\Sigma}$ are as in the proof of sufficiency. In particular, $\mathscr{U}$ is a Banach, hence complete metric space.

The following result is the key step in the proof of sufficiency; as noted in the main text, it implies that, once one-step-ahead VEU preferences have been specified, a *bounded* utility process $U_t(\cdot,\omega)$ that satisfies the recursive relation in Eq. (17) is *uniquely* pinned down. The proof is standard.

Proposition 2 *Fix $\beta \in (0,1)$, a Bernoulli utility function $u : X \to \mathbb{R}$, and an adapted collection of niveloids $(J_{t,\omega})_{t\geq 0,\omega\in\Omega}$, where $J_{t,\omega}$ is defined on $B(\sigma(\mathscr{F}_{t+1}))$ for every node (t,ω).*

Then, for every $f \in F^p$, the map $T^f : \mathscr{U} \to \mathscr{U}$ defined by letting

$$\forall \upsilon = (u_t)_{t\geq 0} \in \mathscr{U}, \qquad T^t_t(\upsilon)(\omega) = u \circ f_t(\omega) + J_{t,\omega}(\beta u_{t+1})$$

has a unique fixed point $\upsilon^* = (u^*_t)_{t\geq 0}$ in $\mathscr{U}$; if $\xi,\xi' \in \mathbb{R}$ are such that $u \circ f_t(\omega) \in [\xi,\xi']$ for all (t,ω), then $u^*_t(\omega) \in [\frac{\xi}{1-\beta}, \frac{\xi'}{1-\beta}]$ for all (t,ω). Finally, for every $\upsilon_0 \in \mathscr{U}$, $u^* = \lim_{N\to\infty}[T_f]^N(\upsilon_0)$, where $[T_f]^N$ denotes the N-fold application of T^f.

Proof: Fix $f \in F^p$. Note first that T^f indeed maps into $\mathscr{U}$, because $u \circ f_t(\omega) \in [\xi,\xi']$ for some $\xi,\xi' \in \mathbb{R}$ and $\min\ a \leq J_{t,\omega}(a) \leq \max\ a$ for all $a \in B(\sigma(\mathscr{F}_{t+1}))$. Next, for all $\upsilon = (u_t)_{t\geq 0}, \upsilon' = (u'_t)_{t\geq 0} \in \mathscr{U}$, and all nodes (t,ω),

$$|T^f_t(\upsilon)(\omega) - T^f_t(\upsilon')(\omega)| = |J_{t,\omega}(\beta u_{t+1}) - J_{t,\omega}(\beta u'_{t+1})| \leq \beta \|u_{t+1} - u'_{t+1}\|$$

because the niveloid $J_{t,\omega}$ has Lipschitz constant 1. Since this is true for all $t \geq 0$ and $\omega \in \Omega$, $\|T_f(\upsilon) - T_f(\upsilon')_t\| \leq \beta \sup_{t\geq 0} \|u_{t+1} - u'_{t+1}\| = \beta\|\upsilon - \upsilon'\|$, i.e. T^f is a contraction. Therefore, by the Contraction Mapping Theorem (e.g Aliprantis and Border, 1994, Theorem 3.48) it has a unique fixed point $u^* \in \mathscr{U}(\xi,\xi')$, that can be approached by iterating T^f.

Now, adapting an argument due to Marinacci and Montrucchio (2010, p. 24), let $\mathscr{U}(\xi,\xi') = \{\upsilon = (u_t)_{t\geq 0} \in \mathscr{U} : \forall (t,\omega),\ u_t(\omega) \in [(1-\beta)^{-1}\xi, (1-\beta)^{-1}\xi']\}$, where ξ,ξ' are as in the statement.

Then $\mathscr{U}(\xi,\xi')$ is a closed subset of $\mathscr{U}$, and hence it is complete; furthermore, if $\upsilon=(u_t)_{t\geq 0}\in\mathscr{U}(\xi,\xi')$, then $T^f_t(\upsilon)(\omega)=u\circ f_t(\omega)+J_{t,\omega}(\beta u_{t+1})\leq\xi'+\beta\frac{\xi'}{1-\beta}=\frac{\xi'}{1-\beta}$; similarly, $T^f_t(\upsilon)(\omega)\geq\frac{\xi}{1-\beta}$. In other words, T^f maps $\mathscr{U}(\xi,\xi')$ into itself. Hence, T^f has a unique fixed point $\upsilon^{**}\in\mathscr{U}(\xi,\xi')$; but since $\mathscr{U}(\xi,\xi)\subset\mathscr{U}$, $\upsilon^{**}=\upsilon^*$. ■

VEU-like representation of utility. Return to the proof of necessity. Since by assumption $\left(U_t(f,\cdot)\right)_{t\geq 0}$ is bounded and recursive for every $f\in F^p$, by Prop. 2 it is the unique such representation of $(\succcurlyeq_{t,\omega})$.

Now let $x,x'\in X$ be such that $x'\succcurlyeq f_{t'}(\omega')\succcurlyeq x$ for all (t',ω'), and let $\phi=\frac{1}{2}u(x)+\frac{1}{2}u(x')$. Define $\upsilon_0=(\phi)$ (the constant function equal to ϕ). Since $\phi\in u(X)$, $\upsilon_0\in\mathscr{U}(u(x),u(x'))$, so that $([T^f(0)]^N)\in\mathscr{U}(u(x),u(x'))$ for all N, as shown in the proof of Prop. 2. But this implies that $\hat{I}_{t,\omega}(\beta([T^f(0)]^{N-1})_{t+1})=I_{t,\omega}(\beta([T^f(0)]^{N-1})_{t+1})$, which is a VEU functional ($\hat{I}_{t,\omega}$ need not be).

For every $n\geq 0$, define the adapted processes $(a^n_t),(u^n_t)$ as follows: for every (t,ω),

$$
\begin{aligned}
a^0_t(f,\omega)&=0\\
u^0_t(f,\omega)&=u\circ f_t(\omega)+\beta\phi\\
a^n_t(f,\omega)&=E_{t,\omega}[\beta a^{n-1}_{t+1}(f,\cdot)]+A_{t,\omega}(\beta E_{t,\omega}[\zeta^{t,\omega}\cdot u^{n-1}_{t+1}(f,\cdot)])\\
u^n_t(f,\omega)&=\sum_{\tau=t}^{t+n}\mathrm{E}_{t,\omega}[\beta^{\tau-t}u\circ f_\tau]+\beta^{n+1}\phi+a^n_t(f,\omega).
\end{aligned}
$$

I now inductively rewrite u^n_t and at the same time verify that the definition of a^n_t is well-posed (in particular, $\beta\mathrm{E}_{t,\omega}[\zeta^t_{t+1}\cdot u^{n-1}_{t+1}(f,\cdot)]$ is in the domain of $A_{t,\omega}$). For $n=0$, $a^0_t(f,\omega)=0$, which is obviously well-defined; moreover, note that, for all τ,ω', $u^0_\tau(f,\omega')=u\circ f_\tau(\omega')+\beta\phi\leq u(x')+\beta\frac{u(x')}{1-\beta}=\frac{u(x')}{1-\beta}$ and similarly $u^0_\tau(f,\omega')\geq\frac{u(x')}{1-\beta}$. By assumption, $I_{t,\omega}$ is well-defined for functions with values in $\frac{\beta}{1-\beta}(u(X))$, hence in particular for $\beta u^0_{t+1}(f,\cdot)$; thus, $A_{t,\omega}(\beta\mathrm{E}_{t,\omega}[\zeta^{t,\omega}\cdot u^0_{t+1}(f,\cdot)])$ is indeed well-defined. Inductively, assume that $u^{n-1}_\tau(f,\omega')\in[\frac{\beta}{1-\beta}(x),\frac{\beta}{1-\beta}(u(x'))]$ for all τ,ω'; then $a^n_t(f,\cdot)$ is well-defined; furthermore, rewrite $u^n_t(f,\omega)$ as follows, breaking up the summation and substituting for $a^n_t(f,\cdot)$:

$$\begin{aligned} u^n_t(f,\omega) =& u\circ f_t(\omega) + \beta \sum_{\tau=t+1}^{(t+1)+(n-1)} \mathrm{E}_{t,\omega}[\beta^{\tau-t-1} u\circ f_\tau] + \beta^n \phi + E_{t,\omega}[\beta a^{n-1}_{t+1}(f,\cdot)] + A_{t,\omega}(\beta E_{t,\omega}[\zeta^{t,\omega}\cdot u^{n-1}_{t+1}(f,\cdot)]) = \\ =& u\circ f_t(\omega) + \mathrm{E}_{t,\omega}[\beta u^{n-1}_{t+1}(f,\cdot)] + A_{t,\omega}(\mathrm{E}_{t,\omega}[\zeta^{t,\omega}\cdot \beta u^{n-1}_{t+1}(f,\cdot)]) = \\ =& u\circ f_t(\omega) + I_{t,\omega}[\beta u^{n-1}_{t+1}(f)]; \end{aligned} \tag{21}$$

It then follows that $u^n_t(f,\omega) \geq u(x) + \beta\frac{u(x)}{1-\beta} = \frac{u(x)}{1-\beta}$ and similarly $u^n_t(f,\omega) \leq \frac{u(x')}{1-\beta}$, by monotonicity of $I_{t,\omega}$ and the induction hypothesis. This completes the induction step.

Eq. (21) also shows that, letting $v^n = \left(u^n_t(f,\cdot)\right)_{t\geq 0}$,

$$u^n_t(f,\cdot) = T^f(v^{n-1}) = \ldots = [T^f(u_0)]^n,$$

and therefore $u^n_t(f,\cdot) \to U_t(f,\cdot)$. Since $\sum_{\tau=t}^{t+n} \mathrm{E}_{t,\omega}[\beta^{\tau-t} u\circ f_\tau] + \beta^{n+1}\phi \to \sum_{\tau\geq t} \mathrm{E}_{t,\omega}[\beta^{\tau-t} u\circ f_\tau]$, $a^n_t(f,\cdot)$ also has a well-defined limit, denoted $a_t(f,\cdot)$, and one can write

$$U_t(f,\cdot) = \sum_{\tau\geq t} \mathrm{E}_{t,\omega}[\beta^{\tau-t} u\circ f_\tau] + a_t(f,\cdot). \tag{22}$$

Finally, observe that, since $\mathrm{E}_{t,\omega}[\zeta^{t,\omega}\cdot\beta\phi] = 0$,

$$\begin{aligned} a^n_t(f,\omega) =& E_{t,\omega}[\beta a^{n-1}_{t+1}(f,\cdot)] + A_{t,\omega}\left(\beta E_{t,\omega}\left[\zeta^{t,\omega}\cdot\left(\sum_{\tau=t+1}^{(t+1)+(n-1)} \mathrm{E}_{t+1}[\beta^{\tau-(t+1)} u\circ f_\tau] + a^{n-1}_{t+1}(f,\cdot)\right)\right]\right) = \\ =& E_{t,\omega}[\beta a^{n-1}_{t+1}(f,\cdot)] + A_{t,\omega}\left(E_{t,\omega}\left[\sum_{\tau=t+1}^{t+n} \zeta^{t,\omega}\cdot\beta^{\tau-t} u\circ f_\tau\right] + \mathrm{E}_{t,\omega}\left[\zeta^{t,\omega}\cdot\beta a^{n-1}_{t+1}(f,\cdot)\right]\right). \end{aligned} \tag{23}$$

Verification of the axioms. I now employ Eqs. (22) and (23) to show that the axioms hold. For the basic preference axioms, focus on $\succcurlyeq_0$ (the argument for other conditional preferences is analogous). Note first that, if $f,g\in F^p$ are such that $u\circ f - u\circ g$ is a deterministic process $(\gamma_t)_{t\geq 0}$, then inductively, using Eq. (23), $a^n_t(f,\omega) = a^n_t(g,\omega)$; hence, the same is true in the limit. Similarly, if $f_t(\omega)\geq g_t(\omega)$ for all (t,ω), then inductively, using Eq. (21), $u^n_0(f,\omega)\geq u^n_0(g,\omega)$, and so again $U_0(f,\omega)\geq U_0(g,\omega)$. These observations imply that, if $\hat\Omega = \mathscr{T}\times\Omega$ and $\hat F^b$ is the set of bounded acts on $\hat\Omega$ corresponding to adapted sequences in F^p, the functional $\hat I : u\circ\hat F^b \to \mathbb{R}$ defined by $\hat I(u\circ\hat f) = (1-\beta)U_0(f,\omega)$ (any ω), is a normalized niveloid [if $x\in X$, then $U_0(x,\omega) = \frac{u(x)}{1-\beta}$, so the adjustment factor $(1-\beta)$ is required]. Hence, the corresponding preference $\hat\succcurlyeq_0$ on

$\hat{F}^b$ satisfies the atemporal Weak Order, Monotonicity, Continuity, Non-Degeneracy and weak constant Independence axioms in Maccheroni et al. (2006a), which implies that $\succsim_0$ satisfies Axioms 1–5.

By induction using Eq. (23), $a_0^n((x_t)_{t\geq 0},\omega)=0$ for all deterministic consumption streams $(x_t)_{t\geq 0}$, so the same is true in the limit. This implies Axioms 8.

Again, by induction, if $(f,\bar{f})$ are complementary then inductively, using Eq. (23), $a_0^n(f,\omega)= a_0^n(\bar{f},\omega)$, and so $a_0(f,\omega)=a_0(\bar{f},\omega)$; this implies that Axioms 9 and 10 hold. Axiom 7 also holds, because EU preferences satisfy this axiom and complementary plans are evaluated using their baseline EU evaluation.

Because $\succsim_0$ satisfies Axioms 9 and 10, the corresponding preference $\hat{\succsim}_0$ satisfies Complememtary Independence and Complementary Translation Invariance, so Lemma 5 and the argument on p. 842 of S09 imply that $\hat{\succsim}_0$ satisfies Monotone Continuity, so that $\succsim_0$ satisfies Axiom 6 (see the proof of sufficiency for details on how to map between the two domains).

Axioms 11, 12, 15 and 14 are immediate; for the first two, again recall that $a_0((x_t)_{t\geq 0},\omega)=0$ for all consumption streams $(x_t)_{t\geq 0}$. For 13 (IMP), observe that, if f^*,f,f^n,f^{**},x are as in the Axiom and ϕ is as in the above construction of the process $u_t(f,\cdot)$, we have $a_t^m(f^n,\omega)=0= a_t^0(f,\omega)$ for all ω, $t\geq n$ and $m\geq 0$ by induction on m and t; therefore, $a_0^{m-n}(f^n,\omega)=a_0^n(f,\omega)$ for $m\geq n$, so $a_0(f^n,\omega)=a_0^n(f,\omega)$. Therefore,

$$
\begin{aligned}
u_0(f^n,\omega) &= \sum_{t=0}^{n}\mathrm{E}_0[\beta^t u\circ f_t]+\beta^{n+1}\frac{u(x)}{1-\beta}+a_0(f^n,\omega)=\\
&= \sum_{t=0}^{n}\mathrm{E}_0[\beta^t u\circ f_t]+\beta^{n+1}\phi+\beta^{n+1}\left(\frac{u(x)}{1-\beta}-\phi\right)+a_0^n(f,\omega)=\\
&= u_0^n(f,\omega)+\beta^{n+1}\left(\frac{\beta u(x)}{1-\beta}-\phi\right)\to u_0(f,\omega),
\end{aligned}
$$

which implies that IMP holds.

Sharp representation and uniqueness. Any recursive VEU representation $\big(u,\beta,p,(n_{t,\omega},\zeta^{t,\omega},A_{t,\omega})_{t,\omega}\big)$ yields a VEU representation $\big(\frac{\beta}{1-\beta}u,p(\cdot|\mathscr{F}_t(\omega)),n_{t,\omega},\zeta^{t,\omega},A_{t,\omega}\big)$ on F_{t+1} at each (t,ω); if this representation is not sharp, Theorem 1 in S09 shows that a sharp representation $\big(\frac{\beta}{1-\beta}\bar{u},p(\cdot|\mathscr{F}_t(\omega)),\bar{n}_{t,\omega},\bar{\zeta}^{t,\omega},\bar{A}_{t,\omega}\big)$ can be constructed. In particular, since preferences over X must be preserved, one may as well assume that $\bar{u}=u$, so that $\bar{A}_{t,\omega}(\mathrm{E}_{t,\omega}[\bar{\zeta}^{t,\omega}\cdot a])=A_{t,\omega}(\mathrm{E}_{t,\omega}[\zeta^{t,\omega}\cdot a])$ for all $a\in\frac{\beta}{1-\beta}u\circ F_{t+1}$. But then, one can indifferently construct the sequences u_t^n,a_t^n as in the proof of necessity in terms of $\bar{A}_{t,\omega}$

and $\bar{\zeta}^{t,\omega}$, or in terms of the original $A_{t,\omega}$ and $\zeta^{t,\omega}$: hence, the utility indices $\bar{U}_t$ generated using the sharp VEU representations at each node (t,ω) coincide with the original ones, U_t. Hence, $\left(u,\beta,p,(\bar{n}_{t,\omega},\bar{\zeta}^{t,\omega},\bar{A}_{t,\omega})_{t,\omega}\right)$ is a sharp recursive VEU representation of the same preferences. Also, since acts $f_{t+1}\in F_{t+1}$ can be seen as maps $\hat{f}_{t+1}:\mathscr{F}_{t+1}\to X$ on a finite "state space" $\mathscr{F}_{t+1}$, Theorem 1 in S09 implies that $\bar{n}_{t,\omega}\le|\mathscr{F}_{t+1}|-1$.

The uniqueness claim follows directly from S09, again after noticing that every recursive VEU representation induces a VEU representation on F_{t+1} at each (t,ω).

B Proof of Theorem 2

Sufficiency. Note first that, if $f,\bar{f}$ and $g,\bar{g}$ are as in the statement of the axiom, then one can exchange the role of $(f,\bar{f})$ and $(g,\bar{g})$, using π^{-1} as the permutation of $\{0,\dots,t-1\}$, to conclude that $f\succcurlyeq_0\bar{f}$ if and only if $g\succcurlyeq_0\bar{g}$.

As shown in the proof of Theorem 1, for any complementary pair $(h,\bar{h})$, $h\succcurlyeq_0\bar{h}$ iff $\mathrm{E}_p[\sum_{t\ge0}\beta^t u\circ h_t]\ge\mathrm{E}_p[\sum_{t\ge0}\beta^t u\circ\bar{h}_t]$. Assume wlog that $0\in\operatorname{int}u(X)$, and let $z\in u^{-1}(0)$; also fix $\epsilon>0$ such that $[-\epsilon,2\epsilon]\subset u(X)$ and let $y\in u^{-1}(\epsilon)$.

Now construct $(f,\bar{f})$ and $(g,\bar{g})$ as follows: f,g are as in Axiom 16; $\bar{f}_\tau=\bar{g}_\tau=z$ for $\tau\neq t$; and $\frac{1}{2}u\circ f_t+\frac{1}{2}u\circ\bar{f}_t=\frac{1}{2}u\circ g_t+\frac{1}{2}u\circ\bar{g}_t\equiv\gamma_t$, a parameter that shall be determined momentarily. Then $f\succcurlyeq_0\bar{f}$ iff $\mathrm{E}_p[u\circ f_t]\ge2\gamma_t-\mathrm{E}_p[u\circ f_t]$, i.e. iff $\mathrm{E}_p[u\circ f_t]\ge\gamma_t$; similarly, $g\succcurlyeq_0\bar{g}$ iff $\mathrm{E}_p[u\circ g_t]\ge\gamma_t$.

Now let $\gamma_t=\mathrm{E}_p[u\circ f_t]$; I claim that this is possible, i.e. that $2\gamma_t-u\circ f_t(\omega)\in u(X)$ for all ω. To see this, note that $\mathrm{E}_p[u\circ f_t]\in[0,\epsilon]$, so $\gamma_t\in[0,2\epsilon]$; hence, $2\gamma_t-u\circ f_t\in[-\epsilon,2\epsilon]\subset u(X)$ by the choice of $\epsilon>0$.

This completes the definition of $f,\bar{f},g,\bar{g}$; by the above arguments, $f\sim_0\bar{f}$, and so Axiom 16 implies that $g\sim_0\bar{g}$ as well. But then

$$p(X_0=x_0,\dots,X_{t-1}=x_{t-1})=\frac{1}{\epsilon}\mathrm{E}_p[u\circ f_t]=\frac{1}{\epsilon}\mathrm{E}_p[u\circ g_t]=p(X_0=x_{\pi(0)},\dots,X_{t-1}=x_{\pi(t-1)}).\quad(24)$$

Since this holds for all $t>0$ and $\pi\in\mathscr{P}^t$, this implies that p is exchangeable, as e.g. per the definition of Hewitt and Savage (1955) (who actually use the term "symmetric"); the result then follows. For completeness, I provide the details.

Now fix a finite permutation of $\mathscr{T}=\{t:t\ge0\}$, i.e. a map $\pi^\infty:\mathscr{T}\to\mathscr{T}$ that is one-to-one and such that $\pi^\infty(t) \neq t$ for finitely many $t \in \mathcal{T}$. Then there is $T \geq 0$ such that $t > T$ implies $\pi^\infty(t) = t$; furthermore, for all $t \leq T$, $\pi^\infty(t) \leq T$: otherwise, if $t' = \pi^\infty(t) > T$ for some $t \leq T$, we would have $\pi^\infty(t') = t' = \pi^\infty(t)$ and $t' > t$, a contradiction. Hence, for every such π^∞ there is $T \geq 0$ and $\pi \in \mathcal{P}^T$ such that $\pi^\infty(t) = \pi(t)$ for $t \leq T$, and $\pi^\infty(t) = t$ for $t > T$.

Let $\mathcal{C} = \{(\prod_{\tau=0}^t C_\tau) \times \mathcal{X}^\infty : t \geq 0, C_\tau \subset \mathcal{X}\, \forall \tau\}$, the class of "cylinders." Note that $\mathcal{C}$ generates Σ (it contains all sets E_t as in Eq. (11), and every $A \in \mathcal{C}$ is a finite union of such sets E_t), and

$$p\left(\left(\prod_{\tau=0}^t C_\tau\right) \times \mathcal{X}^\infty\right) = p(X_0 \in C_0, \ldots, X_t \in C_t) = \sum_{(x_0,\ldots,x_t)\in\mathcal{X}^{t+1} : x_\tau \in C_\tau \forall \tau} p(X_0 = x_0, \ldots, X_t = x_t)$$

For $A \in \Sigma$, let $\pi^\infty[A] = \{(x_t)_{t\geq 0} : (x_{\pi^\infty(t)})_{t \geq 0} \in A\}$. In particular, for $t \geq T$,

$$\pi^\infty\left[\left(\prod_{\tau=0}^t C_\tau\right) \times \mathcal{X}^\infty\right] = \left\{(x_\tau)_{\tau\geq 0} : (x_{\pi(0)}, \ldots, x_{\pi(T)}, x_{T+1}, \ldots, x_t) \in \prod_{\tau=0}^t C_\tau\right\}$$

and therefore

$$\begin{aligned} p\left(\pi^\infty\left[\left(\prod_{\tau=0}^t C_\tau\right) \times \mathcal{X}^\infty\right]\right) &= \sum_{\substack{(x_0,\ldots,x_T,\ldots,x_t)\in\mathcal{X}^{t+1} : \\ x_{\pi(\tau)} \in C_\tau \forall 0\leq\tau\leq T,\, x_\tau\in C_\tau \forall T+1\leq\tau\leq t}} p(X_0 = x_0, \ldots, X_t = x_t) = \\ &= \sum_{\substack{(x_0,\ldots,x_T,\ldots,x_t)\in\mathcal{X}^{t+1} : \\ x_{\pi(\tau)} \in C_\tau \forall 0\leq\tau\leq T,\, x_\tau\in C_\tau \forall T+1\leq\tau\leq t}} p(X_0 = x_{\pi(0)}, \ldots, X_T = x_{\pi(T)}, \ldots, X_t = x_t) = \\ &= \sum_{(x_0,\ldots,x_t)\in\mathcal{X}^{t+1} : x_\tau \in C_\tau \forall \tau} p(X_0 = x_0, \ldots, X_t = x_t) = \\ &= p\left(\left(\prod_{\tau=0}^t C_\tau\right) \times \mathcal{X}^\infty\right). \end{aligned}$$

The second equality follows by considering $\pi' \in \mathcal{P}^t$ with $\pi'(\tau) = \pi(\tau) = \pi^\infty(\tau)$ for $\tau \leq T$, and $\pi'(\tau) = \tau = \pi^\infty(\tau)$ for $\tau > T$, and applying Eq. (24). Finally, the third equality follows by simply relabeling $x_{\pi(\tau)}$ as x_τ for $0 \leq \tau \leq T$, as we have both $X_\tau = x_{\pi(\tau)}$ and $x_{\pi(\tau)} \in C_\tau$.

Hence, $p(\pi^\infty A) = p(A)$ for all $A \in \mathcal{C}$ with $T \geq t$ non-trivial components; since some C_τ's can be equal to $\mathcal{X}$, this is true for all $A \in \mathcal{C}$. Then, by standard arguments (e.g. Aliprantis and Border, 2007, Theorem 10.10), this is true for all $A \in \sigma(\mathcal{C}) = \Sigma$.

Necessity. Consider $t > 0$, $(x_0, \ldots, x_{t-1})$, $y, z, f, \bar{f}, g, \bar{g}$ as in the Axiom. Letting $\frac{1}{2} u \circ f_\tau + \frac{1}{2} u \circ$

$\bar{f}_\tau = \gamma_\tau$ for all τ, we have $f \succsim_0 \bar{f}$ iff

$$\sum_{\tau \neq t} \beta^\tau u(z) + \beta^t \{u(z) + [u(y) - u(z)] p(X_0 = x_0, \ldots, X_{t-1} = x_{t-1})\} \geq$$
$$\geq \sum_{\tau \neq t} \beta^\tau [2\gamma_\tau - u(z)] + \beta^t \{2\gamma_t - u(z) - [u(y) - u(z)] p(X_0 = x_0, \ldots, X_{t-1} = x_{t-1})\}$$

and therefore iff

$$\beta^t [u(y) - u(z)] p(X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) \geq \sum_{\tau \geq 0} \beta^\tau [\gamma_\tau - u(z)].$$

Similarly, $g \succsim_0 \bar{g}$ iff

$$\beta^t [u(y) - u(z)] p(X_0 = x_{\pi(0)}, \ldots, X_{t-1} = x_{\pi(t-1)}) \geq \sum_{\tau \geq 0} \beta^\tau [\gamma_\tau - u(z)].$$

If $p(X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) = \int_{\Delta(\mathscr{X})} \prod_{\tau=0}^{t-1} \ell(x_\tau)\, d\mu(\ell)$, then clearly Eq. (24) holds, so that $f \succsim_0 \bar{f}$ iff $g \succsim_0 \bar{g}$, as required. ■

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