

# Non-Equilibrium Steady States for gradient systems with ports

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In gradient flows, the steady states are given by the critical points of the driving functional. Hence, the associated fluxes vanish.

If a gradient system is coupled to the environment via so-called ports, then steady states may have non-zero fluxes and are called Non-Equilibrium Steady States (NESS). We consider so-called port gradient systems  $(X, \mathcal{E}, \mathcal{R}, \mathbf{P})$ , which generate the port gradient-flow equations

$$\dot{u}(t) = \partial_{\xi} \mathcal{R}^*(u(t), -D\mathcal{E}(u(t))) - \mathbf{P}y(t), \quad -\mathbf{P}^*D\mathcal{E}(u(t)) = \eta \in Y^*.$$

Here the linear port mapping  $\mathbf{P} : Y \rightarrow X$  introduces time-dependent fluxes  $\mathbf{P}y(t)$  into the equation, while  $\eta \in Y^*$  is a constant constraint.

NESS for this equation can be characterized as null-saddle points of the the B-function

$$\mathfrak{B}_{\mathcal{E}, \mathcal{R}}(u, \xi) = \mathcal{R}^*(u, -D\mathcal{E}(u)) - \mathcal{R}^*(u, \xi),$$

where  $\mathfrak{B}$  is maximized with respect to  $\xi$  subject to  $\mathbf{P}^*\xi = \eta$  and minimized with respect to  $u$  subject to  $-\mathbf{P}^*D\mathcal{E}(u) = \eta$ .

We discuss a few examples of this characterization and show how it appears naturally when performing the EDP-convergence of slow-fast gradient systems, where the fast part stays in NESS connected via ports to the slow system.

Prigogine's dissipation principle of 1947 states that under suitable conditions, NESS are minimizers of the dissipation. We will discuss how the saddle-point formulation provides a mathematically rigorous generalization of this principle.

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