Bargaining with Rational Inattention

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Abstract

I study a one-sided offers bargaining game in which a fully rational seller is making repeated offers to a rationally inattentive buyer (Sims, 1998). The quality of the good is random and is known to the seller. The buyer needs to pay attention to both the quality of the good and the seller's offers. I show that the buyer attains half of the uncertain portion of the surplus as attention costs become negligible and offers are frequent. With infrequent offers and positive attention costs an equilibrium exists both in the finite and the infinite horizon games. This equilibrium involves the buyer paying more for, but also obtaining a higher surplus from, higher quality goods. Trade occurs with delay that is decreasing with the quality of the good and persists even when offers are frequent. Finally, I show that revealing the quality of the good to the buyer reduces both the buyer's surplus and overall efficiency.

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1 Introduction

Consider a seller who is making repeated offers to a buyer in an attempt to sell an indivisible good. It is well known that if the buyer perfectly observes the good's price and quality the unique equilibrium involves immediate agreement with the seller obtaining all of the gains from trade. This stark outcome stands in contrast with our day to day experience. In most transactions, the buyer is much more likely to be satisfied or disappointed with her purchase than indifferent about it. Similarly, merchants rarely manage to instantly sell their merchandise to every potential buyer that comes their way. This gap between the full information model and reality has been the subject of much of the bargaining literature.

My main contribution is to incorporate limited attention into non-cooperative bargaining. Many studies show that the way people allocate their attention can have a substantial economic impact. For instance, a study conducted by Chetty et al. (2009) found that shoppers often overpay for products because they fail to pay attention to sales tax. In the used cars market, Lacetera et al. (2012) showed that buyers only pay attention to the left-most digit of the odometer. The goal of my paper is to evaluate the effects of partial inattention on the outcomes one-on-one buyer-seller interactions.

I do so by substituting the fully informed buyer with one who is *rationally inattentive* (Sims, 1998). Such a buyer needs to pay attention to information in order to take advantage of it. Paying attention to more information results in better decision making, but also involves more effort, which is costly. Being rational, the buyer achieves the optimal balance, paying attention only to those pieces of information that are worth the effort.

The rational inattention model serves as a natural way of extending utility maximization to include costly attention. The flexibility of the model, its focus on information, and its emphasis on optimality make rational inattention especially suitable for this purpose. Indeed, rational inattention abstracts from the specific process behind attention just as utility maximization abstracts away from the agent's optimization procedure. Thus, we use the rational inattention model as a device for studying how incentives shape the bargaining outcomes of a partially inattentive agent.

In my model, the rationally inattentive buyer bargains with a fully rational seller over a good of random quality. The quality of the good, v, is observable to the seller and is drawn once and for all at the beginning of game. Each period, the seller makes an offer. The buyer then chooses which discrete signal structure to use; that is, how much attention to pay to v, the seller's past offers and the current proposal. The signal structure, the buyer's prior, and the seller's possibly random offer together determine the buyer's attention cost. Once the buyer chooses her signal structure for the period, nature draws a signal conditional on the seller's current proposal, past offers, and the good's quality. Upon observing the signal, the buyer updates her prior and chooses whether to accept or reject. If she accepts, trade occurs and the game ends. Otherwise, the game proceeds to the next period.

I focus my analysis on equilibria that satisfy three conditions. The first condition is that the buyer is *attentive*. That is, there are no periods in which the buyer automatically rejects every offer regardless of the history. The second condition disciplines the buyer's response to off-equilibrium offers. Specifically, for each off-equilibrium offer, I require the buyer's strategy to be a limit of best responses to some sequence of perturbations that put positive probability on that offer. This condition is in the spirit of the perfect equilibrium of Selten (1975), and is needed to avoid non-credible attention threats. I motivate and explain these conditions in section 3. Theorem 1 shows that the set of equilibria that satisfies these conditions in the finite horizon version of the game is non-empty. In the infinite horizon version, I focus on equilibria to that also arise as limits of finite horizon equilibria. I prove that such a limit exists, and is indeed an equilibrium of the infinite horizon game in Theorem 2.

My first major result establishes that in an environment with frequent offers, the buyer obtains a significant portion of the surplus even when attention costs are negligible. More precisely, let v be the realized quality of the good, and take v_l to be the lowest possible quality the good can attain. When offers are frequent, Theorem 4 in section 6 establishes that as the cost of attention go to zero, trade is efficient and the buyer's expected surplus, $E[U_b]$, becomes

$$E[U_b] = \frac{1}{2}(E[v] - v_l)$$

Theorem 4 is based on intuitive properties of the rational inattention cost function. For one, the function is convex in the buyer's signals. As such, the buyer's optimal signal structure equates each signal's marginal cost to its marginal benefit. A signal's marginal benefit is the buyer's expected gains from trade conditional on observing the signal. A signal's marginal costs depend on its informativeness. A kind of signal that is considered extremely informative is one that leads to a belief that some events have zero probability. Because no amount of Bayesian updating can make a zero belief positive, such signals have infinite marginal costs. The possibility of infinite marginal costs allows inattention to remain a factor at the margin even as total attention costs become negligible. When this happens the buyer's marginal benefit from signals, and therefore her expected surplus, stay positive.

My second major result is that with significant attention costs there is delay in agreement even when offers are frequent (Proposition 3). Delay emerges due to the seller's equilibrium optimization problem. In the one-shot game, this problem is similar to that of a monopolist facing a logit demand function (see Matějka and McKay, 2012). As such, the seller's offer will be the one that equates his marginal revenue to his marginal cost. The seller's dynamic problem has a similar structure, but with the seller's marginal cost being the opportunity cost of forgoing next period's profits. It turns out that these future profits in equilibrium do not depend on the seller's current offer. Because of this, future profits enter the seller's current objective function as a fixed cost. When offers are frequent, future profits loom larger, motivating the seller to increase his offer. However, in equilibrium, offers cannot be too high, or else the buyer will not pay attention. Therefore, the seller's marginal revenue must decrease. The buyer's costly attention, though, limits the change in the buyer's demand with respect to higher offers. As a result, the only way to lower the seller's marginal revenue is by decreasing the level of demand; that is, the probability of agreement. The result is delay that persists even when offers are made arbitrarily frequently.

In the addition to the above two results, I show that in equilibrium a rationally inattentive buyer gets what she pays for (Proposition 2). More precisely, both the buyer's surplus and the price of the good are increasing with the good's quality. These features are accompanied by the buyer overpaying for the good when the quality is low, and underpaying when the quality is high. The intuition behind these features comes from the buyer obtaining imperfect information about the good's quality. If the buyer had no information, the seller's price would have been equal to the good's expected value for the buyer. As such, the buyer will overpay for low quality goods, underpay for high quality goods, the buyer's surplus would increase with the good's quality. With full information, the price of the good always equals to v, making prices increasing with quality. However, with full information the buyer always obtains a surplus of zero. The equilibrium with a rationally inattentive buyer lies between the full and the no information extremes.

I conclude by exploring the effects of uncertain quality on bargaining outcomes. I show that revealing the quality of the good to the buyer results in a unique equilibrium (Proposition 4). This equilibrium preserves the delay that arises in the baseline model, but leads to effortless attention on the equilibrium path and to the buyer getting zero surplus. The reason the buyer's attention is effortless is because the seller uses a deterministic strategy. Since in equilibrium the buyer knows both the seller's strategy and v, the buyer's knowledge includes all there is to know about the seller's offers. As such, the buyer's signals carry no information in equilibrium, resulting in zero attention costs. However, the fact that attention is effortless on the equilibrium path does not mean that the buyer perfectly observes the seller's offers. In particular, my equilibrium refinement requires the buyer to take into account the marginal cost of noticing *any* offer, including zero probability ones. As such, the buyer only partially adjusts the probability of agreement in reaction to zero probability offers.

Proposition 5 asserts that revealing the quality of the product to the buyer reduces both overall efficiency and the buyer's surplus. From the buyer's perspective, being ignorant of v results in a variation in the value of the seller's offers, which generates positive attention costs. Since attention costs are strictly convex and the buyer is attentive, she earns a strictly positive surplus. As for overall efficiency, keeping the buyer ignorant of v reduces delay, but creates positive attention costs. Still, Proposition 5 shows that the reduction in delay more than compensates for the increase in the cost of attention. Hence, my analysis suggests that more information can be harmful in the presence of costly attention.

Related Literature

The current paper sits in the intersection of rational inattention and bargaining. The rational inattention literature finds its origins in Sims (1998). A large portion of this literature is based on the linear-quadratic framework (e.g. Sims (2003), Mackowiak and Wiederholt (2009), Van Nieuwerburgh and Veldkamp (2010) and Dessein et al. (2013)). In these models, the rational inattentive agent optimizes over a continuous variable, has a quadratic objective function and the exogenous uncertainty is normally distributed. Under these assumptions, it is optimal for the agent to use a normally distributed signal structure. My model differs from this literature in that the buyer chooses a discrete action. Moreover, my buyer needs to pay attention to the seller's offers which are determined in equilibrium and therefore are not normally distributed.

A strand of the rational inattention literature that is more relevant to my analysis is one that deals with agents whose action is discrete. One example is Woodford (2009) who studies a rationally inattentive firm that chooses when to review its current pricing strategy. Using a similar framework, Yang (2014) studies coordination games with rationally inattentive players in a global games setup. Dasgupta and Mondria (2014) apply the discrete action framework to analyze the decisions of importers. In the context of individual behavior, the studies of Caplin and Dean (2013), Matějka and McKay (2013), Oliveira et al. (2013) and Woodford (2014) analyze the observable implications of rational inattention on choice among discrete alternatives.

As explained earlier, my main contribution is to consider a seller who is making *repeated offers* to a rationally inattentive buyer. Yang (2013), Matějka and McKay (2012) and Martin (2012) also consider one or more rational sellers making offers to one or more rationally inattentive buyers. However, unlike my model, the aforementioned papers study models in which the seller is making a single, take-it or leave-it, offer. Thus, theirs are static models.

A dynamic model is essential for my analysis. Without repeated offers, it would be impossible to study inefficiencies that arise due to bargaining frictions, such as delay. Moreover, the possibility of repeated offers is crucial for the buyer to obtain a positive surplus when attention costs are negligible. To put it differently, one can show that in the one period version of my model all of the surplus goes to the seller as the cost of attention goes to zero. Hence, my model suggests that dynamics play an important role in understanding the effect of rational inattention on bargaining.

In addition to being dynamic, my model differs from the models of Yang (2013), Matějka and McKay (2012) and Martin (2012) in other respects. While I focus on bargaining, Matějka and McKay (2012) focus on competition between multiple sellers. In their model, each seller attempts to sell their good to a rationally inattentive buyer with a unit demand by making simultaneous take-it or leave-it offers. The quality of each seller's good is random and known to all sellers. Similar to my model, the buyer needs to simultaneously pay attention to each good's price and quality. The authors calculate an equilibrium and conduct comparative statics. Thus, while I focus on a *single* seller who is making *repeated offers*, Matějka and McKay (2012) study multiple sellers, each making a single offer. In other words, mine is a model of dynamic bargaining while theirs is a model of static competition.

Martin (2012) studies a one-shot model in which a seller attempts to sell a single good of random quality to a rationally inattentive buyer. The quality of the good is either low or high, and the seller is restricted to one of two possible prices. My model, therefore, differs from that of Martin (2012) in that I allow for repeated interactions, a continuous range of offers and a more general distribution of qualities. Another difference is that in Martin (2012) the buyer gets to observe the seller's offer perfectly at zero cost, and only needs to pay attention to the good's quality. Letting the buyer observe the good's price before choosing her attention strategy results in multiple equilibria due to the buyer's ability to threaten with beliefs. I avoid some of this multiplicity thanks to my refinement and my assumption that the buyer also needs to pay attention to the seller's offer.

Yang (2013) considers a slightly different set-up to study security design. In his model, the seller makes a take it or leave it offer in the form of an asset based security. Both the seller and the buyer are uninformed about the asset's future dividends when the offer is made. The buyer gets to observe the seller's offer, and may use her attention to learn about the asset's future dividends. Yang (2013) shows that the seller will offer the buyer a debt contract to minimize attention costs. My model differs from that of Yang (2013) not just by the virtue of being dynamic but also in that I abstract from the structure of the seller's offer. In my model, the seller's offers are one dimensional. However, this is without loss of generality as long as the player's utility is linear in money. Moreover, unlike in Yang (2013), my seller has private information about the value of the good. Finally, I assume that the buyer needs to pay attention also to the content of the seller's offer, while in Yang (2013) the buyer gets to observe that content for free. Thus, while I wish to study the outcomes of bargaining, Yang (2013) is concerned with the structure of securities.

In the bargaining literature, the natural starting point are models that use the one-sided repeated offers bargaining protocol. Most of the papers using this protocol considered bargaining with one-sided incomplete information in a private values set up (e.g. Fudenberg et al. (1985), Gul et al. (1986), Ausubel and Deneckere (1989)). Unlike the model studied in my paper, such models involve an informed buyer and an uninformed seller. A classic result in this literature is the Coase conjecture. This result states that there is no delay when offers are frequent and the gains from trade are positive with probability 1. My model differs from this literature in that I assume that it is the seller, not the buyer, that has private information. Moreover, my model exhibits delay even when offers are frequent and gains from trade are known to be strictly positive.

A model more closely related to mine is the one due to Deneckere and Liang (2006). They consider an uninformed buyer making repeated offers to an informed seller. However, the information available to the seller is also relevant for the buyer. Thus, their model is one of interdependent values. Deneckere and Liang (2006) show that under certain conditions, the equilibrium involves bursts of trade followed by periods of delay. In my model, trade occurs continuously rather than in bursts, and it is the informed party that makes the offers. Moreover, I assume that the buyer is rationally inattentive and that values are private.

Several studies looked at one-sided repeated offers bargaining models in which both parties have private information about the gains from trade (Cramton, 1984; Cho, 1990). When offers are frequent, such models often result in no trade or a large multiplicity of equilibria with various predictions (Ausubel and Deneckere, 1992). A similar multiplicity was first pointed out by Rubinstein (1985) who studied an alternative offers model in which the discount rate of one of the players was private information. He showed that one can support a large set of equilibrium outcomes by constructing belief-based threats off the equilibrium path.

Another model in which the informed party gets to make offers is the one due to Gul and Sonnenschein (1988). Theirs is an alternating offers bargaining model between a buyer and a seller who is uncertain about the buyer's valuation of the good. They show that taking the time between offers to zero results in immediate trade in all equilibria satisfying their refinement. As in Gul and Sonnenschein (1988), my refinement does not identify a unique equilibrium. However, my model generates delay and a potentially negative ex-post surplus to the buyer, outcomes that cannot arise in the analysis of Gul and Sonnenschein (1988).

My work also relates to Abreu and Gul (2000). They study a bargaining model with general two sided offers in which each player is uncertain about the rationality of the other. In particular, players may be irrational and insist on receiving a fixed portion of the surplus. Thus, their model is one of two-sided offers, no uncertainty about gains from trade, and irrationality. In contrast, mine is a model of one-sided offers in which one side is uncertain about the gains from trade, the seller is fully rational and buyer is rationally inattentive.

2 The Cost of Attention

I study bargaining between a fully rational seller and a buyer with limited attention. Each period, the seller makes an offer which the buyer either accepts or rejects; if she accepts, the game ends, otherwise the period ends and the seller makes a new offer in the next period. Unlike in standard bargaining models, in my model the buyer makes her decision to accept or reject with less than perfect information about the seller's offers. How much information the buyer has depends on her attentiveness; that is, on how much effort she devotes to understanding both the value of the object and the seller's offers. The buyer knows that attention is costly and therefore, allocates her attention rationally, which I interpret to mean optimally. The model of rational inattention that describes my buyer is due to Sims (1998). My contribution is the application of the model to non-cooperative bargaining.

2.1 The Extensive Form

The extensive form game is as follows: before the seller makes her first offer, she observes the quality of the good, v; a random variable that takes on values according to the distribution μ_0 from a finite set $V = \{v_l, \dots, v_h\}$, where $v_l \leq v \leq v_h$ for all $v \in V$. The seller makes the buyer offers in periods $m = 1, \dots$. The number of periods can be finite or infinite. An offer is a number, $x_m \in X = [0, \bar{x}]$[1], where $\bar{x} > v_h$. I interpret x as a reduced form of the seller's offers. For example, a payment plan offered by a car dealer to a potential buyer will be represented by its expected present value. In other words, x serves as a summary of the monetary value that the offer transfers from the buyer to the seller.

Each period, the buyer decides whether to accept or reject the current offer. If the buyer accepts an offer at some period m, she gets the good, pays the seller x_m and the game ends. Otherwise, the game continues to the next period. If no agreement is reached, the game ends with no transaction taking place. The seller's payoff when the buyer accepts an offer x_m in period m is:

$$U_s := e^{-r\Delta(m-1)} x_m \tag{1}$$

where $r > 0$ is the (common) discount rate and Δ is the time difference between offers. If no transaction takes place, the seller's payoff is 0, independent of the

[1] The upper bound, $\bar{x}$, plays no role in the equilibrium strategies, and is needed only to ensure compactness of the seller's strategy space.

good's quality.

Let $h \in X^m := X \times \cdots \times X$ denote an m-period history of offers and for $Y \subset R^n$, let $\Delta(Y)$ denote the set of all Borel probability measures on Y. Then, a *behavioral strategy*, σ, for the seller is a sequence of functions[2] $\sigma_m : X^{m-1} \times V \to \Delta(X)$ for $m = 1, \ldots$. Thus, $\sigma_m(h, v)$ is the random offer that the seller makes after history $h \in X^{m-1}$ given that the value of the object is $v \in V$.

The buyer is rationally inattentive and as such, neither observes the seller's offers nor the quality of the good. Instead, she chooses a *period m signal structure*, which is a likelihood function: $l_m : S \times X^m \times V \to [0,1]$, where $S = \{0, 1, 2, \ldots\}$ is the discrete set of possible signal realizations. The signal structure l_m satisfies:

(1) $\sum_{s \in S} l_m(s, h, v) = 1$ for every h, v, and

(2) $l_m(s, \cdot, \cdot)$ is measurable for all s.

The first of these conditions ensures that $l_m(\cdot, h, v)$ is a probability distribution for all h, v; the second is a technical condition necessary for evaluating payoffs. After the seller chooses x_m and the buyer chooses l_m, nature draws the signal s with probability $l_m(s, x^m, v)$. The buyer observes only the signal s, not the value of object v nor the seller's current or past offers. Based on s the buyer decides whether to accept or reject the seller's offer. I assume the seller observes neither l_m nor s.

The signal structures represent the buyer's choice to pay more or less attention to the good's quality and the seller's offers. The flexibility of the signal structure captures the possibility of focusing on some information that is easy to process but is only a proxy of v and h. For instance, a buyer of a used car may pay attention only to the left most digit of the odometer (Lacetera et al., 2012). Similarly, shoppers may compare prices without checking which prices include sales tax and which do not (Chetty et al., 2009).

The rational inattention modeling approach makes three implicit assumptions about the way the buyer allocates her attention. First, it assumes the buyer has a large collection of easily accessible data at her disposal which is related the good's quality and the seller's offer. Second, her attention is selective in that she picks and

[2] Define $X^0 := \{\varnothing\}$.

chooses which parts of the data to pay attention to and which to ignore. Third, the buyer is aware of the stochastic relationship between the data, the good's quality and the seller's offers, and uses this stochastic relationship to optimallty allocate her attention.

The buyer's payoff depends on when she gets the the good, its value and the cost of attention that she incurs. More precisely, the buyer's payoff if she accepts an offer x_m in period m, and uses the signal structures l_j in each period $j \leq m$ is:

$$U_b = e^{-r\Delta(m-1)}\left(v - x_m\right) - \sum_{j=1}^{m} e^{-r\Delta(j-1)} \kappa\, \mathbf{I}\left(l_j, \mu_j\right) \tag{2}$$

where $\kappa \in (0, v_l)$ is a constant, μ_m is the buyer's belief at the start of period m about the value of the object and the seller's current and past offers. The term $\kappa\,\mathbf{I}$ is the flow cost of attention; as I explain in the next subsection, $\mathbf{I}$ is the average decrease in the entropy of the buyer's beliefs. I interpret $\mathbf{I}$ as a measure of the buyer's level of attention and hence, κ is the constant marginal cost of attention.

An outcome of the game is the period in which agreement is reached, m, the accepted offer, x_m, the signal structures that the buyer has chosen in each period, $(l_1, \ldots, l_m)$, and the buyer's beliefs at the beginning of each period, $(\mu_1, \ldots, \mu_m)$. The seller's payoffs are simply his transaction payoffs, while the buyer's payoff is her transaction payoffs minus the discounted sum of attention costs in each period. In the next subsection, I define $\mathbf{I}$.

2.2 Shannon's measure of mutual information

Shannon (1948) was the first to suggest the use of entropy to measure information. The idea is to measure the amount of information there is to learn about a random variable by the entropy of its distribution. Learning the outcome of the variable, therefore, corresponds to obtaining information equal to its distribution's entropy. Shannon (1948) also suggested a way of measuring how much one can learn about a variable by observing a random signal. His answer was the expected difference between the entropy of the variable's unconditional and conditional distributions upon observing the signal. This quantity is now known as Shannon's measure of mutual information (Cover and Thomas, 2006).

Formally, let $\mu \in \Delta(Y)$ be a prior on Y; that is, μ is a Borel probability measure on $Y \subset R^k$. In period m of my bargaining game, Y is $X^m \times V$. Take $l : S \times Y \to [0, 1]$ to be a signal structure where $S = \{0, 1, \ldots\}$, l is measurable in its second argument and $l(\cdot, y)$ is a discrete probability. First, assume μ has a finite support $\{y_1, \ldots, y_n\}$. Then, $\mathbf{H}$, the entropy of μ is:

$$\mathbf{H}(\mu) = -\sum_{i=1}^{n} \mu(y_i) \ln \mu(y_i)$$

To ensure continuity, I let $0 \ln 0 = 0$, $b \ln \frac{b}{0} = \infty$ if $b > 0$, and $0 \ln \frac{0}{0} = 0$.[3] In information theory, entropy is interpreted as a measure of the information one can learn about a random variable. Here, I interpret it as the level of exertion needed to understand or process the information in question. That is, it is the level of attention that the buyer needs to fully understand the offer and the value of the good.

Let π be the prior distribution of the signal:

$$\pi(s) = \int_Y l(s, y) \mathrm{d}\mu$$

Then, given the signal $s \in S_{\mu l} := \{s \in S \,|\, \mu \cdot l(s) > 0\}$, the posterior on Y is:

$$\mu^s(E) = \frac{\int_E l(s, y) \mathrm{d}\mu}{\int_Y l(s, y) \mathrm{d}\mu}$$

for any Borel set $E \subset Y$. Then, *Shannon's measure of mutual information*, $\mathbf{I}(l, \mu)$, is the expected change in entropy between the prior μ and the posterior given μ and the signal structure l. Hence,

$$\mathbf{I}(l, \mu) = \sum_{s \in S_{\mu l}} [\mathbf{H}(\mu) - \mathbf{H}(\mu^s)] \, \pi(s) \tag{3}$$

[3]For analytical convenience, I measure entropy in nats; that is, I am using the natural logarithm in the formula above rather than to more common $\log_2$.

which is the average change in entropy between the prior and the posterior distribution that results from seeing the signal structure l.

By assuming that the buyer's information cost is proportional to Shannon's measure of mutual information, I am assuming that the buyer already understands the prior joint distribution of the offers and the value but can pay further attention to these variables and understand more. Thus, she incurs attention costs *at the margin.*

For the general case; that is, if μ is not discrete, one can define Shannon's measure of mutual information as:

$$\mathbf{I}(l, \mu) = \sum_{s \in S_{l\mu}} \int l(s, y) \ln \left(\frac{l\,(s, y)}{\pi\,(s)} \right) \mathrm{d}\mu \tag{4}$$

which becomes the same as equation 3 when μ is discrete.

Note that the cost of attention depends on the buyer's prior. To illustrate, suppose the buyer only needs to pay attention to the seller's offers, and that the value of the good is 2. Let l be the signal structure that sends 0 if the seller's first offer is strictly above 2, and 1 otherwise. If the buyer's prior about the seller's first offer is uniform over $[0, 2]$, then l will send 1 for sure. In this case, l is completely uninformative and therefore has a cost of 0. In contrast, l's cost would have been positive had the buyer's prior been a uniform distribution over $[1, 3]$. Hence, the informativeness, and therefore the attention cost of every signal structure depends on the buyer's prior information.

The fact that prior information influences attention costs captures the idea that one needs to pay less attention to familiar information. For example, it is easier to understand papers that use more common methodologies than it is to understand papers that employ innovative techniques. In the model, familiar information is one that has already been absorbed into the buyer's prior. Examining a familiar piece of information is therefore equivalent to acquiring a signal that results in little to no updating i.e. barely provides the buyer with any new information. Such signals have very low attention costs, which expresses the ease with which the buyer can pay attention to familiar information.

2.3 Recommendation Strategies

My first goal is to show that an optimal strategy for the buyer can be found within a class of simple strategies that I call *recommendation strategies.* A recommendation strategy is defined by two properties: there are only two signals; call them 0 and 1, and the buyer does not randomize; she accepts for sure if and only if she observes 1. Recommendation strategies can be described by a sequence of mappings: $\beta = (\beta_m)_{m\geq 1}$, where $\beta_m(x^m, v) \in [0,1]$ is the probability that the buyer receives an accept recommendation. Thus, for every m, β_m is some (measurable) mapping from $X^m \times V$ into $[0,1]$.

Equation 4 of subsection 2.2 implies that Shannon's measure of mutual information between β_m and the prior distribution, μ_m over $X^m \times V$ has the following convenient form:

$$\mathbf{I}(\beta_m, \mu_m) = \int \beta_m \ln\left(\frac{\beta_m}{\int \beta_m \mathrm{d}\mu_m}\right) + (1-\beta_m)\ln\left(\frac{1-\beta_m}{\int (1-\beta_m)\mathrm{d}\mu_m}\right)\mathrm{d}\mu_m$$

The following proposition ensures that I can focus on recommendation strategies.

Proposition 1. *For every strategy for the buyer there exists an outcome equivalent recommendation strategy β with weakly lower attention costs.*

Proof. See appendix. □

For the one-period version of my game, Proposition 1 is easy to prove. One can show that $\mathbf{I}(l,\mu) \geq \mathbf{I}(l',\mu)$ whenever l is more informative than l' in the sense of Blackwell (1953) (see Cover and Thomas (2006), for example). Replacing any l with the resulting distribution over actions induced by the buyer's strategy creates a new signal structure that is less informative than l, without changing the terms or probability of agreement. Since a less informative signal costs less, Proposition 1 follows.

The argument above is not enough for proving Proposition 1 for multi-period bargaining games. With multiple periods, information gained in period 1 can be useful in period 2. If, for some reason, the cost of processing information in period

2 were higher than in period 1, it might make sense to process that information in period 1 rather than wait until period 2. To prove Proposition 1 for the general case, I invoke the *chain rule for mutual information* which states that the expected sum of information that is gained by observing two signals consecutively is equal to the amount of information that results from observing both signals simultaneously. To use this property, I view each signal structure as including two different signals: an action recommendation and a residual. The chain rule then assures me that processing this residual simultaneously with any future signal instead of processing it with today's recommendation does not increase the total cost of information. Therefore, one can delay the processing of this residual to the time in which this residual is used.

I appeal to Proposition 1, and assume henceforth that the buyer only uses recommendation strategies[4].

3 Recommendation Perfect Equilibria

In this section I define, characterize and prove existence of an equilibrium satisfying a refinement which I call *recommendation perfect equilibrium.* This refinement's purpose is to address some of the unique issues that arise when the buyer is rationally inattentive. Theorem 1 establishes that recommendation perfect equilibria exist in the finite horizon game. Moreover, the theorem shows that in equilibrium players use simple strategies; i.e. strategies that do not depend on the past. In the infinite horizon game I focus on limits of finite horizon equilibria. Theorem 2 establishes that such a limit exists, is an equilibrium of the infinite horizon game, and involves simple strategies. These simple strategies have a useful characterization which I present in Lemma 1.

[4]In fact, I will treat the buyer as if she is using a pure recommendation strategy. This is without loss of generality since the buyer's objective function is concave and the seller does not observe the buyer's signal structure.

3.1 Attentive Strategies and Credible Best Responses

In this subsection I provide a formal definition of recommendation perfect equilibrium and state the existence result for my finite horizon game. My refinement is composed of two parts. First, I assume that the buyer is *attentive*, meaning that there are no periods in which the buyer automatically rejects every offer regardless of the history. Second, I impose a perfection requirement similar to that of Selten (1975). More precisely, for each off-equilibrium offer, I require the buyer's strategy to be a limit of best responses to some sequence of perturbations that put positive probability on that offer. This condition is needed to avoid non-credible attention threats that may arise when the buyer is rationally inattentive.

The first issue I address in my refinement is the possibility of the buyer automatically rejecting every offer. To illustrate what I mean by automatic rejections, take any sequential equilibrium and adjust it in the following way. In period 1 have the buyer reject every offer, regardless of its content. At the same time, have the seller's first offer always be equal to $\bar{x}$. From period 2 onwards, let the players play according to the original equilibrium as if period 1 never happened. Clearly, this is a sequential equilibrium. In fact, I can extend this kind of logic to obtain the following observation[5]:

Observation There is a sequential equilibrium without trade after any history.

I wish to avoid periods in which the buyer automatically rejects the seller's offers regardless of their content. Formally, I say that the buyer's strategy β is *attentive* if for every period m there exists some price history and some quality of the good, (x^m, v), such that $\beta_m(x^m, v) > 0$. Assumptions of similar flavor are often made in bargaining models. For example, Rubinstein (1985) assumes that the uninformed player never makes irrelevant offers. Similarly, Gul and Sonnenschein (1988) assume away the possibility of periods in which all offers rejected for sure and are only made to allow one party to send a signal to the other.

[5] One can actually show that for every set of periods $A \subset \{1, 2, \ldots\}$ there exists an equilibrium where trade occurs in period m if and only if $m \in A$.

There are reasons to want to avoid trade break downs based on the buyer being inattentive. First, the model describes two parties that are engaged in active bargaining. It is unreasonable to assume that one party can completely ignore the other when the two are directly facing each other. Second, one of my goals is to show that inattention can lead to delay. Clearly delay can be created by interspersing periods of trade shut downs resulting from automatic rejections. What is more interesting is to know whether inattention can cause delay even when the buyer is at least somewhat attentive to the seller's offers.

A second and more subtle issue that arises in my model is that a rationally inattentive buyer can make non-credible attention threats. Such threats involve the buyer committing to pay close attention to off-equilibrium offers. With suitably chosen off-path beliefs, one can sustain a very large number of sequential equilibria. These threats are possible because the rational inattention cost function does not depend on off-path signals.

Non-credible attention threats matter in my model because the buyer is paying attention to another player's choice variable. In most of the current rational inattention literature, agents are paying attention only to exogenous or aggregate variables. For example, in Yang (2013) agents need to pay attention to the fundamental value of an asset, while in Mackowiak and Wiederholt (2009) producers need to pay attention to macroeconomic outcomes. Such variables are outside the control of any other agent. As such, the variable's equilibrium distribution is not influenced by possible attention threats[6].

Consider the one-shot version of my model in which the seller makes the buyer a take-it or leave-it offer. I will show that one can construct an extreme equilibrium in which the buyer obtains a large surplus. Assume that $V = \{2, 4\}$, $\kappa = 1$ and that both qualities can occur with strictly positive probability. Suppose further that the seller offers 2 for sure regardless of the realized quality of the good. Let μ be the buyer's beliefs given the seller's strategy, σ. One can show that a necessary

[6]One exception is Matějka and McKay (2012), who focus on an equilibrium with particular assumptions on the buyer's behavior towards zero probability prices. Their assumptions preclude the possibility of non-credible attention threats.

and sufficient condition for β to be optimal for the buyer in this setting is to have $\beta(x,v) = 1$ μ-almost surely[7]. Therefore, the strategy defined by $\beta(x,v) = 1$ if x equals to 2 and 0 otherwise is optimal for the buyer. Clearly, it is also optimal for the seller to offer 2 for sure given β. Thus, (μ, β, σ) is a sequential equilibrium. Turns out that by using a similar construction one can support the seller offering for sure any x in $[2, 2+\delta]$, where $\delta > 0$ depends on the probability of $v = 2$.

In the above equilibrium the rejects for sure any offer that is above 2. In this strategy the buyers reacts very differently to zero probability offers compared to positive probability ones. Such an extreme change in behavior towards zero probability offers is non-credible. In particular, the buyer never chooses to react in this way towards offers that are made with strictly positive probability. I formalize this idea in my definition of a credible best response for the buyer.

Let $E_m[U_b|\mu_m, \beta, \sigma]$ be the buyer's expected utility conditional on arriving to period m, the buyer's beliefs over $X^m \times V$ being μ_m and future play being conducted according to (β, σ). I'll say that the beliefs μ and strategies (β, σ) are *consistent* if μ is updated according to Bayes rule whenever possible.

The following definition formalizes my requirement that the buyer's strategy be credible. In particular, for every (x^m, v) I identify a belief perturbation μ^* that puts positive probability on (x^m, v). I then use μ^* to create a sequence of perturbations of μ_m. To create this sequence one mixes μ^* into μ_m by putting a diminishing weight on μ^*. As written in the definition below, the buyer's strategy is credible if it is a limit of best responses to at least on such sequence of perturbations for every (x^m, v).

Definition 1. For a consistent (μ, β, σ), β is a *credible best response to* σ given μ if:

1. β maximizes $E_m[U_b|\mu_m, \beta, \sigma]$ for all m.

2. For every (x^m, v) there is a $\mu^* \in \Delta(X^m \times V)$ with $\mu^*(x^m, v) > 0$ and a $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ with $\mu^n = \epsilon^n \mu^* + (1 - \epsilon^n)\mu_m$, $\epsilon^n \downarrow 0$ and $\beta^n \to \beta$, such that β^n maximizes $E_m[U_b|\mu^n, \beta^n, \sigma]$ for all n.

[7] I solve for the buyer's general optimal strategy in the dynamic game in appendix C. To obtain that $\beta(x,v) = 1$ μ-almost surely is optimal here one can also use the results of Woodford (2008), Yang (2014) and Matějka and McKay (2013).

The first part of Definition 1 is standard. This part requires that the buyer's strategy maximizes her expected utility after every history. The second part of the definition rules out non-credible attention threats. Similar to Selten (1975)'s perfect equilibrium, I require the buyer's strategy to be robust to mistakes. In my formulation, the buyer is aware of possible mistakes in her *beliefs*. While I do so for analytical convenience, the difference between mistakes in beliefs and mistakes in strategies is insubstantial in my setup. This is because by the time the buyer chooses her period m signal structure, the seller's m-th offer has already been determined. Therefore all that matters for the buyer is her beliefs over that offer, i.e. μ_m. As such, it does not matter whether we use beliefs that are consistent with perturbed strategies, or whether we perturb beliefs directly.

To understand how the above definition rules out non-credible attention threats, consider my previous example. Suppose we perturb the buyer's belief from the example by adding a probability of ϵ that the seller offers 6 whenever the quality of the good is 4. Let μ^{ϵ} denote the buyer's perturbed beliefs. Assume further for concreteness that the probability of $v = 4$ is $\frac{1}{2}$. Calculating the buyer's expected utility from using β given μ^{ϵ} (see equation 2 from section 2.1) gives[8]:

$$E\left[U_b|\beta,\mu^{\epsilon}\right] = 1-\epsilon-\frac{1}{2}\ln\left(\frac{2}{2-\epsilon}\right)-\frac{1}{2}\left((1-\epsilon)\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)+\epsilon\ln\left(\frac{2}{\epsilon}\right)\right)$$

where $1-\epsilon$ is the buyer's expected transaction payoffs, while the remainder is the buyer's attention costs.

Compare β to the following alternative strategy: $\beta'(x,v) = 1$ for all x and v. That is, the buyer accepts every x offered by any quality with probability 1. The buyer's expected transaction payoff under β' is $1-2\epsilon$. Moreover, since β' is completely uninformative, it has an attention cost of 0. Therefore the buyer's expected utility from β' given μ^{ϵ} is:

$$E\left[U_b|\beta',\mu^{\epsilon}\right] = 1-2\epsilon$$

[8]In the one shot game μ is sufficient for calculating the buyer's expected utility. This is because μ is a distribution over both V and the seller's offer. In the multi-period version I need σ to specify the seller's future play.

Therefore, β' is strictly better for the buyer than β if and only if:

$$\frac{1}{\epsilon}\ln\left(\frac{2}{2-\epsilon}\right)+\frac{1}{\epsilon}\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)-\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)+\ln\left(\frac{2}{\epsilon}\right)>2$$

As ϵ goes to zero, an application of L'Hopital's rule reveals that the left hand side goes to infinity. Therefore, for all small enough ϵ, the buyer prefers β' over β. One can prove that this kind of logic will extend to all perturbations involving the buyer offering 6.

I now present the definition of a perfect recommendation equilibrium. This definition is somewhat different than the more familiar notions of perfect equilibrium due to Selten (1975) and proper equilibrium due to Myerson (1978). In perfect and proper equilibria one needs to perturb all information sets simultaneously using a single set of full support trembles. In my definition, I perturb information sets one at a time and allow different off-path histories to be evaluated using different trembles. Moreover, in the definitions of Selten (1975) and Myerson (1978) one introduces perturbation to both players. In my formulation, I introduce the perturbations only on buyer's side, since only she can make attention threats.

Definition 2. A consistent (μ, β, σ) is a *perfect recommendation equilibrium* if:

1. β is a credible best response to σ given μ.

2. σ is a best response to β after every history.

If, in addition, β is attentive, then I say that (μ, β, σ) is an *attentive perfect recommendation equilibrium.*

From now on I will reserve the term equilibrium to mean attentive perfect recommendation equilibrium, unless specified otherwise. In the next subsection, I establish that an equilibrium exists, and that it admits a recursive structure.

I now turn to state Theorem 1, which shows two things. First, there exists an equilibrium in the finite horizon version of the game. Second, equilibrium strategies are *simple*. For the seller, a strategy is *simple* if it prescribes a single deterministic offer, $z_{m,v}$, for every period m and every v. A v type seller makes this offer in period

m regardless of the seller's realized offers in periods $m^{'} < m$. As for the buyer, her strategy is *simple* if for every m, the probability that the buyer accepts an offer x_m made by a v type seller is $b_m(x, v)$, regardless of the seller's offers in previous periods.

Theorem 1. *There exists an equilibrium of the finite horizon game, and every such equilibrium is in simple strategies.*

Proof. See appendix. □

Given the theorem, I will often identify equilibrium strategies β and σ by their corresponding simple counterparts, b and z. To put it differently, I will often write $b_m(x_m, v)$ instead of $\beta_m(x_1, \ldots, x_m, v)$, and say that the seller uses the strategy z rather than σ.

The proof of Theorem 1 is partially constructive and partially dependent on a fixed point argument. The main difficulty is to ensure that b is attentive. The observation made at the beginning of this section showed that the game admits a fully inattentive equilibrium. Requiring β to be a credible best response to σ is insufficient to rule such an equilibrium out. As such, to prove the theorem I derive a set of necessary and sufficient conditions for (μ, β, σ) to be an attentive equilibrium. I then use a fixed point argument to show that there is some (μ, β, σ) that satisfies these conditions. To derive these conditions, I first prove that equilibrium strategies must admit a specific recursive structure, which I present below.

At this stage the reader may wonder about uniqueness of equilibrium. The following corollary states that in the one-shot game the equilibrium is unique.

Corollary 1. *There exists a unique equilibrium in the one-shot game.*

Proof. See appendix. □

When there are more than two periods, one can obtain multiple equilibria. Intuitively, the multiplicity comes from the interdependency of current and future periods. Future periods are influenced by the buyer's posterior over the quality of the good at the end of the current period. However, behavior at the current period, and therefore the buyer's posterior, depend on both player's continuation values, which depend on the future. Combined these can result in multiple equilibrium paths.

3.2 Infinite Horizon Bargaining

In the infinite horizon game I focus my analysis on equilibria which arise as a limit of finite horizon equilibria. Such equilibria exist and are simple, just like finite horizon equilibria (Theorem 2). These equilibria also satisfy some useful structural properties, which I present in Lemma 1.

Early papers in the bargaining literature also focused on limits of finite horizon equilibria (e.g. Cramton (1984) and Sobel and Takahashi (1983)). I do so in my analysis to exclude the players' strategies from exhibiting complicated history dependence. Other studies often avoid complicated dependencies on the past by focusing on stationary equilibria (see for example Gul et al. (1986), Gul and Sonnenschein (1988), Ausubel and Deneckere (1989) and Gul (2001)). Stationarity in its standard form will not be possible under rational inattention. The fact that a rationally inattentive buyer does not get to perfectly observe past offers creates a rigidity in the buyer's strategy that precludes stationary play. Focusing on equilibria that can be approximated by finite horizon play allows us to recover some of the simplicity lost by allowing for non-stationary strategies.

The following theorem establishes that finite horizon equilibria satisfy a form of sequential compactness. Therefore one can attain a sequence of equilibria that converges as the horizon becomes infinite. The theorem also states that any infinite horizon limit of finite horizon equilibria is, in fact, an equilibrium of the infinite horizon game. Finally, the theorem establishes that the strategies in the resulting infinite horizon equilibrium are simple.

Theorem 2. *For any sequence of finite horizon equilibria with the horizon going to infinity, there exists a convergent sub-sequence. Moreover, every infinite horizon limit of finite horizon equilibria is simple and is an equilibrium of the infinite horizon game.*

Proof. See appendix. □

The proof of Theorem 2 is rather technical. The equilibrium in the finite horizon game satisfies properties similar to the ones stated in Lemma 1 below. Using these properties, one can connect the convergence of $b_m(x,v)$ and $z_{m,v}$ to the convergence of $z_{m+1,v}$ and $b_{m+1}(z_{m+1,v},v)$. Since these are members of a countable product of compact subsets of $\mathbb{R}$, one can assure the existence of a converging subsequence. Similar to Theorem 1, the tricky part of the proof is to ensure that the buyer's limit strategy is attentive. However, this ends up being ensured by the structure of the finite horizon equilibria. Once attentiveness is established, I prove optimality of the buyer's limiting strategy via sufficient conditions derived in appendix C. Optimality of the seller's strategy is then attained via standard continuity at infinitey arguments.

The equilibrium in the infinite horizon game satisfies some structural properties, which are used throughout the analysis. To present these properties, let (μ, b, z) be an equilibrium of the game with infinite periods. Denote the marginal of μ_m over V by $\bar{\mu}_m$. Given an equilibrium, (μ, b, z), take $b_{m,v}$ to be the probability that the buyer accepts the v-seller's period m offer conditional on arriving to period m. That is, let $b_{m,v} := b_m(z_{m,v},v)$. Define π_m as the prior probability that the buyer accepts the m-th offer conditional on arriving to period m, i.e.

$$\pi_m := \sum_v \bar{\mu}_{m,v} b_{m,v} \tag{5}$$

and let $w_{m,v}$ be the seller's expected profits in equilibrium conditional on v and on arriving to m in period m terms. Note that, since the buyer's strategy is simple, $w_{m,v}$ is well-defined and does not depend on the seller's past offers.

Lemma 1 does two things. First, it shows that the equilibrium involves inefficiency due to delay, which This delay is implied by π_m being strictly below 1 for all m. Second, the lemma provides a characterization of the player's equilibrium strategies in the infinite horizon game.

Lemma 1. *Let (μ, b, z) be the equilibrium in the infinite horizon game. Then for every $m = 1, \ldots$ and every v:*

$$0 < \pi_m < 1 \text{ and } w_{m,v} = z_{m,v} - \kappa \tag{6}$$

moreover, for every m, x and v:

$$\left(\frac{b_m(x,v)}{1-b_m(x,v)}\right) = \left(\frac{\pi_m}{1-\pi_m}\right) e^{\frac{1}{\kappa}\left(v-x+\kappa\sum_{j=m+1}^{\infty} e^{-r\Delta(j-m)}\ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right)\right)} \tag{7}$$

$$\left(\frac{b_{m,v}}{1-b_{m,v}}\right) = \left(\frac{z_{m,v}-\kappa}{\kappa}\right) - e^{-r\Delta}\left(\frac{z_{m+1,v}-\kappa}{\kappa}\right) \tag{8}$$

The derivation of the buyer's optimal strategy is somewhat lengthy, and is delegated to appendix C. For a partial intuition, consider the following way of rewriting equation 7 when x equals $z_{m,v}$:

$$v - z_{m,v} - \kappa \ln\left(\frac{b_{m,v}}{\pi_m}\right) = -\kappa \sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right) \tag{9}$$

The above equation comes from the buyer's first order condition for $b_{m,v}$. On the left hand side there is the buyer's marginal utility from accepting the m-th offer conditional on $(z_{m,v}, v)$. Accepting conditional on $(z_{m,v}, v)$ gives the buyer a transaction utility of $v - z_{m,v}$. However, accepting involves seeing an *accept* signal, which has a cost. When considering whether to increase or decrease the probability of accepting, what matters is the cost of increasing the signal's probability at the margin. Turns out that the marginal attention costs conditional on v depend only on the change in the probability the buyer assigns to v[9]. Following an accept signal, these beliefs change from $\bar{\mu}_{m,v}$ to $\bar{\mu}_{m,v}(b_{m,v}/\pi_m)$, resulting in a marginal attention cost of $\kappa \ln(b_{m,v}/\pi_m)$.

[9] In general, the buyer's marginal attention costs depend on the change in the probability of the entire history of offers and good quality, (x^m, v). The reason it depends only on v is come from the seller using a simple strategy.

On the right hand side of equation 9 is the marginal utility from rejecting the current and all future offers conditional on $(z_{m,v}, v)$. Note that rejecting forever gives the buyer a transaction utility of zero. Because of this, only the total present value of the marginal attention costs from observing an infinite sequence of *reject* signals influences the buyer's marginal utility. As in the case of *accept*, the marginal attention costs of rejecting forever conditional on $(z_{m,v}, v)$ depend only on the change in the probability the buyer assigns to v. This leads to the expression on the right hand side of equation 9.

Equation 9 therefore says that the buyer's marginal utility from accepting is equal to that of rejecting forever. Intuitively, the equality holds because every period the buyer chooses to observe both an accept and a reject signal with positive probability. As such, the marginal utility of accepting the m-th offer is equal to the marginal utility of seeing *reject* in period m and moving to period $m+1$. However, upon ariving to period $m+1$ the buyer is again indifferent at the margin between an *accept* signal and a *reject* signal. Continuing along this sequence of indfferences leads to equation 9.

The conditions that characterize the seller's side are easier to derive. Since the buyer's strategy is simple, the value of the seller's period $m+1$ problem conditional on v does not depend on the seller's offer in period m. Therefore, the value of the seller's problem in period m is:

$$w_{m,v} = \max_x b_m(x,v)\,x + (1 - b_m(x,v))\,e^{-r\Delta} w_{m+1,v} \tag{10}$$

using equation 7 one can derive this problem's first order condition:

$$x - e^{-r\Delta} w_{m+1,v} = \frac{\kappa}{1 - b_m(x,v)} \tag{11}$$

This condition can be rearranged into $w_{m,v} = z_{m,v} - \kappa$, where $z_{m,v}$ is the solution to the seller's problem. One can then substitute $w_{m+1,v} = z_{m+1,v} - \kappa$ and rearrange to obtain equation 8.

4 Getting what you pay for, Rip-offs and Bargains

In this section I highlight two features shared by all equilibria of the bargaining game with rational inattention (Proposition 2). The first feature is that the buyer gets what she pays for. In other words, the buyer both pays more for, and gets a higher surplus from, higher quality products. The second feature is that the buyer gets cheated on low quality products and gets good value when buying goods of high quality. Thus, there are rip-offs at the bottom, and bargains at the top. These features are captured in the following proposition.

Proposition 2. *For every m, prices, acceptance probability, and the surplus remaining for the buyer are all strictly increasing in v. That is, $z_{m,v}$, $b_{m,v}$ and $v - z_{m,v}$ are increasing in v. Moreover, the buyer makes a negative surplus on v_l and a positive surplus on v_h, i.e.*

$$v_l - z_{m,v_l} < 0 < v_h - z_{m,v_h}$$

Proof. Full proof is in the appendix. Here we take as given that $b_{m,v}$ is increasing in v for all m. Repeated substitution of equation 8 into itself gives:

$$\frac{z_{m,v} - \kappa}{\kappa} = \sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \left(\frac{b_{j,v}}{1 - b_{j,v}} \right)$$

Since $b_{j,v}$ is increasing in v for all j, the above implies that $z_{m,v}$ is increasing in v. To obtain that $v - z_{m,v}$ is increasing in v, note that one can rewrite equation 7 as:

$$e^{\frac{1}{\kappa}(v - z_{m,v})} = \left(\frac{b_{m,v}(1 - \pi_m)}{\pi_m (1 - b_{m,v})} \right) e^{\sum_{j=m+1}^{\infty} e^{-r\Delta(j-m)} \ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)}$$

$v - z_{m,v}$ being increasing follows. □

Proposition 2 stands in contrast with the classic full information model. When the buyer is fully informed about the seller's product and offers, the seller obtains all of the surplus. Thus, in the full information model, the buyer is always indifferent about the transaction. In contrast, Proposition 2 says that a rationally inattentive buyer is much more likely to be disappointed or satisfied with her purchase than indifferent about it. Moreover, a rationally inattentive buyer feels ripped off after buying cheap products, and is satisfied with expensive ones. Thus, prices are representative not only of a good's quality but also the buyer's satisfaction, features that are abscent from the classic full information bargaining game.

5 Frequent Offers and Delay

This section shows that introducing costly attention into bargaining results in delay which is independent of the time between offers. Theorem 2 already implies that delay occurs in equilibrium in an environment with infrequent offers. However, as pointed out by Gul and Sonnenschein (1988), the delay that arises in an environment with infrequent offers can be misleading. In particular, restricting the time between offers to be positive may conflate the time of agreement with the number of offers needed to reach it. I therefore study what happens in my model as the time between offers goes to zero. Proposition 3 establishes that the model's delay persists even when offers are made infinitely frequently. Moreover, the proposition shows that this delay is decreasing with the good's quality. Thus, the more there is to lose from delay, the smaller it is.

Let $B(\Delta, \kappa)$ be the bargaining game with infinite horizon where the time between offers is $\Delta > 0$ and the marginal cost of attention is κ. Given $\Delta > 0$, take $\mathcal{T}(\Delta) = \{\Delta, 2\Delta, \ldots\}$ to be the set of calendar times of the periods of $B(\Delta, \kappa)$. Thus, each period m corresponds to a calendar time of Δm. A sequence of time between offers, $\{\Delta_n\}_{n=1}^{\infty}$, is a *refining sequence* if $\Delta_n \downarrow 0$ and $\mathcal{T}(\Delta_n) \subset \mathcal{T}(\Delta_{n+1})$ for all n. The current section is concerned with the distribution of the calendar time of agreement, defined below.

Definition 3. A function $F : \mathbb{R} \times V \to [0, 1]$ a *timing distribution function* of $B(\Delta, \kappa)$ if there exists an equilibrium (μ, b, z) of $B(\Delta, \kappa)$ such that for every v, $F_v(t)$ is the probability that trade occurred on or before calendar time t conditional on the quality being equal to v.

Theorem 3 establishes that the timing distribution functions of $B(\Delta, \kappa)$ converge as Δ goes to zero and κ remains constant. I interpret a lower Δ as an increase in the rate in which new information accumulates, but not necessarily as an increase in the rate in which new information is absorbed. By fixing κ I assume that absorbing the same amount of information at any given moment results in the same cost of attention, regardless of Δ. This assumption is inline with the chain rule of information, which states that observing multiple consecutive signals costs the same as observing all signals simultaneously (see section 2.3). Thus, what matters is the amount of information the buyer absorbs, not the number of signals she uses to absorb it.

Theorem 3. *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a refining sequence, and take $\{F^n\}_{n=1}^{\infty}$ to be a sequence of corresponding timing distribution functions. Then there is a sub-sequence $\{\Delta_{n_k}\}_{k=1}^{\infty}$ and a cumulative distribution function for every v, F_v, such that $F_v^{n_k}(t) \to F_v(t)$ for all t and v.*

Proof. See appendix. □

Given Theorem 3, I will say that a function $F : \mathbb{R} \times V \to [0,1]$ is a *frequent offers timing function* of $B_0(\kappa)$ if there exists a refining sequence $\{\Delta_n\}_{n=1}^{\infty}$ and a corresponding sequence of timing distribution functions $\{F^n\}_{n=1}^{\infty}$ of $B(\Delta_n, \kappa)$ such that $F_v^n(t) \to F_v(t)$ for all v and t.

The following proposition establishes that rational inattention leads to delay in an environment with frequent offers. Moreover, it shows that delay is decreasing with the quality of the good and that there is always some probability of trade occuring. The proof of the proposition is rather involved, and is tightly connected to the proof of Theorem 3. The key step in the proof of Theorem 3 is to approximate each F^n by a distribution G^n that is absolutely continuous over time for every v. In particular, G^n is chosen in a way that agrees with F^n for every $t \in \mathcal{T}(\Delta_n)$ and every v. To construct G^n, I use $b_{m,v}$ to create a time-dependent hazard rate for each v, $\lambda_{t,v}$. These hazard-rates can be shown to be uniformly bounded by a number that remains finite as Δ goes to zero. Utilizing this bound I can embed the hazard rates in an L_2 space with the appropriate measure over $\mathbb{R}_+$. I then evoke the sequential version of the Banach-Alaoglo theorem to generate a weakly convergent sub-sequence of the said hazard rates. This results in an absolutely continuous limit F_v for all v, as stated in Proposition 3 below.

Proposition 3. *Let F be a frequent offers timing function of $B_0(\kappa)$. Then:*

1. *F_v is an absolutely continuous for every v and satisfies $F_v(0) = 0$.*

2. *$F(t, v)$ is strictly increasing in v and in t for all $t > 0$.*

Proof. See appendix. □

One can get an intuition for Proposition 3 by examining the game in which the buyer knows v. For that, let $B_v(\Delta)$ be the bargaining game in which the quality of the good is equal to v with probability 1. Proposition 4 below characterizes and proves uniqueness of the equilibrium of $B_v(\Delta)$. The equilibrium in $B_v(\Delta)$ involves inefficient delay which persists even when offers are made infinitely frequently, just like the equilibrium of $B(\Delta, \kappa)$. Moreover, the proposition shows that as the time between offers goes to zero, the distribution of agreement time in $B_v(\Delta)$ converges to the distribution of the first arrival from a Poisson process with a rate of $r\left(\frac{v-\kappa}{\kappa}\right)$.

Proposition 4. *There exists a unique equilibrium in the game $B_v(\Delta)$. In this equilibrium:*

1. *The seller offers v every period with probability 1 regardless of the history.*

2. *The buyer accepts v with probability:*

$$\pi^*_{\Delta,v} = \frac{\left(1 - e^{-r\Delta}\right)(v - \kappa)}{\left(1 - e^{-r\Delta}\right)(v - \kappa) + \kappa} \tag{12}$$

3. *Let $F_{v,\Delta}$ to be the cdf of the time of agreement in equilibrium. Then as $\Delta_n \to 0$:*

$$F_{v,\Delta_n}(t) \to 1 - e^{-r\left(\frac{v-\kappa}{\kappa}\right)t}$$

Proof. Note that since the seller's quality is known to the buyer one has $\bar{\mu}_{m,v} = \bar{\mu}_{m+1,v} = 1$ and $b_{m,v} = \pi_m$ for all m. Equation 7 from lemma 1 then implies that

$z_{m,v} = v$ for all m. Equation 8 from lemma 1 then establishes that $b_{m,v}$ equals to $\pi^*_{\Delta,v}$ for all Δ. Finally, note that $1 - F_{v,\Delta_n}(t)$ equals $\left(1 - \pi^*_{\Delta_n,v}\right)^{t/\Delta_n}$. Part 3 of the proposition then follows from L'Hoptial's rule. □

Delay arises in B_v in order to ensure that the seller's offer remains equal to v. Intuitively, if the seller were to make an offer strictly below v, then the buyer's best response is to accept the seller's offer for sure. As suggested by the example in section 3.1, the only credible way the buyer can do so is to accept for sure *every offer*. But if this were the case, the seller would surely make offers much higher than v. If the seller's price was strictly above v, then the buyer's best response is to surely reject the seller's offer. Again, the only way the buyer can do so in a credible fashion is to reject for sure every offer in period m. That, however, will mean that the buyer is inattentive (section 3.1), which cannot happen in my equilibrium refinement. Therefore, the seller must be charging v. The seller's first order condition (see equation 11) then implies that $b_{m,v}$ must be equal to $\pi^*_{\Delta,v}$, thereby leading to delay.

When the quality of the good is unknown to the buyer, a seller of a v quality good may set a price different than v. Still, if period m prices are either too high or too low, the buyer will choose either to automatically reject (i.e. $\pi_m = 0$) or accept every offer for sure (i.e. $\pi_m = 1$). Since we focus on attentive equilibria, neither of these can occur in equilibrium. Making sure that prices stay within an acceptable range then generates delay similarly to how ensuring $z_{m,v} = v$ created delay in B_v.

6 Negligible Attention Costs and Surplus Splitting

In many situations, it seems unreasonable that the cost of attention is large compared to the size of the economic surplus. Such situations may appear especially likely in an environment with frequent offers, which can be thought of as representing a face-to-face interaction between the buyer and the seller. In these environments, one may presume that the outcome would be very similar to that of the game with a fully rational buyer. Theorem 4 below shows that this is not the case.

I will say that $\left(\bar{U}_s, \bar{U}_b\right)$ are *frequent offer utilities* of $B_0(\kappa)$ if there exists a refining sequence $\{\Delta_n\}_{n=1}^{\infty}$ with a corresponding sequence of equilibria $\{(\mu^n, b^n, z^n)\}_{n=1}^{\infty}$ such that $E[U_i^n]$ converges to $\bar{U}_i$ for $i \in \{s, b\}$. Thus, $\bar{U}_i$ is the expected utility of player i in some frequent offers environment. To establish existence of frequent offer utilities, I state the following corollary of Theorem 3.

Corollary 2. *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a refining sequence, and take $\{(\mu^n, b^n, z^n)\}_{n=1}^{\infty}$ to be a sequence of corresponding equilibria. Then there exists a sub-sequence $\{(\mu^{n_k}, b^{n_k}, z^{n_k})\}_{k=1}^{\infty}$ such that $E[U_i^{n_k}]$ converges for all $i \in \{s, b\}$.*

Theorem 4 states that the buyer and the seller split the uncertain portion of the surplus when offers are frequent and attention costs are negligible. The seller still appropriates the sure portion of the surplus, which is v_l. However, the rest of the surplus is split evenly between the two players. In addition, in the zero κ limit there is no inefficiency. Thus, no surplus is lost neither due to delay nor due to costly attention.

Theorem 4. *Let $\left\{\left(\bar{U}_s^n, \bar{U}_b^n\right)\right\}_{n=1}^{\infty}$ be a sequence of frequent offer utilities of $B_0(\kappa_n)$ with $\kappa_n \to 0$. Then:*

$$\lim_{n\to\infty} \bar{U}_s^n = \frac{1}{2}\left(E[v] + v_l\right)$$
$$\lim_{n\to\infty} \bar{U}_b^n = \frac{1}{2}\left(E[v] - v_l\right)$$

Proof. See appendix. □

When κ vanishes, efficiency is restored because the equilibrium becomes similar to the one in the full information model. Namely, as κ goes to zero, any sequence of acceptable offers results in immediate agreement. Similarly, an unacceptable sequence of offers leads to disagreement until acceptable offers are made. Standard bargaining arguments then give that trade is immediate with the seller making offers that are just at the threshold between what is acceptable and what is not.

Which offers sit at the buyer's margin between accepting and rejecting depend on the buyer's marginal cost of attention. More specifically, these offers depend on the marginal attention costs of rejecting all of the seller's future offers (Lemma 1). When offers are frequent, this marginal cost converges to zero for v_l and remains negative for all other values of v. Because the marginal attention costs are negative at the limit, the buyer's marginal utility from rejecting is positive. Thus, the buyer's marginal attention costs effectively increase the value she assigns to future periods.

The reason marginal costs remain negative as κ vanishes is that rejecting forever can immediately and completely misinform the buyer. In any equilibrium, as time goes by the buyer's belief that the good's quality is v_l goes to 1. The buyer's beliefs evolve in this way due the buyer accepting good offers more frequently than she accepts bad ones. Since the worst offers are made when the seller has the lowest quality product (Proposition 2), the buyer's belief that the good's quality is v_l increases over time. As κ goes to zero, this process accelerates, leading the buyer to believe that the good's quality is v_l with probability 1 within a blink of an eye. Thus, if the buyer were to reject the offers of a seller with $v \neq v_l$ for even a second, she would immediately believe that v has a probability of 0. Since no quantity of Bayesian updating can make a zero belief positive, such signals would be *infinitely misinformative*. Therefore, if κ was to remain positive, the marginal cost of rejecting forever would have converged to negative infinity. This convergence competes with the convergence of κ to zero, eventually leading to a strictly negative limit.

To get a better sense of how infinite marginal attention costs can lead to a result like Theorem 4, consider the following one-period version of equation 9 from section 3.2:

$$v - z_{1,v} - \kappa \ln\left(\frac{b_{1,v}}{\pi_1}\right) = -\kappa \ln\left(\frac{1 - b_{1,v}}{1 - \pi_1}\right) \tag{13}$$

For illustrative purposes, suppose that the probability of a reject signal conditional on v satisfies:

$$b_{1,v} = 1 - (1 - \pi_1)\, e^{-(v - v_l)/2\kappa}$$

Then as κ goes to zero, $b_{1,v}$ converges to 1 for all v, meaning that π_1 converges to 1 as well. In addition, for every level of κ the right hand side of equation 13 is equal to $(v - v_l)/2$. Therefore, as κ goes to 0 equation 13 becomes:

$$v - z_{1,v} = \frac{1}{2}(v - v_l)$$

Thus, a v seller's offer converges to $(v + v_l)/2$, and the buyer's acceptance probability converges to 1.

In equilibrium the buyer does not use a lone signal as described above. However, as κ goes to zero in a frequent offers setting the effect of the buyer's signal sequence in the game's first few moments converges to that of the above single signal. Consequently, trade becomes efficient and the buyer obtains half of uncertain portion of the surplus.

7 More Information, Lower Surplus

The current section explores the efficiency implications of revealing the good's quality to the buyer. To do so, I compare the equilibrium payoffs in the standard game to the payoffs when the buyer knows v. As shown in Proposition 4 in section 5, revealing v to the buyer results in a unique equilibrium. Corollary 3 below establishes that this equilibrium involves the seller appropriating all of the surplus. Moreover, the corollary shows that revealing v to the buyer results in an inefficiency of size κ. The corollary follows from Proposition 4 and Lemma 1.

Corollary 3. *The expected utilities of the buyer and the seller in the equilibrium of* $B_v(\Delta)$ *are* 0 *and* $v - \kappa$*, accordingly.*

Proof. By Proposition , $z_{m,v} = v$ for all v. Therefore, by equation 5 of Lemma 1, the seller's expected utility is equal to $v - \kappa$. To get that the buyer's expected utility is zero, note that we have both $b_{m,v} = \pi_m$ for all m and $z_{m,v} = v$ for all v. The first implies that the buyer's attention costs are zero, while the second means that her expected transaction utility is also zero. Thus, the buyer's surplus is zero. □

Corollary 7 suggests that in B_v, attention is effortless in equilibrium. The reasion attention is effortless is because the seller uses a deterministic strategy. Since in equilibrium the buyer knows both the seller's strategy and v, the buyer's knowledge includes all there is to know about the seller's offers, resulting in zero attention costs.

The fact that attention is effortless on the equilibrium path does not mean that the buyer perfectly observes the seller's offers. Indeed, perfectly observing the seller's offers would constitute a non-credible attention threat. Such threats are ruled out by my equilibrium refinement. In particular, my refinement requires the buyer to take into account the potential marginal cost of paying attention to off-equilibrium offers. These marginal costs induce the buyer to only partially adjust her acceptance probability in reaction to zero probability offers. The interaction between this partial adjustment and the seller's incentives that leads to delay in equilibrium, which remains when the buyer knows the quality of the good.

Proposition 5 wishes to compare the welfare in the original game, $B(\Delta, \kappa)$, to the welfare obtained by revealing the good's quality to the buyer. For this, consider a hypothetical game in which both the buyer and the seller get to observe v before the bargaining stage. By Corollary 3 this hypothetical game will give the seller an expected utility of $E[v] - \kappa$ while the buyer's ex-ante surplus will be equal to zero. Hence, the total surplus when the quality is revealed to the buyer is given by the expected value of v minus κ. Proposition 5 below shows that both the buyer's surplus and overall efficiency are strictly higher when the quality of the good is not announced to the buyer. Therefore, revealing information in an environment with rational inattention could have negative consequences on efficiency and the utility of the inattentive individuals.

Proposition 5. *There exists $\delta > 0$ and $\tau > 0$ such that for every equilibrium the total expected surplus is larger than $E[v] - \kappa + \tau$ and the buyer's expected utility is strictly larger than δ.*

Proof. See appendix. □

Proposition 5 asserts that revealing the quality of the product to the buyer results in reduced efficiency and lower surplus for the buyer across all equilibria. From the buyer's perspective, being ignorant of v results in a variation in the value of the seller's offers. This variation generates positive attention costs. Since attention costs are strictly convex and the buyer is attentive, she earns a strictly positive surplus.

As for overall efficiency, keeping the buyer ignorant of v creates positive attention costs but reduces the overall efficiency due to delay. The size of the overall inefficiency, however, turns out to be convex in the seller's expected profits conditional on v. When the buyer is uncertain about the quality of the good, the distribution of the seller's conditional expected profits becomes more concentrated, thereby reducing the overall inefficiency. Thus, keeping the buyer in the dark with respect to the quality of the good results in less delay that more than compensated for the buyer's positive attention costs. Thus, when attention is costly, improving the information of the inattentive buyer

8 Concluding Remarks

In this paper I showed that introducing rational inattention into one-sided repeated offers bargaining gives rise to many features reminiscent of real-world transactions. A rationally inattentive buyer earns a strictly positive surplus, even when attention costs are negligible (Theorem 4). When attention costs are positive, trade occurs with delay that is decreasing with the value of the good (Proposition 3). The resulting delay is accompanied by the buyer being unhappy with cheap, low quality products, and pleased with expensive products of higher quality (Proposition 2). Finally, I've shown that in the presence of rational inattention, ignorance is bliss in the sense that both the buyer's and the total surplus are higher when the buyer does not know the good's quality (Proposition 5).

My starting point was perhaps the simplest of dynamic bargaining models: One-sided repeated offers with full information. It remains an open question how much of my insights survive the transition to other, more complex bargaining environments. Examples of such environments include the buyer having private information, two-sided offers, existence of an the outside option, etc. In addition, there are some questions that remain open even with respect to the simple bargaining protocol studied in my paper. For example, in the infinite horizon model, I focused on an equilibrium which was the limit of finite horizon equilibria. This refinement resulted in equilibria that were completely forward looking and depended very little on past offers (Theorem 2). It remains to be seen whether it is possible to obtain qualitatively different results in more complicated equilibria .

In addition to exploring variations on the bargaining environment and solution concept, it would be of interest to study the implications of more structured models of attention on bargaining. As mentioned in the introduction, I view rational inattention as extending utility maximization to include costly attention. Thus, just as people study how deviation from utility maximization influence economic outcomes, so it would be of interest to study the interaction of bargaining and other ways of modeling limited attention.

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A Information Theory Preliminaries

Lemma 2 (Log-Sum inequality)**.** *Let* $(a_i)_{i=1}^n$ *and* $(b_i)_{i=1}^n$ *be non-negative numbers. Then:*

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right) \ln \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

with equality if and only if $\frac{a_i}{b_i}$ *is constant.*

Proof. The function $f(c) = c \ln c$ is strictly convex since $f^{''}(c) = \frac{1}{c} > 0$. Set $c_i = \frac{a_i}{b_i}$ and set $\alpha_i = \frac{b_i}{\sum_{i=1}^{n} b_i}$. Then by Jensen's inequality:

$$\sum_{i=1}^{n} \frac{a_i}{\sum_j b_j} \ln \frac{a_i}{b_i} = \sum_i \alpha_i f(c_i) > f\left(\sum_i \alpha_i c_i\right) = \frac{\sum_{i=1}^{n} a_i}{\sum_{j=1}^{n} b_j} \ln \left(\frac{\sum_{i=1}^{n} a_i}{\sum_{j=1}^{n} b_j}\right)$$

the lemma follows. □

Lemma 3 (Chain rule for mutual information)**.** *Let* $(Z, \mathcal{F}, p)$ *be some probability space, and let* $l : S^2 \times Z \to [0,1]$ *be such that* $l(s_1, s_2; \cdot)$ *is measurable for all* s_1, s_2 *and* $\sum_{(s_1,s_2)} l(s_1, s_2; z) = 1$ *for all* z*. Define* $p_{l s_1}$ *by setting:*

$$p_{l s_1}(E) = \frac{\int_E \sum_{s_2} l(s_1, s_2; z) \, \mathrm{d}p(z)}{\int_Z \sum_{s_2} l(s_1, s_2; z) \, \mathrm{d}p(z)}$$

take:

$$\bar{l}_1(s_1; z) = \sum_{s_2} l(s_1, s_2; z)$$

and for every s_1 *let* $l_{|s_1} : S \times Z \to [0,1]$ *be:*

$$l_{2|s_1}(s_2; z) = \frac{l(s_1, s_2; z)}{\sum_{s_2^{'}} l\left(s_1, s_2^{'}; z\right)}$$

then:

$$\mathbf{I}(l; p) = \mathbf{I}\left(\bar{l}_1; p\right) + \sum_{s_1} \left(\int_Z \bar{l}_1(s_1; z) \, \mathrm{d}p\right) \mathbf{I}\left(l_{2|s_1}; p_{l s_1}\right)$$

Proof. Note that:

$$\begin{aligned}
\mathbf{I}\left(l_{2|s_1};p_{l s_1}\right) &= \int_Z \sum_{s_2} l_{2|s_1}\left(s_2;z\right)\ln\left(\frac{l_{2|s_1}\left(s_2;z\right)}{\int l_{2|s_1}\left(s_2;z\right)\mathrm{d}p_{l s_1}}\right)\mathrm{d}p_{l s_1} \\
&= \int_Z \sum_{s_2} l_{2|s_1}\left(s_2;z\right)\ln\left(\frac{\left(\frac{l(s_1,s_2;z)}{\sum_{s_2} l(s_1,s_2;z)}\right)}{\int\left(\frac{l\left(s_1,s_2;z\right)}{\int_Z\sum_{s_2} l\left(s_1,s_2;z\right)\mathrm{d}p\left(z\right)}\right)\mathrm{d}p}\right)\mathrm{d}p_{l s_1} \\
&= \int_Z \sum_{s_2} l_{2|s_1}\left(s_2;z\right)\begin{pmatrix}\ln\left(\frac{l\left(s_1,s_2;z\right)}{\int l\left(s_1,s_2;z\right)\mathrm{d}p}\right) \\ -\ln\left(\frac{\bar{l}\left(s_1;z\right)}{\int_Z \bar{l}\left(s_1;z\right)\mathrm{d}p}\right)\end{pmatrix}\mathrm{d}p_{l s_1} \\
&= \int_Z \sum_{s_2} l_{2|s_1}\left(s_2;z\right)\ln\left(\frac{l\left(s_1,s_2;z\right)}{\int l\left(s_1,s_2;z\right)\mathrm{d}p}\right)\mathrm{d}p_{l s_1} \\
&\quad -\int_Z \ln\left(\frac{\bar{l}\left(s_1;z\right)}{\int_Z \bar{l}\left(s_1;z\right)\mathrm{d}p}\right)\mathrm{d}p_{l s_1}
\end{aligned}$$

and therefore,$\sum_{s_1}\left(\int_Z \bar{l}_1\left(s_1;z\right)\mathrm{d}p\right)\mathbf{I}\left(l_{2|s_1};p_{l s_1}\right)$ is equal to:

$$\begin{aligned}
&= \sum_{s_1}\left(\int_Z \bar{l}_1\left(s_1;z\right)\mathrm{d}p\right)\int_Z\sum_{s_2} l_{2|s_1}\left(s_2;z\right)\ln\left(\frac{l\left(s_1,s_2;z\right)}{\int l\left(s_1,s_2;z\right)\mathrm{d}p}\right)\mathrm{d}p_{l s_1} \\
&\quad -\sum_{s_1}\left(\int_Z \bar{l}_1\left(s_1;z\right)\mathrm{d}p\right)\int_Z \ln\left(\frac{\bar{l}\left(s_1;z\right)}{\int_Z \bar{l}\left(s_1;z\right)\mathrm{d}p}\right)\mathrm{d}p_{l s_1} \\
&= \int_Z\sum_{s_1}\sum_{s_2}\bar{l}\left(s_1;z\right)l_{2|s_1}\left(s_2;z\right)\ln\left(\frac{l\left(s_1,s_2;z\right)}{\int l\left(s_1,s_2;z\right)\mathrm{d}p}\right)\mathrm{d}p \\
&\quad -\int_Z\sum_{s_1}\bar{l}\left(s_1;z\right)\ln\left(\frac{\bar{l}\left(s_1;z\right)}{\int_Z \bar{l}\left(s_1;z\right)\mathrm{d}p}\right)\mathrm{d}p \\
&= \mathbf{I}\left(p;l\right)-\mathbf{I}\left(p;\bar{l}_1\right)
\end{aligned}$$

thereby implying the desired equality. □

B Sufficiency of recommendation strategies

This part of the appendix shows that it is sufficient to focus on recommendation strategies. The section begins by introducing required notation and the definition of general strategies for the buyer. Then, the section defines what it means for two strategies to be outcome equivalent and when we call a strategy for the buyer as a recommendation strategy. I then present the proof that recommendation strategies are sufficient by showing that for each strategy we can construct a sequence of outcome equivalent strategies with lower information costs that converge to a recommendation strategy.

Let Y is some compact subspace of a finite dimensional Euclidean space. Defining $Z_{m-1} = X^m \times Y$, we take σ_0 to be a borel probability measure over $Z_0 = Y$, and $(\sigma_m)_{m=1}^M$ to be a sequence of probability transition kernels: $\sigma_m : Z_{m-1} \to \Delta(X)$ that give the conditional probability over the m time fundamental X_m given $z_{m-1} = \left(x^{m-1}, y\right)$. For $n < m$ and $z_m \in Z_M$, we will take $z_m(n)$ to denote the projection of z_m on Z_n. Throughout we will focus on the peirod 0 problem without loss of generality. For $M \in \mathbb{N} \cup \{\infty\}$, we let $\sigma \in \Delta\left(Y \times X^M\right)$ denote the probability measure derived by repeatedly application of $(\sigma_m)_{m=0}^M$. Similarly, we will write: $\sigma(\cdot|z_n) \in \Delta(Z_{M-n})$ for the probability distribution over $\{z_n\} \times X^{M-n}$ resulting from repeated application of $(\sigma_m)_{m=n+1}^M$. For the sake of brevity, we write $\sigma(E_m)$ instead of $\sigma\left(E_m \times X^{M-m}\right)$ for any $E_m \in \mathcal{B}(Z_m)$.

A strategy for the buyer consists of a pair (λ, β), such that $\lambda = (\lambda_m)_{m=0}^M$ and $\beta = (\beta_m)_{m=0}^M$, where $\lambda_m : Z_m \times S^{m-1} \to \Delta(S)$ and $\beta_m : S^m \to [0,1]$. Thus, $\lambda_m\left(s_m|z_m, s^{m-1}\right)$ is the probability of observing signal s_m conditional on past signals being s^{m-1} and the state up to time m being z_m, while $\beta_m(s^m)$ is the probability of the buyer stopping at period m conditional on having observed the signals s^m. Given (λ, β), we let $\mu_{\lambda,\beta}(\cdot|s^m)$ be the posterior distribution over Z_M conditional on having reached any period $n \geq m$ and having observed signals s^m from λ.

Definition 4. Two strategies (λ, β) and $\left(\lambda^{'}, \beta^{'}\right)$ are *outcome equivalent* if for every σ and every m, the probability that the buyer stops at period m is the same under both (λ, β) and $\left(\lambda^{'}, \beta^{'}\right)$.

Definition 5. A strategy (λ, β) is a *recommendation strategy* if for every m:

$$\lambda_m\left(s_m|z_m, s^{m-1}\right) > 0$$

implies $s_m \in \{0, 1\}$, and $\beta_m(s^m) = 0$ if $s_m = 0$ and $\beta_m(s^m) = 1$ otherwise.

We will prove here the following proposition:

Proposition 6. *For every (λ, β) there exists an outcome equivalent recommendation strategy $\left(\lambda^{'}, \beta^{'}\right)$ that is weakly better for the buyer than (λ, β).*

To prove the proposition fix some (λ, β). We will construct the strategy (λ^*, β^*) in the following way. Let $\lambda_0^*(0|z_0) = \sum_{s_0} \lambda_0(s_0|z_0)(1 - \beta_0(s_0))$ and $\lambda_0^*(1|z_0) = \sum_{s_0} \lambda_0(s_0|z_0)\beta_0(s_0)$. Set $\beta_0^*(0) =$

0 and $\beta_0^*(1) = 1$. Define the period 0 likelihood l_0 by:

$$l_0(s_0|z_0) = \frac{\lambda_0(s_0|z_0)(1-\beta_0(s_0))}{\lambda_0^*(0|z_0)}$$

and let $\lambda_1^*(s_1, s_0|z_1, 0) = \lambda_1(s_1|z_1, s_0)\, l_0(s_0|z_0)$. Note that we can let λ_1^* send signals in $S \times S$ since S is countably infinite and therefore there exists a bijection from S to $S \times S$. For every $m \geq 1$, let $\beta_m^*(s^m, 0) = \beta_m(s^m)$ and for any $m \geq 2$ take $\lambda_m^*\left(s_m|z_m, s^{m-1}, 0\right) = \lambda_m\left(s_m|z_m, s^{m-1}\right)$.

Lemma 4. (λ, β) *and* (λ^*, β^*) *are outcome equivalent*

Proof. Obviously the probability of stopping in period 0 is the same under both. As for stopping in period 1, the probability under (λ^*, β^*) given some z_1 is:

$$\lambda_0^*(0|z_0) \sum_{s_0, s_1} \lambda_1(s_1|z_1, s_0)\, l_0(s_0|z_0)(1-\beta_1(s_0, s_1)) =$$
$$\sum_{s_0, s_1} \lambda_1(s_1|z_1, s_0)\, \lambda_0(s_0|z_0)(1-\beta_0(s_0))(1-\beta_1(s_0, s_1))$$

which is the same probability of stopping under (λ, β). It therefore follows that the probability at any period m given any z_m is the same under both, hence the two are outcome equivalent. □

Lemma 5. *The expected present value of the cost of attention is lower under* (λ^*, β^*) *than under* (λ, β).

Proof. Clearly, $\mu_{\lambda^*, \beta^*}(\cdot|s^m, 0) = \mu_{\lambda, \beta}(\cdot|s^m)$ for every $m \geq 1$, and therefore

$$\mathbf{I}\left(\lambda_m\left(\cdot|s^{m-1}\right); \mu_{\lambda, \beta}\left(\cdot|s^{m-1}\right)\right) =$$
$$\mathbf{I}\left(\lambda_m^*\left(\cdot|s^{m-1}, 0\right); \mu_{\lambda^*, \beta^*}\left(\cdot|s^{m-1}, 0\right)\right)$$

We will now show that:

$$\begin{aligned} &\mathbf{I}(\lambda_0^*; \sigma) \\ &+e^{-r\Delta} \int \lambda_0^*(0|z_0)\, \mathrm{d}\sigma \\ &\times \mathbf{I}\left(\lambda_1^*(\cdot|0), \mu_{\lambda^*, \beta^*}(\cdot|0)\right) \quad \leq \quad \mathbf{I}(\lambda_0; \sigma) \\ &\qquad + e^{-r\Delta} \int \begin{pmatrix} \sum_{s_0} \lambda_0(s_0|z_0)(1-\beta_0(s_0)) \\ \times \mathbf{I}\left(\lambda_1(\cdot|s_0), \mu_{\lambda, \beta}(\cdot|s_0)\right) \end{pmatrix} \mathrm{d}\sigma \end{aligned} \tag{14}$$

Let:

$$\begin{aligned}\lambda_0\left(s|z_0,\lambda_0^*=1\right) &= \frac{\beta_0\left(s\right)\lambda_0\left(s|z_0\right)}{\sum_{s_0}\beta_0\left(s_0\right)\lambda_0\left(s_0|z_0\right)} \\ &= \frac{\beta_0\left(s\right)\lambda_0\left(s|z_0\right)}{\lambda_0^*\left(1|z_0\right)} \equiv l_1\left(s|z_0\right)\end{aligned}$$

and:

$$\begin{aligned}\lambda_0\left(s|z_0,\lambda_0^*=0\right) &= \frac{\left(1-\beta_0\left(s\right)\right)\lambda_0\left(s|z_0\right)}{\sum_{s_0}\left(1-\beta_0\left(s_0\right)\right)\lambda_0\left(s_0|z_0\right)} \\ &= \frac{\left(1-\beta_0\left(s\right)\right)\lambda_0\left(s|z_0\right)}{\lambda_0^*\left(0|z_0\right)} = l_0\left(s|z_0\right)\end{aligned}$$

then by the chain rule of mutual information (Lemma 3) and non-negativity of mutual information:

$$\begin{aligned}\mathbf{I}\left(\lambda_0;\sigma\right) &= \mathbf{I}\left(\lambda_0^*;\sigma\right)+\int\sum_{s_0^*=0,1}\lambda_0^*\left(s_0^*|z_0\right)\mathbf{I}\left(l_{s_0^*};\mu_{\lambda^*,\beta^*}\left(\cdot|s_0^*\right)\right)\mathrm{d}\sigma \\ &\geq \mathbf{I}\left(\lambda_0^*;\sigma\right)+\int\lambda_0^*\left(0|z_0\right)\mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right)\mathrm{d}\sigma\end{aligned}$$

while $\mathbf{I}\left(\lambda_1^*\left(\cdot|0\right),\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right)$ is equal to:

$$\begin{aligned}&= \mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right) \\ &\quad +\sum_{s_0}\int\begin{pmatrix} l_0\left(s_0|z_0\right)\times \\ \mathbf{I}\left(\lambda_1\left(\cdot|s_0\right);\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\mu_{\lambda^*,\beta^*}\left(z_0|0\right) \\ &= \mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right) \\ &\quad +\int\begin{pmatrix}\left(\frac{\lambda^*\left(0|z_0\right)}{\int\lambda^*\left(0|z_0\right)\mathrm{d}\sigma}\right)\times \\ \sum_{s_0}\begin{pmatrix} l_0\left(s_0|z_0\right) \\ \times\mathbf{I}\left(\lambda_1\left(\cdot|s_0\right);\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\end{pmatrix}\mathrm{d}\sigma\end{aligned}$$

which is equal to:

$$\begin{aligned}&= \mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(0\right)\right) \\ &\quad +\int\begin{pmatrix}\left(\frac{\sum_{s_0}\lambda_0\left(s_0|z_0\right)\left(1-\beta_0\left(s_0\right)\right)}{\int\lambda^*\left(0|z_0\right)\mathrm{d}\sigma}\right) \\ \times\mathbf{I}\left(\lambda_1\left(\cdot|s_0\right);\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\sigma\end{aligned}$$

therefore:

$$\begin{aligned}
&\mathbf{I}\left(\lambda_0;\sigma\right) \\
&+e^{-r\Delta}\int\begin{pmatrix}\sum_{s_0}\lambda_0\left(s_0|z_0\right)\left(1-\beta_0\left(s_0\right)\right)\times\\ \mathbf{I}\left(\lambda_1\left(\cdot|s_0\right),\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\sigma \quad\geq\quad \mathbf{I}\left(\lambda_0^*;\sigma\right) \\
&\qquad +\int\lambda_0^*\left(0|z_0\right)\mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(0\right)\right)\mathrm{d}\sigma \\
&\qquad +e^{-r\Delta}\int\begin{pmatrix}\sum_{s_0}\lambda_0\left(s_0|z_0\right)\left(1-\beta_0\left(s_0\right)\right)\times\\ \mathbf{I}\left(\lambda_1\left(\cdot|s_0\right),\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\sigma
\end{aligned}$$

which is weakly larger than:

$$\begin{aligned}
&\geq\mathbf{I}\left(\lambda_0^*;\sigma\right) \\
&+e^{-r\Delta}\int\lambda_0^*\left(0|z_0\right)\mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(0\right)\right)\mathrm{d}\sigma \\
&+e^{-r\Delta}\int\begin{pmatrix}\sum_{s_0}\lambda_0\left(s_0|z_0\right)\left(1-\beta_0\left(s_0\right)\right)\times\\ \mathbf{I}\left(\lambda_1\left(\cdot|s_0\right),\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\sigma
\end{aligned}$$

which is equal to:

$$\begin{aligned}
= &\quad \mathbf{I}\left(\lambda_0^*;\sigma\right)+e^{-r\Delta}\int\lambda_0^*\left(0|z_0\right)\mathrm{d}\sigma\times \\
&\left(\begin{array}{c}\mathbf{I}\left(l_0;\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right)+\\ \int\begin{pmatrix}\sum_{s_0}l_0\left(s_0|z_0\right)\times\\ \mathbf{I}\left(\lambda_1\left(\cdot|s_0\right);\mu_{\lambda,\beta}\left(\cdot|s_0\right)\right)\end{pmatrix}\mathrm{d}\mu_{\lambda^*,\beta^*}\left(z_0|0\right)\end{array}\right) \\
= &\quad \mathbf{I}\left(\lambda_0^*;\sigma\right) \\
&+e^{-r\Delta}\begin{pmatrix}\int\lambda_0^*\left(0|z_0\right)\mathrm{d}\sigma\\ \times\mathbf{I}\left(\lambda_1^*\left(\cdot|0\right),\mu_{\lambda^*,\beta^*}\left(\cdot|0\right)\right)\end{pmatrix}
\end{aligned}$$

as required. □

Note that together, lemmas 4 and 5 imply that (λ^*,β^*) give the buyer a utility at least as high as (λ,β) while achieving the same distribution over outcomes.

Construct now the following sequence: $\left(\lambda^0,\beta^0\right)=(\lambda^*,\beta^*)$. For every n, construct (λ^n,β^n) by taking $\left(\lambda^{n-1},\beta^{n-1}\right)$ and replacing the strategies starting at periods n and $n+1$ in a similar way that λ_0^*, β_0^*, λ_1^* and β_1^* replaced λ_0, β_0, λ_1 and β_1. It is easy to verify that (λ^n, β^n) converges to some limit $(\lambda^\infty, \beta^\infty)$, and that $(\lambda^\infty, \beta^\infty)$ is a recommendation strategy. Moreover, by 4 and 5 that $(\lambda^\infty, \beta^\infty)$ is both outcome equivalent to, and is better for the buyer than,(λ, β).

C Optimal stopping with rational inattention

In this section we characterize the optimal recommendation strategy for the buyer under general settings. The generalization is modest in that we allow for a more general space of outcomes and for a more general relation between these outcomes and the buyer's payoffs. Thus, the results provided here are given for a general stopping problem under rational inattention in which the unknown evolves according to a sequence of probability transition kernels, and is not limited to the one-sided bargaining setup studied in the paper. The specialization of the results to the setup in the paper should be pretty straightforward. This is despite the notation in this section being completely self contained. To make it as easy as possible to move from this section to the paper, I have attempted to keep the notation of this section as similar as possible to that of the paper.

C.1 Two formulations of the optimal stopping problem

Let Y is some compact subspace of a finite dimensional Euclidean space. Defining $Z_{m-1} = X^m \times Y$, we take σ_0 to be a borel probability measure over $Z_0 = Y$, and $(\sigma_m)_{m=1}^M$ to be a sequence of probability transition kernels: $\sigma_m : Z_{m-1} \to \Delta(X)$ that give the conditional probability over the m time fundamental X_m given $z_{m-1} = \left(x^{m-1}, y\right)$. For $n < m$ and $z_m \in Z_M$, we will take $z_m(n)$ to denote the projection of z_m on Z_n. Throughout we will focus on the peirod 1 problem without loss of generality. For $M \in \mathbb{N} \cup \{\infty\}$, we let $\sigma \in \Delta\left(Y \times X^M\right)$ denote the probability measure derived by repeatedly application of $(\sigma_m)_{m=1}^M$. Similarly, we will write: $\sigma(\cdot|z_n) \in \Delta(Z_{M-n})$ for the probability distribution over $\{z_n\} \times X^{M-n}$ resulting from repeated application of $(\sigma_m)_{m=n+1}^M$. For the sake of brevity, we write $\sigma(E_m)$ instead of $\sigma\left(E_m \times X^{M-m}\right)$ for any borel set $E_m \subset Z_m$.

A binary stopping strategy is characterized by a collection of mappings $(\beta_m)_{m=0}^M$ where $\beta_m : Z_m \to [0,1]$ is measurable for each m. $\beta_m(z_m)$ represents the conditional probability that the buyer decides to stop at period m given that m has been reached and the fundamental is z_m. We take $\mathbf{B}$ to be the set of all such mappings defined up to σ-almost sure equality. For a fixed β, let $\beta_{m,n} = (\beta_m, \beta_{m+1}, \ldots, \beta_n)$ for $n \geq m$. We will write l_{β_m} as short hand for the m likelihood defined by $l_{\beta_m}(0; z_m) = 1 - \beta_m(z_m)$ and $l_{\beta_m}(1; z_m) = \beta_m(z_m)$, abuse notation by writing for every $p \in \Delta(Z_M)$: $\mathbf{I}(\beta_m; p)$ instead of $\mathbf{I}(l_{\beta_m}; p)$. Understanding that β_m is a function of $z_m \in Z_M$, we will often use $\beta_m(z_M)$ instead of $\beta_m(z_M(m))$. Moreover, when there is no risk of confusion about z_M, we will sometimes write β_m instead of $\beta_m(z_m)$.

We now turn to defining the buyer's objective function. As in the main paper, for any sequence of constants $c^\infty \in \mathbb{R}^\infty$, we will let $\prod_{j=n}^N c_j = 1$ whenever $N < n$. Let $v(z_m)$ be the value of the buyer from stopping at perod m given fundamental z_m. We assume that v is Borel measurable and takes value in $[\underline{v}, \bar{v}]$ for $-\infty < \underline{v} < 0 < \bar{v} < \infty$, thereby implying that v is integrable. For $z_M \in Z_M$, we will take $v_m(z_M) = v(z_M(m))$, and again abuse notation and write v_m instead of $v_m(z_M)$ whenever there is no risk of confusion.

Let $\mu_{\beta,m}$ be the buyer's posterior over Z_M conditional on having used β and reaching the beginning of period m. For any $m \geq 0$, define:

$$U_m(\beta;\sigma) = \int_{Z_M} \prod_{n=1}^{m-1} (1-\beta_n)\left(\beta_m v_m - \kappa \mathbf{I}(\beta_m;\mu_{\beta,m})\right) \mathrm{d}\sigma$$

Taking:

$$\mathcal{U}(\beta;\sigma) = \sum_{j=1}^{M} e^{-r\Delta(j-1)} U_j(\beta;\sigma)$$

We can then define the buyer's time 1 problem as:

$$\max_{\beta\in\mathbf{B}} \mathcal{U}(\beta;\sigma) \quad (15)$$

To state our theorem, we need to introduce the concept of the buyer's quasi-value. Letting $\pi_m = \int \beta_m \mathrm{d}\mu_{\beta,m}$ For a given z_m, let:

$$\begin{aligned} U_m^*(\beta|z_m) &= \beta_m v_m - \kappa\beta_m \ln(\beta_m/\pi_m) \\ &\quad -\kappa(1-\beta_m)\ln((1-\beta_m)/(1-\pi_m)) \end{aligned}$$

It is not too difficult to see that we can rewrite the buyer's objective function as:

$$\mathcal{U}(\beta;\sigma) = \sum_{m=1}^{M} \int \left(\prod_{j=1}^{m-1}(1-\beta_j)\right) e^{-r\Delta(m-1)} U_m^*(\beta|z_m)\,\mathrm{d}\sigma$$

This way of writing the buyer's objective function suggests a useful way of thinking of the buyer's continuation values. In particular, we will define the quasi-value of the buyer conditional on arriving to period $n \geq m$ and knowing z_m:

$$\mathcal{U}_n(\beta;\sigma|z_m) = \int \left(\sum_{j=n}^{M} e^{-r\Delta(j-n)} \prod_{k=n}^{j-1}(1-\beta_k)\, U_j^*\left(\beta|x^M, v\right)\right) \mathrm{d}\sigma\left(z^M|z_m\right)$$

as the conditional 'value' of the buyer conditional on z_m and arriving to period n. With this notation in hand, we will turn to prove the following theorem:

Theorem 5. *β solves the problem 15 only if:*

$$\beta_m(z_m) = \frac{\pi_m e^{\frac{1}{\kappa}v_m(z_m)}}{\pi_m e^{\frac{1}{\kappa}v_m(z_m)} + (1-\pi_m)\, e^{\frac{1}{\kappa}e^{-r\Delta}\mathcal{U}_{m+1}(\beta;\sigma|z_m)}} \quad (16)$$

Moreover, if β satisfies the above conditions with $\pi_m \in (0,1)$ for all m as well as:

$$\lim_{m\to\infty}\int\left(e^{-r\Delta(m-n)}\ln\left(\frac{1-\beta_m}{1-\pi_m}\right)\right)\mathrm{d}\sigma\left(z^M|z_n\right)=0 \tag{17}$$

σ-almost surely, then β_m is optimal.

To solve this program, we will solve an equivalent program in which the buyer chooses the ex-ante probability that she will stop in period m conditional on z_m. More specifically, take $\mathbf{H}$ to be the set of all measurable mappings $h : Z_M \to [0,1]^M$ defined up to σ-almost sure equality that satisfy: (1) $\sum_{m=0}^{M} h_m(z_M) \leq 1$ and (2) $z_M(m) = z_M^{'}(m)$ implies $h_m(z_M) = h_m\left(z_M^{'}\right)$. Then we are interested in maximizing a functional equivalent to the one in 15 defined over $\mathbf{H}$ instead of $\mathbf{B}$. It is easy to see that $\mathbf{H}$ is convex. We think of $h_m(z_M)$ as the probability that the buyer stops at period m conditional on the fundamental up to and including period m being equal to $z_M(m) = \left(x^{m+1}, y\right)$. When the choice of $z_M \in Z_M$ is clear, we will sometime drop z_M and simply write h_j instead of $h_j(z_M)$.

Fixing some $h \in \mathbf{H}$ and some m, we can define the corresponding conditional stopping strategies:

$$\begin{aligned}\beta_m^h(z_M) &= \frac{h_m}{1-\sum_{n=0}^{m-1}h_n}\\ 1-\beta_m^h(z_M) &= \frac{1-\sum_{n=0}^{m}h_n}{1-\sum_{n=0}^{m-1}h_n}\end{aligned}$$

whenever $\sum_{n=0}^{m-1} h_n < 1$ and have $\beta_m^h(z_M) = 0$ otherwise. Note that $\beta^h \in \mathbf{B}$, which allows us to write: $\mu_{h,m} = \mu_{\beta^h,m}$ and $\mathbf{I}(h_m;\mu_{h,m}) = \mathbf{I}\left(\beta_m^h;\mu_{\beta^h,m}\right)$. Take:

$$\tilde{U}_m(h;\sigma) = \int_{Z_M}\left(h_m(z_m)\,v(z_m) - \kappa\left(1-\sum_{n=0}^{m-1}h_n(z_m)\right)\mathbf{I}(h_m;\mu_{h,m})\right)\mathrm{d}\sigma$$

and define the functional:

$$\tilde{\mathcal{U}}(h;\sigma) = \sum_{m=0}^{M} e^{-r\Delta m}\tilde{U}_m(h;\sigma)$$

then we will consider the following ex-ante program:

$$\max_{h\in\mathbf{H}}\tilde{\mathcal{U}}(h;\sigma) \tag{18}$$

To establish the equivalence of these two programs, for every $\beta \in \mathbf{B}$ take h^β to be defined by:

$$h^\beta(z_M) = \prod_{n=0}^{m-1}\left(1-\beta_n(z_M)\right)\beta_m(z_M)$$

The the following facts are easy to verify and we therefore ommit their proof.

Fact 1. *For every $h \in \mathbf{H}$ and $\beta \in \mathbf{B}$: (1)$h^{\beta^h} = h$; (2) For every z_M for which $\prod_{n=0}^{m-1}(1-\beta_n(z_M)) > 0$: $\beta_n^{h^\beta}(z_M) = \beta_n(z_M)$. (3) $\mathcal{U}(\beta;\sigma) = \tilde{\mathcal{U}}(h^\beta;\sigma)$; (4) $\tilde{\mathcal{U}}(h;\sigma) = \mathcal{U}(\beta^h;\sigma)$.*

Given the above equivalence, we will focus on finding a solution to the program 18 from which we will back out the solution to 15. The main advantage of focusing on maximizing $\tilde{\mathcal{U}}$ rather than $\mathcal{U}$ will be that $\tilde{\mathcal{U}}$ is concave. We now turn to proving this fact.

C.2 $\tilde{U}$ is concave

Lemma 6. *Let $\varphi : Z_M \to [0,1]$ be some measurable function. Then the function: $f_{z_M}(\varphi) = \varphi(z_M)\ln\left(\frac{\varphi(z_M)}{\int \varphi d\sigma}\right)$ is convex in φ for all z_M.*

Proof. Fix φ and $\varphi^{'}$. Then by the log-sums inequality (2):

$$\alpha\varphi(z_M)\ln\left(\frac{\varphi(z_M)}{\int \varphi d\sigma}\right) + (1-\alpha)\varphi^{'}(z_M)\ln\left(\frac{\varphi^{'}(z_M)}{\int \varphi' d\sigma}\right)$$
$$\geq \left(\alpha\varphi(z_M) + (1-\alpha)\varphi^{'}(z_M)\right)\ln\left(\frac{\alpha\varphi(z_M) + (1-\alpha)\varphi^{'}(z_M)}{\int \alpha\varphi + (1-\alpha)\varphi' d\sigma}\right)$$

as required. □

Lemma 7. *$\tilde{\mathcal{U}}$ is concave in h.*

Proof. Note that we can write:

$$\begin{aligned}
\tilde{U}_m(h;\sigma) &= h_m v_m - \kappa\left(h_m \ln\left(\frac{\beta_m^h}{\int \beta_m^h d\mu_{h,m}}\right) + \left(1-\sum_{j=0}^m h_j\right)\ln\left(\frac{1-\beta_m^h}{\int (1-\beta_m^h) d\mu_{h,m}}\right)\right) \\
&= h_m v_m - \kappa\left(\begin{array}{c} h_m \ln\left(\frac{h_m}{\int h_m d\sigma}\right) + \left(1-\sum_{j=0}^m h_j\right)\ln\left(\frac{1-\sum_{j=0}^m h_j}{\int \left(1-\sum_{j=0}^m h_j\right) d\sigma}\right) \\ -\left(1-\sum_{j=0}^{m-1} h_j\right)\ln\left(\frac{1-\sum_{j=0}^{m-1} h_j}{\int \left(1-\sum_{j=0}^{m-1} h_j\right) d\sigma}\right)\end{array}\right)
\end{aligned}$$

Thus, letting:

$$\begin{aligned}
\tilde{U}_m^*(h;z_M) &= h_m v_m - \kappa h_m \ln\left(\frac{h_m}{\int h_m d\sigma}\right) \\
&\quad -\kappa\left(1-e^{-r\Delta}\mathbf{1}_{[m<M]}\right)\left(1-\sum_{j=0}^m h_j\right)\ln\left(\frac{1-\sum_{j=0}^m h_j}{\int \left(1-\sum_{j=0}^m h_j\right) d\sigma}\right)
\end{aligned}$$

we can rewrite:

$$\tilde{\mathcal{U}}(h;\sigma) = \int_{Z_M} \sum_{m=0}^{M} e^{-r\Delta m} \tilde{U}_m^*(h; z_M)\, \mathrm{d}\sigma \tag{19}$$

Concavity then follows from equation 19 and lemma 19. □

C.3 Necessary Conditions

Note that throughout this subsection we fix σ. Therefore, we will simply write $\tilde{\mathcal{U}}(h)$ instead of $\tilde{\mathcal{U}}(h;\sigma)$.

Lemma 8. *Let $f : [0,1] \to \mathbb{R}$ be concave. Then for all $0 \le \gamma \le \lambda < 1$:*

$$\frac{f(1) - f(\lambda)}{1-\lambda} \le \frac{f(1) - f(\gamma)}{1-\gamma}$$

Proof. Set $\alpha = \frac{1-\lambda}{1-\gamma}$. By concavity: $f(\lambda) \ge \alpha f(\gamma) + (1-\alpha) f(1)$. Therefore: $f(\lambda) \ge f(1) - \alpha(f(1) - f(\gamma))$, or: $\alpha(f(1) - f(\gamma)) \ge f(1) - f(\lambda)$. Dividing both sides by $1-\lambda$ gives the desired inequality. □

Lemma 9. *Suppose h maximizes $\tilde{\mathcal{U}}(h)$. Then both:*

$$\ln\left(\frac{1 - \sum_{k=0}^{m} h_j}{\int (1 - \sum_{k=0}^{m} h_j)\, \mathrm{d}\sigma}\right)$$

and

$$\sum_{j=m}^{M} e^{-r\Delta j} \ln\left(\frac{1 - \sum_{k=0}^{j} h_k}{\int \left(1 - \sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right)$$

are σ-integrable for all m.

Proof. We will begin by showing that $\sum_{j=m}^{M} e^{-r\Delta j} \ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int(1-\sum_{k=0}^{j} h_k)\mathrm{d}\sigma}\right)$ is integrable for $m = 0$. Since $\sum_{j=0}^{M} e^{-r\Delta j}\left(1 - \sum_{k=0}^{j} h_k\right) \ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int(1-\sum_{k=0}^{j} h_k)\mathrm{d}\sigma}\right) \ge 0$ is integrable, this implies that $\sum_{j=0}^{M} e^{-r\Delta j}\left(\sum_{k=0}^{j} h_k\right) \ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int(1-\sum_{k=0}^{j} h_k)\mathrm{d}\sigma}\right)$ is not.

Take h^0 to be such that $h_j^0 = 0$ for all j. Then by optimality of h and lemma 8 we have that for all $0 \le \gamma \le \lambda < 1$:

$$\begin{aligned} 0 \le \frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\lambda h + (1-\lambda) h^0\right)}{1-\lambda} &\le \frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\gamma h + (1-\gamma) h^0\right)}{1-\gamma} \\ &\le \tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(h^0\right) \end{aligned}$$

where the last inequality comes from setting $\gamma = 0$. Note that:

$$\frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\lambda h + (1-\lambda) h^0\right)}{1-\lambda} =$$

$$\int_{Z_M} \sum_{j=0}^{M} e^{-r\Delta j} \left(\begin{array}{c} h_j v_j - \kappa h_j \ln\left(\frac{h_j}{\int h_j \mathrm{d}\sigma}\right) - \kappa \left(\frac{1-\mathbf{1}_{[j<M]} e^{-r\Delta}}{1-\lambda}\right) \times \\ \times \left(\begin{array}{c} \left(1 - \sum_{k=0}^{j} h_j\right) \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int \left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) \\ - \left(1 - \lambda \sum_{k=0}^{j} h_k\right) \ln\left(\frac{\left(1-\lambda\sum_{k=0}^{j} h_k\right)}{\int \left(1-\lambda\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) \end{array} \right) \end{array} \right) \mathrm{d}\sigma$$

using the fact that for every $\alpha \in [0,1]$:

$$0 \leq -\left(\alpha \ln \alpha + (1-\alpha) \ln (1-\alpha)\right) \leq \ln 2$$

and that $\ln \alpha$ is concave, we obtain:

$$\begin{aligned} 0 \leq -\int h_j \ln\left(\frac{h_j}{\int h_j \mathrm{d}\sigma}\right) \mathrm{d}\sigma &= -\left(\int h_j \ln h_j \mathrm{d}\sigma - \left(\int h_j \mathrm{d}\sigma\right) \ln\left(\int h_j \mathrm{d}\sigma\right)\right) \\ &= -\int \left(h_j \ln h_j - \left(\int h_j \mathrm{d}\sigma\right) \ln\left(\int h_j \mathrm{d}\sigma\right)\right) \mathrm{d}\sigma \\ &\leq \ln 2 \end{aligned}$$

Thus, since v_j is bounded, we can rewrite:

$$\frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\lambda h + (1-\lambda) h'\right)}{1-\lambda} =$$

$$\int_{Z_M} \sum_{j=0}^{M} e^{-r\Delta j} \left(h_j v_j - \kappa h_j \ln\left(\frac{h_j}{\int h_j \mathrm{d}\sigma}\right)\right) \mathrm{d}\sigma$$

$$-\kappa \int_{Z_M} \sum_{j=0}^{M} \left(1 - e^{-r\Delta} \mathbf{1}_{[j<M]}\right) e^{-r\Delta j} \left(\begin{array}{c} \left(\frac{1-\sum_{k=0}^{j} h_j}{1-\lambda}\right) \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int \left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) \\ - \left(\frac{1-\lambda\sum_{k=0}^{j} h_k}{1-\lambda}\right) \ln\left(\frac{\left(1-\lambda\sum_{k=0}^{j} h_k\right)}{\int \left(1-\lambda\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) \end{array} \right) \mathrm{d}\sigma$$

Letting:

$$\zeta_j\left(z_M;\lambda\right)=\begin{pmatrix} \left(\frac{1-\sum_{k=0}^{j}h_j}{1-\lambda}\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right) \\ -\left(\frac{1-\lambda\sum_{k=0}^{j}h_k}{1-\lambda}\right)\ln\left(\frac{\left(1-\lambda\sum_{k=0}^{j}h_k\right)}{\int\left(1-\lambda\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\end{pmatrix}$$

we obtain that the following inequality:

$$\begin{aligned}
&\frac{1}{\kappa}\int_{Z_M}\sum_{j=0}^{M}e^{-r\Delta j}\left(h_jv_j-\kappa h_j\ln\left(\frac{h_j}{\int h_j\mathrm{d}\sigma}\right)\right)\mathrm{d}\sigma\\
&\geq\int_{Z_M}\sum_{j=0}^{M}\left(1-e^{-r\Delta}\mathbf{1}_{[j<M]}\right)e^{-r\Delta j}\zeta_j\left(z_M;\lambda\right)\mathrm{d}\sigma\\
&\geq\frac{1}{\kappa}\int_{Z_M}\sum_{j=0}^{M}e^{-r\Delta j}\left(h_jv_j-\kappa h_j\ln\left(\frac{h_j}{\int h_j\mathrm{d}\sigma}\right)\right)\mathrm{d}\sigma-\left(U\left(h\right)-U\left(h^{'}\right)\right)
\end{aligned}$$

holds for all $\lambda<1$. This inequality implies that there exists a sequence λ_l with $\lambda_l\to1$ and that $\int_{Z_M}\sum_{j=0}^{M}\left(1-e^{-r\Delta}\mathbf{1}_{[j<M]}\right)e^{-r\Delta j}\zeta_j\left(z_M;\lambda\right)\mathrm{d}\sigma$ converges to some limit $L^{\infty}<\infty$. By the log sums inequality:

$$\begin{aligned}
&\left(1-\sum_{k=0}^{j}h_j\right)\ln\left(\frac{1-\sum_{k=0}^{j}h_j}{1-\int\sum_{k=0}^{j}h_k\mathrm{d}\sigma}\right)\\
&+\left(1-\lambda\right)\left(\sum_{k=0}^{j}h_j\right)\ln\left(\frac{\left(1-\lambda\right)\sum_{k=0}^{j}h_j}{\left(1-\lambda\right)\int\sum_{k=0}^{j}h_k\mathrm{d}\sigma}\right)\quad\geq\quad\left(1-\lambda\alpha\right)\ln\left(\frac{1-\lambda\sum_{k=0}^{j}h_j}{1-\lambda\int\sum_{k=0}^{j}h_k\mathrm{d}\sigma}\right)
\end{aligned}$$

which implies:

$$\zeta_j\left(z_M;\lambda\right)\geq\left(1-\sum_{k=0}^{j}h_j\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\tag{20}$$

for every z_M and all λ, and therefore:

$$\sum_{j=0}^{M}e^{-r\Delta j}\zeta_j\left(z_M;\lambda\right)\geq\sum_{j=0}^{M}e^{-r\Delta j}\left(1-\sum_{k=0}^{j}h_j\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)$$

Which is integrable. Hence by Fatou's lemma:

$$\begin{aligned}\lim_{l\to\infty}\int_{Z_M}\sum_{j=0}^{M}e^{-r\Delta j}\zeta_j\left(z_M;\lambda_l\right)\mathrm{d}\sigma &= \liminf_{l\to\infty}\int_{Z_M}\sum_{j=0}^{M}e^{-r\Delta j}\zeta_j\left(z_M;\lambda_l\right)\mathrm{d}\sigma \\ &\geq \int_{Z_M}\liminf_{l\to\infty}\sum_{j=0}^{M}e^{-r\Delta j}\zeta_j\left(z_M;\lambda_l\right)\mathrm{d}\sigma\end{aligned}$$

However:

$$\begin{aligned}\lim_{l\to\infty}\zeta_j\left(z_M;\lambda_l\right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(1-\lambda\sum_{k=0}^{j}h_k\right)\ln\left(\frac{\left(1-\lambda\sum_{k=0}^{j}h_k\right)}{\int\left(1-\lambda\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\Bigg|_{\lambda=1} \\ &= -\left(\sum_{k=0}^{j}h_k\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right) \\ &\quad+\left(\frac{\int\left(\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}-\frac{\sum_{k=0}^{j}h_k}{1-\sum_{k=0}^{j}h_k}\right)\left(1-\sum_{k=0}^{j}h_k\right)\end{aligned}\tag{21}$$

Since equation 20 holds for all λ, it must also hold in the limit, i.e.

$$\sum_{j=0}^{M}e^{-r\Delta j}\lim_{l\to\infty}\zeta_j\left(z_M;\lambda_l\right)\geq\sum_{j=0}^{M}e^{-r\Delta j}\left(1-\sum_{k=0}^{j}h_j\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)$$

But:

$$\int\left(\frac{\int\left(\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}-\frac{\sum_{k=0}^{j}h_k}{1-\sum_{k=0}^{j}h_k}\right)\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma=0$$

for all j then suggests that:

$$\int\sum_{j=0}^{M}e^{-r\Delta j}\left(\sum_{k=0}^{j}h_k\right)\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma$$

is bounded, a contradiction. Therefore $\sum_{j=0}^{M}e^{-r\Delta j}\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma$ is integrable.

We will now prove that $\ln\left(\frac{\left(1-\sum_{k=0}^{m}h_k\right)}{\int\left(1-\sum_{k=0}^{m}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma$ and $\sum_{j=m+1}^{M}e^{-r\Delta j}\ln\left(\frac{\left(1-\sum_{k=0}^{j}h_k\right)}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma$ are both integrable for all m. Suppose both are for all $j\leq m-1$, but that one of them is not for

$j = m$. Then both must not be integrable since:

$$\sum_{j=0}^{M} e^{-r\Delta j} \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) = \sum_{j=0}^{m-1} e^{-r\Delta j} \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) + \sum_{j=m+1}^{M} e^{-r\Delta j} \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) + e^{-r\Delta m} \ln\left(\frac{(1-\sum_{k=0}^{m} h_k)}{\int (1-\sum_{k=0}^{m} h_k) \mathrm{d}\sigma}\right)$$

is integrable. Using equations 20 and 21 for every j we obtain that:

$$-\left(\sum_{k=0}^{j} h_k\right) \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) + \left(\frac{\int\left(\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma} - \frac{\sum_{k=0}^{j} h_k}{1-\sum_{k=0}^{j} h_k}\right)\left(1-\sum_{k=0}^{j} h_k\right) \geq \left(1-\sum_{k=0}^{j} h_k\right) \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right)$$

which, since

$$\int \left(\frac{\int\left(\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma} - \frac{\sum_{k=0}^{j} h_k}{1-\sum_{k=0}^{j} h_k}\right)\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma = 0$$

means that both:

$$\int \sum_{j=m+1}^{M} e^{-r\Delta j} \ln\left(\frac{\left(1-\sum_{k=0}^{j} h_k\right)}{\int\left(1-\sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma}\right) \mathrm{d}\sigma = \infty$$

and:

$$\int \ln\left(\frac{(1-\sum_{k=0}^{m} h_k)}{\int (1-\sum_{k=0}^{m} h_k) \mathrm{d}\sigma}\right) \mathrm{d}\sigma = \infty$$

a contradiction. □

The above lemma immediately leads to the following result:

Corollary 4. *Assume h maximizes $\tilde{\mathcal{U}}(h)$. Then $\sigma\left\{\sum_{j=0}^{m} h_j = 1\right\} > 0$ implies $\sigma\left\{\sum_{j=0}^{m} h_j = 1\right\} = 1$.*

Lemma 10. *Suppose h maximizes $\tilde{\mathcal{U}}$. Then the function $\ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right)$ is integrable for all m.*

Proof. Suppose, by contradiction, that m is such that $\ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right)$ is not integrable. Since $\ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right) < -\ln\left(\int h_m \mathrm{d}\sigma\right)$, we must have $\int \ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right)\mathrm{d}\sigma = -\infty$. Let h^+ be such that $h_j = 0$ for all $j \neq m$ and $h_m = 1$ for all z_M. Then by concavity of $\tilde{\mathcal{U}}$ and optimality of h, we have for all $0 \leq \lambda < 1$:

$$0 \leq \frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\lambda h + (1-\lambda) h^+\right)}{1-\lambda} \leq \tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(h^+\right) = \tilde{\mathcal{U}}(h) - e^{-r\Delta m}\int v_m \mathrm{d}\sigma$$

letting:

$$\zeta_j(z_M;\lambda) = \frac{1}{1-\lambda}\begin{pmatrix} \left(1-\sum_{k=0}^{j} h_k\right)\ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int\left(1-\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right) \\ -\left(1-\lambda\sum_{k=0}^{j} h_k\right)\ln\left(\frac{1-\lambda\sum_{k=0}^{j} h_k}{\int\left(1-\lambda\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)\end{pmatrix}$$

and:

$$\xi_m(z_M;\lambda) = \frac{1}{1-\lambda}\begin{pmatrix} h_m \ln\left(\frac{h_m}{\int h_m}\right) - \left(\lambda h_m + (1-\lambda)\right)\times \\ \ln\left(\frac{\lambda h_m + (1-\lambda)}{\int\left(\lambda h_m + (1-\lambda)\right)\mathrm{d}\sigma}\right)\end{pmatrix}$$

we have that:

$$\begin{gathered}\frac{\tilde{\mathcal{U}}(h) - \tilde{\mathcal{U}}\left(\lambda h + (1-\lambda) h^+\right)}{1-\lambda} = \\ \int \sum_{j=0}^{M} e^{-r\Delta j} h_j v_j \mathrm{d}\sigma - \kappa \int \sum_{j\neq m} e^{-r\Delta j} h_j \ln\left(\frac{h_j}{\int h_j \mathrm{d}\sigma}\right)\mathrm{d}\sigma \\ -\kappa \int \sum_{j=m}^{M}\left(1-e^{-r\Delta}\mathbf{1}_{[j<M]}\right)\left(1-\sum_{k=0}^{j} h_k\right)\ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int\left(1-\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma \\ -\kappa e^{-r\Delta m}\int \xi_m(z_M;\lambda)\mathrm{d}\sigma - \kappa\left(1-e^{-r\Delta}\right)\int \sum_{j=0}^{m-1} e^{-r\Delta j}\zeta_j(z_M;\lambda)\mathrm{d}\sigma\end{gathered}$$

and therefore we obtain that there are $\underline{L}$ and $\bar{L}$ such that:

$$\underline{L} \leq e^{-r\Delta m}\int \xi_m(z_M;\lambda)\mathrm{d}\sigma + \left(1-e^{-r\Delta}\right)\int \sum_{j=0}^{m-1}\zeta_j(z_M;\lambda)\mathrm{d}\sigma \leq \bar{L}$$

for all λ. Thus, there exists a sequence $\lambda_l \to 1$ such that:

$$\int e^{-r\Delta m}\xi_m\left(z_M;\lambda_l\right) + \left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1} e^{-r\Delta j}\zeta_j\left(z_M;\lambda_l\right) \mathrm{d}\sigma \to L_\infty$$

where $L_\infty \in \left[\underline{L}, \bar{L}\right]$. Using the log-sum inequality (Lemma 2) we obtain:

$$\begin{aligned}
&\left(1-\sum_{k=0}^{j} h_k\right)\ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int\left(1-\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)\\
&+(1-\lambda)\left(\sum_{k=0}^{j} h_k\right)\ln\left(\frac{(1-\lambda)\left(\sum_{k=0}^{j} h_k\right)}{(1-\lambda)\int\left(\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)\\
&\geq\left(1-\lambda\sum_{k=0}^{j} h_k\right)\ln\left(\frac{1-\lambda\sum_{k=0}^{j} h_k}{\int\left(1-\lambda\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)
\end{aligned}$$

implying that for all λ:

$$\sum_{j=0}^{m-1} e^{-r\Delta j}\zeta_j\left(z_M;\lambda\right) \geq \sum_{j=0}^{m-1} e^{-r\Delta j}\left(\sum_{k=0}^{j} h_k\right)\ln\left(\frac{\sum_{k=0}^{j} h_k}{\int\sum_{k=0}^{j} h_k \mathrm{d}\sigma}\right)$$

while using the log-sum inequality (2):

$$\begin{aligned}
&h_m\ln\left(\frac{h_m}{\int h_m\mathrm{d}\sigma}\right)\\
&+(1-\lambda)\left(1-h_m\right)\ln\left(\frac{(1-\lambda)\left(1-h_m\right)}{(1-\lambda)\int\left(1-h_m\right)\mathrm{d}\sigma}\right) \quad\geq\quad \left(\lambda h_m+(1-\lambda)\right)\ln\left(\frac{\lambda h_m+(1-\lambda)}{\int\left(\lambda h_m+(1-\lambda)\right)\mathrm{d}\sigma}\right)
\end{aligned}$$

gives:

$$\xi_m\left(z_M;\lambda\right) \geq \left(1-h_m\right)\ln\left(\frac{\left(1-h_m\right)}{\int\left(1-h_m\right)\mathrm{d}\sigma}\right) \tag{22}$$

we can therefore use Fatou's lemma to obtain:

$$\begin{aligned}
L_\infty &\geq \int\liminf_{l\to\infty}\left(e^{-r\Delta m}\xi_m\left(z_M;\lambda\right)+\left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1} e^{-r\Delta j}\zeta_j\left(z_M;\lambda\right)\right)\mathrm{d}\sigma\\
&\geq \int e^{-r\Delta m}\left(1-h_m\right)\ln\left(\frac{\left(1-h_m\right)}{\int\left(1-h_m\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma\\
&\quad+\left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1} e^{-r\Delta j}\int\left(\sum_{k=0}^{j} h_k\right)\ln\left(\frac{\sum_{k=0}^{j} h_k}{\int\sum_{k=0}^{j} h_k\mathrm{d}\sigma}\right)\mathrm{d}\sigma > -\infty
\end{aligned}$$

However, for every z_M:

$$\begin{aligned}\lim_{l\to\infty}\xi_m\left(z_M;\lambda_l\right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(\lambda h_m+(1-\lambda)\right)\ln\left(\frac{\lambda h_m+(1-\lambda)}{\int\left(\lambda h_m+(1-\lambda)\right)\mathrm{d}\sigma}\right)\right]\Bigg|_{\lambda=1}\\ &= \left(h_m-1\right)\ln\frac{h_m}{\int h_m\mathrm{d}\sigma}+h_m\left(\frac{h_m-1}{h_m}-\frac{\int h_m\mathrm{d}\sigma-1}{\int h_m\mathrm{d}\sigma}\right)\end{aligned}$$

and:

$$\begin{aligned}\lim_{l\to\infty}\zeta_j\left(z_M;\lambda_l\right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(1-\lambda\sum_{k=0}^{j}h_k\right)\ln\left(\frac{1-\lambda\sum_{k=0}^{j}h_k}{\int\left(1-\lambda\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\right]\Bigg|_{\lambda=1}\\ &= -\sum_{k=0}^{j}h_k\ln\left(\frac{1-\sum_{k=0}^{j}h_k}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\\ &\quad+\left(1-\sum_{k=0}^{j}h_k\right)\left(\frac{\int\sum_{k=0}^{j}h_k\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}-\frac{\sum_{k=0}^{j}h_k}{1-\sum_{k=0}^{j}h_k}\right)\end{aligned}$$

By lemma 9,

$$\sum_{k=0}^{j}h_k\ln\left(\frac{1-\sum_{k=0}^{j}h_k}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)$$

is integrable for every j. In addition, $h_m\ln\left(\frac{h_m}{\int h_m\mathrm{d}\sigma}\right)$ is integrable,

$$\int h_m\left(\frac{h_m-1}{h_m}-\frac{\int h_m\mathrm{d}\sigma-1}{\int h_m\mathrm{d}\sigma}\right)\mathrm{d}\sigma=0$$

and:

$$\int\left(1-\sum_{k=0}^{j}h_k\right)\left(\frac{-\sum_{k=0}^{j}h_k}{1-\sum_{k=0}^{j}h_k}+\frac{\int\sum_{k=0}^{j}h_k\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma=0$$

giving

$$\begin{aligned}\infty &> L_\infty+\left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1}e^{-r\Delta j}\int\sum_{k=0}^{j}h_k\ln\left(\frac{1-\sum_{k=0}^{j}h_k}{\int\left(1-\sum_{k=0}^{j}h_k\right)\mathrm{d}\sigma}\right)\mathrm{d}\sigma\\ &\quad-\int h_m\ln\frac{h_m}{\int h_m\mathrm{d}\sigma}\mathrm{d}\sigma\\ &\geq -\int\ln\left(\frac{h_m}{\int h_m\mathrm{d}\sigma}\right)\mathrm{d}\sigma>-\infty\end{aligned}$$

as required. $\square$

Lemma 10 immediately implies the corollary:

Corollary 5. *Suppose that h maximizes $\tilde{\mathcal{U}}(h;\sigma)$. Then $\sigma\{h_m = 0\} > 0$ implies $\sigma\{h_m = 0\} = 1$.*

Definition 6. For any $h \in \mathbf{H}$, we say that the measurable function $\eta : Z_M \to \mathbb{R}$ is an *h-feasible direction* if there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$: $h + \epsilon\eta \in \mathbf{H}$. We denote the set of h-feasible directions by $\mathbf{H}_h$.

Definition 7. We say that $\tilde{\mathcal{U}}$ is Gateaux diffrentiable at $h \in \mathbf{H}$ if there exists a bounded linear functional$\mathrm{d}\tilde{\mathcal{U}}_h : \mathbf{H}_h \to \mathbb{R}$ such that for every $\eta \in \mathbf{H}_h$:

$$\lim_{\epsilon\to 0}\left|\frac{1}{\epsilon}\left(\tilde{\mathcal{U}}(h+\epsilon\eta) - \tilde{\mathcal{U}}(h) - \mathrm{d}\tilde{\mathcal{U}}_h(\eta)\right)\right| = 0$$

Lemma 11. *Suppose h maximizes $\tilde{\mathcal{U}}$. Let:*

$$\begin{aligned}\Lambda_{h,m}(z_M) &= e^{-r\Delta m} v_m - \kappa e^{-r\Delta m}\ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right)\\ &\quad + \kappa\sum_{j=m}^{M} e^{-r\Delta j}\left(1 - e^{-r\Delta}\mathbf{1}_{[j<M]}\right)\ln\left(\frac{1-\sum_{k=0}^{j} h_k}{\int\left(1-\sum_{k=0}^{j} h_k\right)\mathrm{d}\sigma}\right)\end{aligned}$$

and define $\mathrm{d}\tilde{\mathcal{U}}_h(\eta) = \int_{Z_M}\sum_{m=0}^{M}\Lambda_{h,m}\eta_m \mathrm{d}\sigma$. Then for every h-feasible η, $\mathrm{d}\tilde{\mathcal{U}}_h(\eta)$ is bounded and satisfies:

$$\lim_{\epsilon\to 0}\left|\frac{1}{\epsilon}\left(\tilde{\mathcal{U}}(h+\epsilon\eta) - \tilde{\mathcal{U}}(h) - \mathrm{d}\tilde{\mathcal{U}}_h(\eta)\right)\right| = 0 \tag{23}$$

Proof. Fix some h-feasible direction η. Assume without loss of generality that $h + \eta \in \mathbf{H}$, and let $\lambda_\epsilon = 1 - \epsilon$, while defining $h^\lambda = h + (1-\lambda)\eta$. Then:

$$\begin{aligned}0 \geq \frac{1}{\epsilon}\left(\tilde{\mathcal{U}}(h+\epsilon\eta) - \tilde{\mathcal{U}}(h)\right) &= \frac{1}{1-\lambda_\epsilon}\left(\tilde{\mathcal{U}}\left(h^{\lambda_\epsilon}\right) - \tilde{\mathcal{U}}(h)\right)\\ &= \int_{Z_M}\sum_{m=0}^{M} e^{-r\Delta m}\left(\frac{U_m^*\left(h^{\lambda_\epsilon}; z_M\right) - U_m^*(h; z_M)}{1-\lambda_\epsilon}\right)\mathrm{d}\sigma\end{aligned}$$

since U_m^* is concave (6), we have by 8:

$$\begin{aligned}\left(\frac{U_m^*\left(h^{\lambda}; z_M\right) - U_m^*(h; z_M)}{1-\lambda}\right) &= -\left(\frac{U_m^*(h; z_M) - U_m^*\left(h^{\lambda}; z_M\right)}{1-\lambda}\right)\\ &\geq -\left(U_m^*(h; z_M) - U_m^*\left(h^{1}; z_M\right)\right)\end{aligned}$$

and therefore:

$$\left(\frac{U_m^*\left(h^\lambda; z_M\right) - U_m^*\left(h; z_M\right)}{1-\lambda}\right) + \left(U_m^*\left(h; z_M\right) - U_m^*\left(h^1; z_M\right)\right) \geq 0$$

for all z_M. Moreover, lemma 8 implies that $(1-\lambda)^{-1}\left(U_m^*\left(h^\lambda; z_M\right) - U_m^*\left(h; z_M\right)\right)$ is increasing with λ. Therefore, by the monotone convergence theorem:

$$\begin{aligned}
&\lim_{\lambda\to 1}\int_{Z_M}\sum_{m=0}^{M} e^{-r\Delta m}\left(\frac{U_m^*\left(h^\lambda; z_M\right) - U_m^*\left(h; z_M\right)}{1-\lambda}\right)\mathrm{d}\sigma \\
&= \int_{Z_M}\sum_{m=0}^{M} e^{-r\Delta m}\lim_{\lambda\to 1}\left(\frac{U_m^*\left(h^\lambda; z_M\right) - U_m^*\left(h; z_M\right)}{1-\lambda}\right)\mathrm{d}\sigma
\end{aligned}$$

note that: $\left(\frac{h_m^\lambda v_m - h_m v_m}{1-\lambda}\right) = v_m\eta_m$. In addition:

$$\begin{aligned}
&\lim_{\lambda\to 1}(1-\lambda)^{-1}\left(h_m \ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right) - h_m^\lambda \ln\left(\frac{h_m^\lambda}{\int h_m^\lambda \mathrm{d}\sigma}\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[h_m^\lambda \ln\left(\frac{h_m^\lambda}{\int h_m^\lambda \mathrm{d}\sigma}\right)\right]\Bigg|_{\lambda=1} \\
&= -\eta_m \ln\left(\frac{h_m}{\int h_m \mathrm{d}\sigma}\right) + \left(\frac{h_m \int \eta_m \mathrm{d}\sigma}{\int h_m \mathrm{d}\sigma}\right) - \eta_m
\end{aligned}$$

and:

$$\begin{aligned}
&\lim_{\lambda\to 1}(1-\lambda)^{-1}\left(\begin{array}{c}\left(1-\sum_{j=0}^{m} h_j\right)\ln\left(\frac{\left(1-\sum_{j=0}^{m} h_j\right)}{\int\left(1-\sum_{j=0}^{m} h_j\right)\mathrm{d}\sigma}\right) - \\ \left(1-\sum_{j=0}^{m} h_j^\lambda\right)\ln\left(\frac{1-\sum_{j=0}^{m} h_j^\lambda}{\int\left(1-\sum_{j=0}^{m} h_j^\lambda\right)\mathrm{d}\sigma}\right)\end{array}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(1-\sum_{j=0}^{m} h_j^\lambda\right)\ln\left(\frac{1-\sum_{j=0}^{m} h_j^\lambda}{\int\left(1-\sum_{j=0}^{m} h_j^\lambda\right)\mathrm{d}\sigma}\right)\right]\Bigg|_{\lambda=1} \\
&= \left(\sum_{j=0}^{m}\eta_j\right)\ln\left(\frac{1-\sum_{j=0}^{m} h_j}{\int\left(1-\sum_{j=0}^{m} h_j\right)\mathrm{d}\sigma}\right) \\
&\quad + \sum_{j=0}^{m}\eta_j - \frac{\left(1-\sum_{j=0}^{m} h_j\right)\int\sum_{j=0}^{m}\eta_j \mathrm{d}\sigma}{\int\left(1-\sum_{j=0}^{m} h_j\right)\mathrm{d}\sigma}
\end{aligned}$$

thus, since:

$$\int \left(\frac{h_m \int \eta_m \mathrm{d}\sigma}{\int h_m \mathrm{d}\sigma} \right) - \eta_m \mathrm{d}\sigma = 0$$

and:

$$\int \sum_{j=0}^{m} \eta_j - \frac{\left(1 - \sum_{j=0}^{m} h_j\right) \int \sum_{j=0}^{m} \eta_j \mathrm{d}\sigma}{\int \left(1 - \sum_{j=0}^{m} h_j\right) \mathrm{d}\sigma} \mathrm{d}\sigma = 0$$

we obtain that equation 23 holds. Boundedness of $\mathrm{d}\tilde{\mathcal{U}}_h(\eta)$ for all feasible η follows from concavity of $\tilde{\mathcal{U}}$ and lemma 8 which imply:

$$\begin{aligned} 0 &\geq \lim_{\lambda \to 1} \frac{1}{1-\lambda} \left(\tilde{\mathcal{U}}\left(h^{\lambda}\right) - \tilde{\mathcal{U}}(h) \right) \\ &\geq \tilde{\mathcal{U}}\left(h^{1}\right) - \tilde{\mathcal{U}}(h) \geq \underline{v} - \left(\frac{\ln 2}{1 - e^{-r\Delta}} \right) - \bar{v} \end{aligned}$$

thereby concluding the proof. □

Define:

$$\begin{aligned} \pi_h(m) &= \int \beta_m^h \mathrm{d}\mu_{h,m} \\ &= \frac{\int h_m \mathrm{d}\sigma}{\int \left(1 - \sum_{j=0}^{m-1} h_j\right) \mathrm{d}\sigma} \end{aligned}$$

For the next proof, it is useful to note that one can rewrite:

$$\begin{aligned} \Lambda_{h,m}(z_M) &= e^{-r\Delta m} v_m - \kappa e^{-r\Delta m} \ln \left(\frac{h_m}{\int h_m \mathrm{d}\sigma} \right) \\ &\quad + \kappa \sum_{j=m}^{M} e^{-r\Delta j} \left(1 - e^{-r\Delta} \mathbf{1}_{[j<M]}\right) \ln \left(\frac{1 - \sum_{k=0}^{j} h_k}{\int \left(1 - \sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma} \right) \\ &= e^{-r\Delta m} v_m - \kappa e^{-r\Delta m} \ln \left(\frac{h_m}{\int h_m \mathrm{d}\sigma} \right) + \kappa e^{-r\Delta m} \ln \left(\frac{1 - \sum_{j=0}^{m} h_j}{\int \left(1 - \sum_{j=0}^{m} h_j\right) \mathrm{d}\sigma} \right) \\ &\quad + \kappa \sum_{j=m+1}^{M} e^{-r\Delta j} \left(\ln \left(\frac{1 - \sum_{k=0}^{j} h_k}{\int \left(1 - \sum_{k=0}^{j} h_k\right) \mathrm{d}\sigma} \right) - \ln \left(\frac{1 - \sum_{k=0}^{j-1} h_k}{\int \left(1 - \sum_{k=0}^{j-1} h_k\right) \mathrm{d}\sigma} \right) \right) \\ &= e^{-r\Delta m} v_m - \kappa e^{-r\Delta m} \ln \left(\frac{\beta_m^h}{\pi_h(m)} \right) + \kappa \sum_{j=m}^{M} e^{-r\Delta j} \ln \left(\frac{1 - \beta_j^h}{1 - \pi_h(j)} \right) \end{aligned}$$

Lemma 12. *Suppose h maximizes $\tilde{\mathcal{U}}$. Then:*

$$\beta_m^h(z_m) = \frac{\pi_h(m)\exp\frac{1}{\kappa}v_m(z_m)}{\pi_h(m)\exp\frac{1}{\kappa}v_m(z_m) + (1-\pi_h(m))\exp\frac{1}{\kappa}e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}(h|z_m)}$$

for all m.

Proof. First note that by corollaries 4 and 5 the theorem holds whenever $\sigma\{h_m = 0\} > 0$ or $\sigma\{h_m = 1\} > 0$. Suppose then that $\sigma\{0 < h_m < 1\} = 1$. Fix any z_m in the support of σ, and define h^{z_m} as following: $h^{z_m} = h_j$ if $j < m$ or $j = m$ and $z_M(m) \neq z_m$, $h_m^{z_m}(z_m) = 1 - \sum_{j=0}^{m-1} h_j(z_m)$, $h_j^{z_m}(z_M) = 0$ if $z_M(m) = z_m$ and $j > m$. Let $h^\lambda = \lambda h + (1-\lambda)h^{z_m}$. Then obviously the following:

$$\eta^\lambda = h^\lambda - h = (1-\lambda)(h^{z_m} - h)$$

is a feasible direction for all $\lambda \in [0,1]$. Note that $\eta = \eta^1$ satisfies $\eta_j = 0$ if $j < m$ or $z_M(m) \neq z_m$, $\eta_m(z_m) = 1 - \sum_{j=0}^m h_j(z_m)$, while $\eta_j(z_M) = -h_j(z_M)$ whenever both $j > m$ and $z_M(m) = z_m$. Since η is feasible and h is optimal, we must have that $\mathrm{d}\tilde{U}_h(\eta) \leq 0$. Note that the pertubation $-\eta$ is also feasible, and therefore $0 \geq \mathrm{d}\tilde{\mathcal{U}}_h(-\eta) = -\mathrm{d}\tilde{\mathcal{U}}_h(\eta)$. Hence, we must have $\mathrm{d}\tilde{\mathcal{U}}_h(\eta) = 0$. Therefore:

$$\int \Lambda_{h,m}(z_m)\left(1 - \sum_{j=0}^m h_j(z_m)\right) - \sum_{j=m+1}^M \Lambda_{h,j} h_j \mathrm{d}\sigma(z_M|z_m) = 0$$

which can be rewritten as:

$$\left(\begin{array}{c} \left(1 - \sum_{j=0}^m h_j(z_m)\right)\left(e^{-r\Delta m}v_m - \kappa e^{-r\Delta m}\ln\left(\frac{\beta_m^h}{\pi_h(m)}\right)\right) \\ +\kappa e^{-r\Delta m}\left(1 - \sum_{j=0}^m h_j(z_m)\right)\ln\left(\frac{1-\beta_m^h}{1-\pi_h(m)}\right) \end{array}\right) =$$

$$\left(\begin{array}{c} -\kappa\left(1 - \sum_{j=0}^m h_j(z_m)\right)\int \sum_{j=m+1}^M e^{-r\Delta j}\ln\left(\frac{1-\beta_j^h}{1-\pi_h(j)}\right)\mathrm{d}\sigma(z_M|z_m) \\ +\kappa \int \sum_{j=m+1}^M \sum_{k=j}^M h_j e^{-r\Delta k}\ln\left(\frac{1-\beta_k^h}{1-\pi_h(k)}\right)\mathrm{d}\sigma(z_M|z_m) \\ +\int \sum_{j=m+1}^M h_j\left(e^{-r\Delta j}v_j - \kappa e^{-r\Delta j}\ln\left(\frac{\beta_j^h}{\pi_h(m)}\right)\right)\mathrm{d}\sigma(z_M|z_m) \end{array}\right)$$

But:

$$\int \sum_{j=m+1}^{M} \sum_{k=j}^{M} h_j e^{-r\Delta k} \ln\left(\frac{1-\beta_k^h}{1-\pi_h(k)}\right) \mathrm{d}\sigma(z_M|z_m) =$$

$$\int \sum_{k=m+1}^{M} \left(\sum_{j=m+1}^{k} h_j\right) e^{-r\Delta k} \ln\left(\frac{1-\beta_k^h}{1-\pi_h(k)}\right) \mathrm{d}\sigma(z_M|z_m)$$

and therefore we obtain:

$$\begin{pmatrix} \left(1-\sum_{j=0}^{m} h_j(z_m)\right)\left(e^{-r\Delta m} v_m - \kappa e^{-r\Delta m} \ln\left(\frac{\beta_m^h}{\pi_h(m)}\right)\right) \\ +\kappa e^{-r\Delta m}\left(1-\sum_{j=0}^{m} h_j(z_m)\right) \ln\left(\frac{1-\beta_m^h}{1-\pi_h(m)}\right) \end{pmatrix} =$$

$$\begin{pmatrix} -\kappa \int \sum_{j=m}^{M} \left(1-\sum_{k=0}^{j} h_k\right) e^{-r\Delta j} \ln\left(\frac{1-\beta_j^h}{1-\pi_h(j)}\right) \mathrm{d}\sigma(z_M|z_m) \\ +\int \sum_{j=m+1}^{M} h_j \left(e^{-r\Delta j} v_j - \kappa e^{-r\Delta j} \ln\left(\frac{\beta_j^h}{\pi_h(m)}\right)\right) \mathrm{d}\sigma(z_M|z_m) \end{pmatrix} =$$

$$\int \sum_{j=m+1}^{M} \left(1-\sum_{k=0}^{j-1} h_k\right) e^{-r\Delta j} v_j^*(z_M; h) \,\mathrm{d}\sigma$$

dividing both sides by $e^{-r\Delta m}\left(1-\sum_{j=0}^{m} h_j(z_m)\right)$ gives:

$$v_m - \kappa \ln\left(\frac{\beta_m^h}{\pi_h(m)}\right) + \kappa \ln\left(\frac{1-\beta_m^h}{1-\pi_h(m)}\right) \quad = \quad e^{-r\Delta}\tilde{\mathcal{U}}_n(h|z_m)$$

dividing both sides by κ, exponentiating and solving for β_m^h gives the desired result. □

C.4 Sufficient Conditions

Suppose β satisfies: $\beta_m \in (0,1)$ for all m and equations 16 and 17. We will show that β must be optimal. Note that:

$$\begin{aligned}
\kappa \ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right) &= \kappa \ln\left(\frac{\exp\left(\frac{e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)}{\kappa}\right)}{(1-\pi_m)\exp\left(\frac{e^{-r\Delta}}{\kappa}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)\right)+\pi_m\exp\left(\frac{v_m}{\kappa}\right)}\right) \\
&= e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right) - \kappa\ln\left((1-\pi_m)\exp\left(\frac{e^{-r\Delta}}{\kappa}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)\right)+\pi_m\exp\frac{v_m}{\kappa}\right) \\
&= e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right) - v_m + \kappa\ln\left(\exp\left(\frac{v_m}{\kappa}\right)\right) \\
&\quad -\kappa\ln\left((1-\pi_m)\exp\left(\frac{e^{-r\Delta}}{\kappa}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)\right)+\pi_m\exp\frac{v_m}{\kappa}\right) \\
&= e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right) - \left(v_m - \kappa\ln\left(\frac{\beta_m}{\pi_m}\right)\right) \qquad (24)
\end{aligned}$$

and therefore:

$$\begin{aligned}
U^*_m\left(z_M;h^\beta\right) &= \beta_m v_m - \kappa\beta_m\ln\left(\frac{\beta_m}{\pi_m}\right) - \kappa\left(1-\beta_m\right)\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right) \\
&= \beta_m\left(e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right) - \kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right)\right) - \kappa\left(1-\beta_m\right)\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right) \\
&= -\kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right) + \beta_m e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)
\end{aligned}$$

which implies that:

$$\tilde{\mathcal{U}}_m\left(h^\beta|z_n\right) = \int\left(e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right) - \kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right)\right)\mathrm{d}\sigma\left(z_M|z_n\right)$$

Using the fact that equation 17 holds we can use iterative substitution to obtain:

$$\tilde{\mathcal{U}}_m\left(h^\beta|z_n\right) = -\kappa\int\left(\sum_{j=m}^{M} e^{-r\Delta(j-m)}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)\right)\mathrm{d}\sigma\left(z_M|z_n\right) \qquad (25)$$

σ-almost surely. Rewriting

$$\begin{aligned}
\Lambda_{h^{\beta},m}\left(z_M\right) &= e^{-r\Delta m}v_m - \kappa e^{-r\Delta m}\ln\left(\frac{h_m^{\beta}}{1-\sum_{k=0}^{m}h_k^{\beta}}\right) \\
&\quad +\kappa e^{-r\Delta m}\ln\left(\frac{\int h_m^{\beta}\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{m}h_k^{\beta}\right)\mathrm{d}\sigma}\right) \\
&\quad +\kappa\sum_{j=m+1}^{M}e^{-r\Delta j}\left(\begin{array}{c}\ln\left(\frac{1-\sum_{k=0}^{j}h_k^{\beta}}{1-\sum_{k=0}^{j-1}h_k^{\beta}}\right) \\ -\ln\left(\frac{\int\left(1-\sum_{k=0}^{j}h_k^{\beta}\right)\mathrm{d}\sigma}{\int\left(1-\sum_{k=0}^{j-1}h_k^{\beta}\right)\mathrm{d}\sigma}\right)\end{array}\right) \\
&= e^{-r\Delta m}v_m - \kappa e^{-r\Delta m}\ln\left(\frac{\beta_m}{\pi_m}\right) + \kappa e^{-r\Delta m}\ln\left(\frac{1-\beta_m}{1-\pi_m}\right) \\
&\quad +\kappa\sum_{j=m+1}^{M}e^{-r\Delta j}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)
\end{aligned}$$

Using equations 24 and 25 we can obtain that the following holds for σ-almost every z_m:

$$\begin{aligned}
\int\Lambda_{h^{\beta},m}\left(z_M\right)\mathrm{d}\sigma\left(z_M|z_m\right) &= e^{-r\Delta m}\left(v_m - \kappa\ln\left(\frac{\beta_m}{\pi_m}\right) + \kappa\ln\left(\frac{1-\beta_m}{1-\pi_m}\right)\right) \\
&\quad -\kappa\int\sum_{j=m+1}^{M}e^{-r\Delta j}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)\mathrm{d}\sigma\left(z_M|z_m\right) \\
&= e^{-r\Delta(m+1)}\tilde{\mathcal{U}}_{m+1}\left(h^{\beta}|z_m\right) \\
&\quad -e^{-r\Delta(m+1)}\tilde{U}_{m+1}\left(h^{\beta}|z_m\right) = 0
\end{aligned}$$

Since $\tilde{\mathcal{U}}$ is concave this concludes the proof.

C.5 Quasi-Value Equivalence Lemma

In this section we prove the following lemma:

Lemma 13. *Suppose β is optimal for the buyer given σ, ans satisfies $\pi_m \in (0,1)$ for all m and* *equations , as well as equations 16 and 17. Then:*

$$\begin{aligned}
\tilde{\mathcal{U}}_m\left(h^\beta|z_n\right) &= \int\left(v_m-\kappa\ln\left(\frac{\beta_m}{\pi_m}\right)\right)\mathrm{d}\sigma\left(z_M|z_n\right)\\
&= \int\left(e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)-\kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right)\right)\mathrm{d}\sigma\left(z_M|z_n\right)\\
&= -\kappa\int\left(\sum_{j=m}^{M}e^{-r\Delta(j-m)}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)\right)\mathrm{d}\sigma\left(z_M|z_m\right)\\
&= \int\kappa\ln\left((1-\pi_m)\exp\left(\frac{e^{-r\Delta}}{\kappa}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)\right)+\pi_m\exp\left(\frac{v_m}{\kappa}\right)\right)\mathrm{d}\sigma\left(z_M|z_n\right)
\end{aligned}$$

Proof. Note that:

$$\begin{aligned}
v_m-\kappa\ln\left(\frac{\beta_m}{\pi_m}\right) &= \kappa\ln\left((1-\pi_m)\exp\left(\frac{e^{-r\Delta}}{\kappa}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)\right)+\pi_m\exp\left(\frac{v_m}{\kappa}\right)\right)\\
&= e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}\left(h^\beta|z_m\right)-\kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right)
\end{aligned}$$

which, using equation 25 implies:

$$\begin{aligned}
v_m-\kappa\ln\left(\frac{\beta_m}{\pi_m}\right) &= -\kappa\ln\left(\frac{1-\beta_m}{1-\pi_{m,}}\right)-\kappa e^{-r\Delta}\int\left(\sum_{j=m+1}^{M}e^{-r\Delta(j-(m+1))}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)\right)\mathrm{d}\sigma\left(z_M|z_m\right)\\
&= -\kappa\int\left(\sum_{j=m}^{M}e^{-r\Delta(j-m)}\ln\left(\frac{1-\beta_j}{1-\pi_j}\right)\right)\mathrm{d}\sigma\left(z_M|z_m\right)=\tilde{\mathcal{U}}_m\left(h^\beta|z_m\right)
\end{aligned}$$

where the last equality follows from equation 24. The conclusion then follows from $\tilde{\mathcal{U}}_m\left(h^\beta|z_n\right)=\int\tilde{\mathcal{U}}_m\left(h^\beta|z_m\right)\mathrm{d}\sigma\left(z_M|z_n\right)$ for all $n\leq m$. □

D Equilibrium in finite horizon model

The goal of this section is to prove a characterization theorem for all equilibria of the finite horizon game. It will eventually turn out that an equilibrium exists if and only if an object, which we call an *equilibrium average-ratio path* exists. Moreover, the equilibrium will turn out to be completely characterized by that object. The first subsection, defines, establishes the existence, and proves some properties of this object. We then move to proving that equilibrium strategies are simple. In the next subsection we prove lemma 1, which shows that the equilibria in the finite horizon model admits a recursive structure. In the following subsection we connect the recursive representation back to equilibrium average-ratio paths, which establishes the existence of a finite horizon equilibrium. Finally, we prove some useful properties of a finite horizon equilibrium which are used later in the analysis of equilibria in the infinite horizon.

D.1 Equilibrium Average Ratio Paths

We take $W : \mathbb{R}_+ \to \mathbb{R}_+$ to be Lambert's W function, defined by: $W(ze^z) = z$, or, equivalently, as: $W(z)\, e^{W(z)}$.

Let $c_v = e^{\left(\frac{v-\kappa}{\kappa}\right)}$ and define the functions:

$$z^*(d, v, q) = \frac{d}{1-d} {c_v}^{1-e^{-r\Delta}} q^{e^{-r\Delta}}$$

$$R(d, v, q) = \frac{W(z^*(v, d, q))}{d(1 + W(z^*(v, d, q)))}$$

and:

$$R_c(d, v, q) = (1-d)^{-1} (1 + W(z^*(v, d, q)))^{-1}$$

It will eventually turn out that an equilibrium exists if and only if the following object, called equilibrium average-ratio path exists. We now define this object:

Definition 8. A collection of numbers $a_m \in (0, 1)$, $m = 1, \ldots, M$, $p_{m,v} \in \mathbb{R}_+$, $m = 1, \ldots, M+1$, $v \in V$, and distributions, $\vartheta_m \in \Delta(V)$, $m = 1, \ldots, M$ are an *equilibrium average-ratio path in* $B_M(\mu_0)$ if:

1. $p_{M+1,v} = \exp\left(\frac{v-\kappa}{\kappa}\right)$
2. For every $m = 1, \ldots, M$ and v: $p_{m,v} = R(a_m, v, p_{m+1,v})$
3. $\vartheta_1(v) = \mu_0(v)$ and for every $m = 2, \ldots, M$ and v:

$$\vartheta_m(v) = \frac{\vartheta_{m-1}(v)(1 - a_{m-1} p_{m-1,v})}{\sum_{v'} \vartheta_{m-1}(v') \left(1 - a_{m-1} p_{m-1,v'}\right)}$$

4. For every $m = 1, \ldots, M$: $\sum_v \vartheta_m(v) p_{m,v} = 1$.

D.1.1 Preliminary facts about Lambert's W function

In this section we prove a few facts about the positive part of Lambert's W function, i.e. the function: $W : \mathbb{R}_+ \to \mathbb{R}_+$ defined by: $W(ze^z) = z$, or, equivalently, as: $W(z) e^{W(z)}$.

Lemma 14. $\mathrm{d}W(z)/\mathrm{d}z = \left(e^{W(z)}(1 + W(z))\right)^{-1} = W(z)(z(1 + W(z)))^{-1}$

Proof. By definition: $W(z) e^{W(z)} - z = 0$. The first equality is then implied by the implicit function theorem. Substituting $W(z) = z/e^{W(z)}$ into $\left(e^{W(z)}(1 + W(z))\right)^{-1}$ gives the second equality. ☐

Lemma 15. *W is concave.*

Proof. Calculating the second derivative of W gives:

$$\begin{aligned}
\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} &= \frac{\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right) z(1 + W(z)) - W(z)\left(1 + W(z) + z\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)\right)}{z^2(1 + W(z))^2} \\
&= -\frac{W(z)\left(W(z) + z\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)\right)}{z^2(1 + W(z))^2} < 0
\end{aligned}$$

as required. ☐

D.1.2 Existence of M-Equilibrium Average-Ratio Path

Here we prove the following theorem:

Theorem 6. *For every μ_0 and every M there exists an equilibrium average-ratio path of $B_M(\mu_0)$.*

Lemma 16. *For every v and q, the function $f(d) = R(d; v, q)$ is strictly convex.*

Proof. Rewrite $R(d; v, q)$:

$$\begin{aligned}
R(d; v, q) &= \frac{W\left(\frac{dc_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}}}{1-d}\right)}{d\left(1 + W\left(\frac{dc_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}}}{1-d}\right)\right)} \\
&= \left(\frac{d}{W\left(\frac{\bar{\beta} c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}}}{1-d}\right)} + d\right)^{-1}
\end{aligned}$$

so a sufficient condition for R to be convex is for:

$$\phi(d) = \frac{d}{W\left(\frac{dc_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)}$$

to be concave. Note that the second derivative of ϕ is:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}d^2} = \frac{(1-d)^2\left(1+2W\left(\frac{dc_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right) - \left(1+W\left(\frac{dc_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right)^2}{d(1-d)^2 W\left(\frac{dc_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\left(1+W\left(\frac{dc_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right)^3}$$

so a necessary and sufficient condition for ϕ to being concave is for:

$$(1-d)^2\left(1+2W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right) - \left(1+W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right)^2 < 0$$

for all d in the range. However:

$$\begin{aligned}(1-d)^2\left(1+2W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right) & \\ - \left(1+W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right)^2 &= d(d-2)\left(1+2W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right) \\ &\quad - \left(W\left(\frac{da_v^{1-e^{-r\Delta}}q^{e^{-r\Delta}}}{1-d}\right)\right)^2\end{aligned}$$

which is strictly negative for all $d \in (0,1)$. Therefore ϕ is strictly concave, implying the desired result. □

Lemma 17. *For a fixed v and q, then the unique solution to the equation:$R(d; v, q) = 1$ in $[0,1)$ is:*

$$d^*(v,q) = \frac{\ln\left(\exp\left(\frac{v-\kappa}{\kappa}\right)^{1-e^{-r\Delta}} q^{e^{-r\Delta}}\right)}{1+\ln\left(\exp\left(\frac{v-\kappa}{\kappa}\right)^{1-e^{-r\Delta}} q^{e^{-r\Delta}}\right)}$$

moreover, $R(d; v, q) < 1$ if and only if $d \in (d^*(v,q), 1)$.

Proof. Set $k = \ln\left(c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}}\right)$, one obtains that:

$$\begin{aligned} R\left(d^*\left(v,q\right);v,q\right) &= \frac{W\left(ke^k\right)}{d^*\left(v,\gamma\right)\left(1+W\left(ke^k\right)\right)} \\ &= \left(\frac{k}{1+k}\right)\left(d^*\left(v,q\right)\right)^{-1} = 1 \end{aligned}$$

Note that: $R\left(1;v,q\right)=1$ since R is continuous and:

$$R\left(1;v,q\right) = \lim_{d\to 1} \frac{W\left(\frac{d}{1-d}e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)}q^{e^{-r\Delta}}\right)}{d\left(1+W\left(\frac{d}{1-d}e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)}q^{e^{-r\Delta}}\right)\right)} = 1$$

Suppose then there exist $d < d^{'} < 1$ such that $R\left(d;v,q\right) = R\left(d^{'};v,q\right) = 1$. Let α be such that $d^{'} = \alpha d + (1-\alpha)$. Then by strict convexity of R:

$$R\left(d^{'};v,q\right) = R\left(\alpha d + (1-\alpha);v,q\right) < \alpha R\left(d;v,q\right) + (1-\alpha) R\left(1;v,q\right) = 1$$

a contradiction. This also proves that $R\left(d;v,q\right) < 1$ if and only if $d \in \left(d^*\left(v,q\right),1\right)$. □

Lemma 18. *For every d, v and q: $R\left(d,v,q\right) \le \max\left\{q^{e^{-r\Delta}} c_v^{1-e^{-r\Delta}}, 1\right\}$*

Proof. Note that $R\left(1;v,q\right) = 1$ for all v, q and that $R\left(0;v,q\right) = q^{e^{-r\Delta}} c_v^{1-e^{-r\Delta}}$. The lemma then follows from convexity of R in d. □

Lemma 19. *For every q and v such that $c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}} \ge 1/2$ we have: $R\left(d,v,q\right) \ge 1/2$. If $c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}} < 1/2$, then $R\left(0,v,r\right) < R\left(d,v,q\right)$ for all $d > 0$.*

Proof. Note that $R\left(0,v,q\right) = c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}} \ge 1/2$, while $R\left(1,v,q\right) = 1$. Suppose the minimizer of R is in the interior. Since $d \mapsto R\left(d,v,q\right)$ is convex, a necessary and sufficient condition for d to be a minimizer of R is $\frac{\partial R}{\partial d} = 0$. Taking derivative of R:

$$\begin{aligned} \frac{\partial R}{\partial d} &= -\frac{1}{d^2}\frac{W}{1+W} + \frac{1}{d}\frac{\partial z^*}{\partial d}\left(\frac{W}{z^*\left(1+W\right)^2} - \frac{W^2}{z^*\left(1+W\right)^3}\right) \\ &= -\frac{R}{d} + \frac{z^*}{d^2\left(1-d\right)}\frac{W}{z^*\left(1+W\right)^3} \\ &= -\frac{R}{d} + \frac{R}{d}\frac{R^c}{\left(1+W\right)} \end{aligned}$$

since:

$$\frac{\partial z^*}{\partial d} = \frac{1}{\left(1-d\right)^2} c_v^{1-e^{-r\Delta}} r^{e^{-r\Delta}} = \frac{z^*}{d\left(1-d\right)}$$

Hence:

$$\begin{aligned} -\frac{R}{d}+\frac{R}{d}\frac{R^{c}}{(1+W)} &= 0 \\ \Longleftrightarrow \frac{R}{d}\left(\frac{R^{c}}{(1+W)}-1\right) &= 0 \end{aligned}$$

or: $R^c/(1+W) = 1$. Hence $R^c = (1+W)$, or $W = (1-d)^{-1/2} - 1$ this means that, at the minimum:

$$\begin{aligned} R &= \frac{(1-d)^{-1/2}-1}{d(1-d)^{-1/2}} \\ &= \frac{1-(1-d)^{1/2}}{d} \end{aligned}$$

let $f(d) = \left(1-(1-d)^{1/2}\right)/d$. Then:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}d} &= \frac{\frac{1}{2}(1-d)^{-1/2}d-1+(1-d)^{1/2}}{d^{2}} \\ &= \frac{1-(1-d)^{1/2}}{2d^{2}(1-d)^{1/2}} \end{aligned}$$

which is never 0 in $[0,1]$. Then the minimum of f is either $f(0)$ or $f(1)$. $f(1) = 1$ while:

$$\begin{aligned} f(0) &= \lim_{d\to 0}\frac{\left(1-(1-d)^{1/2}\right)}{d} \\ &= \lim_{d\to 0}\frac{\frac{1}{2}(1-d)^{-1/2}}{1} = \frac{1}{2} \end{aligned}$$

thus, $\min f = 1/2$. Hence, if $R(0,v,q) \geq 1/2$, we must have $R(d,v,q) \geq 1/2$ for all d. Moreover, if $R(0,v,q) = c_v^{1-e^{-r\Delta}} q^{e^{-r\Delta}} < 1/2$, then $R(0,v,q) < R(1,v,q)$ and $R(0,v,q) < R(d,v,q)$ for all $d > 0$, as required. □

Lemma 20. *Suppose that there exists a non-decreasing function* $g : V \to \left[\frac{1}{2}, c_{v_h}\right]$ *such that for every* v*:* $g(v) \leq c_v$*. Then for every* $d \in [0,1)$ *the function* $f(v) = R(d; v, g(v))$ *is strictly increasing and also satisfies* $f(v) \leq c_v$ *for all* v*.*

Proof. For every $v_1 > v_2$ note that $c_{v_1}^{1-e^{-r\Delta}} g(v_1)^{e^{-r\Delta}} > c_{v_2}^{1-e^{-r\Delta}} g(v_2)^{e^{-r\Delta}}$ and therefore $f(v_1) > f(v_2)$. $f(v) \leq c_v$ follows from lemma 18. □

Let $\mathcal{G}$ *d*e the set of all $\gamma \in \left[\frac{1}{2}, c_{v_h}\right]^V$ that are weakly increasing in v and satisfy $\gamma_v \leq c_v$ for all v. For any $\theta \in \Delta(V)$ and $\gamma \in \mathcal{G}$ define the correspondance: $\Phi : \mathcal{G} \times \Delta(V) \rightrightarrows [0,1]$ *d*y taking $\Phi(\gamma, \theta)$

to be the set of all $d \in [0,1]$ that solve:

$$\sum_v \theta(v) R(d; v, \gamma_v) = 1$$

Lemma 21. *For all* γ *and* θ*:*$1 \in \Phi(\gamma, \theta)$.

Proof. By definition:

$$R(1; v, \gamma_v) = \lim_{d \to 1} \frac{W\left(\frac{d}{1-d} a_v^{1-e^{-r\Delta}} r_v^{e^{-r\Delta}}\right)}{d\left(1 + W\left(\frac{d}{1-d} a_v^{1-e^{-r\Delta}} r_v^{e^{-r\Delta}}\right)\right)} = 1$$

therefore $\sum_v \theta(v) R(1; v, r_v) = \sum_v \theta(v) = 1$. □

Lemma 22. *Suppose* $\Phi(\gamma, \theta) \cap [0,1) \neq \varnothing$*. Then there exists a unique* d *in* $\Phi(\gamma, \theta) \cap [0,1)$.

Proof. Assume wlog that there exists $d < d^{'} < 1$ such that $\left\{d, d^{'}\right\} \subset \Phi(\gamma, \theta) \cap [0,1)$. By lemma 21 we have $\sum_v \theta(v) R(1; v, \gamma_v) = 1$. Let α de s.t. $d^{'} = \alpha d + (1-\alpha) 1$. Then:

$$\begin{aligned} 1 &= \alpha \sum_v \theta(v) R(d; v, \gamma_v) + (1-\alpha) \sum_v \theta(v) R(1; v, \gamma_v) \\ &= \sum_v \theta(v) (\alpha R(d; v, \gamma_v) + (1-\alpha) R(1; v, \gamma_v)) \\ &> \sum_v \theta(v) R(\alpha d + (1-\alpha) 1; v, \gamma_v) \\ &= \sum_v \theta(v) R\left(d^{'}; v, \gamma_v\right) \end{aligned}$$

where the inequality follows from lemma 16 and Jensen's inequality. Thus, $d^{'} \notin \Phi(\gamma, \theta)$, a contradiction. □

Lemma 23. $\Phi(\gamma, \theta) \cap [0,1) \neq \varnothing$ *if and only if* $\sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}} \geq 1$*, with* $\Phi(\gamma, \theta) \cap [0,1) \neq \{0\}$ *whenever the inequality is strict.*

Proof. Note that:

$$\begin{aligned} R\left(\frac{\exp(c_{v_h})}{1 + \exp(c_{v_h})}; v_h, c_{v_h}\right) &= \frac{1 + \exp(c_{v_h})}{\exp(c_{v_h})} \frac{W(c_{v_h} \exp(c_{v_h}))}{(1 + W(c_{v_h} \exp(c_{v_h})))} \\ &= \left(\frac{c_{v_h}}{1 + c_{v_h}}\right) / \left(\frac{\exp(c_{v_h})}{1 + \exp(c_{v_h})}\right) < 1 \end{aligned}$$

since $e^x > x$ for all $x > 0$. Since $R(d; v, q)$ is increasing in v and in q this implies:

$$R\left(\frac{\exp(c_{v_h})}{1 + \exp(c_{v_h})}; v, \gamma_v\right) < 1 \tag{26}$$

for all v and r_v. Moreover, if $\sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}} \geq 1$ then:

$$\sum_v \theta(v) R(0; v, r) = \frac{\sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}}}{\exp(W(0))(1+W(0))} = \sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}} \geq 1$$

implies dy the intermediate value theorem that $\Phi(\gamma,\theta) \cap [0,1) \neq \varnothing$, and that $\Phi(\gamma,\theta) \cap [0,1) \neq \{0\}$ whenever $\sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}} > 1$. Suppose now that $\Phi(\gamma,\theta) \cap [0,1) \neq \varnothing$. Take $d \in \Phi(\gamma,\theta) \cap [0,1)$. Then dy $d \mapsto R(d; v, \gamma_v)$ being strictly convex and equation 26 we have that

$$d < \frac{\exp(a_{v_h})}{1+\exp(a_{v_h})}$$

and therefore $\sum_v \theta(v) \frac{\partial R}{\partial d}(d; v, \gamma_v) < 0$ dy convexity of $d \mapsto \sum_v \theta(v) R(d; v, \gamma_v)$ and $\sum_v \theta(v) R(1; v, \gamma_v) = 1$. But convexity of $d \mapsto \sum_v \theta(v) R(d; v, \gamma_v)$ will then imply that $\sum_v \theta(v) R(d; v, \gamma_v)$ is decreasing in the range $[0, d]$, meaning that:

$$1 \leq \sum_v \theta(v) R(0; v, \gamma_v) = \sum_v \theta(v) a_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}}$$

as required. □

Let G be the set of all (γ,θ) pairs that satisfy, $\sum_v \theta(v) c_v^{1-e^{-r\Delta}} \gamma_v^{e^{-r\Delta}} \geq 1$, and let $\phi(\gamma,\theta)$ be the unique element in $\Phi(\gamma,\theta) \cap [0,1)$ for all $(\gamma,\theta) \in G$. Note that G is closed, and that ϕ is continuous since it is a fixed point of a continuous function[10]. For every $d^M \in \left[0, \frac{v_h-\kappa}{v_h}\right]^M$, define $\gamma_M^*\left(d^M\right) = R(d_M; v, c_v)$, and $\gamma_m^* = R\left(d_m; v, \gamma_{m+1,v}^*\left(d^M\right)\right)$ for all $m < M$. Note that $\gamma_m^* \in \mathcal{G}$ dy lemma 20. Then for every m, let $\theta_m^*\left(d^M\right)$:

$$\theta_m^*\left(v; d^M\right) = \frac{\mu_0(v) \prod_{j=0}^{m-1}\left(1 - d_j \gamma_j^*\left(d^M\right)\right)}{\sum_{v'} \mu_0(v) \prod_{j=0}^{m-1}\left(1 - d_j \gamma_j^*\left(d^M\right)\right)}$$

Finally, define the function $\Psi : \left[0, \frac{v_h-\kappa}{v_h}\right]^M \to \left[0, \frac{v_h-\kappa}{v_h}\right]^M$ dy setting: $\Psi_M\left(d^M\right) = \phi\left(c, \theta_M^*\left(d^M\right)\right)$ and letting $\Psi_m\left(d^M\right) = \phi\left(\gamma_M^*\left(d^M\right), \theta_m^*\left(d^M\right)\right)$.

Lemma 24. *Ψ is well defined and $\Psi_{d^M}(m) > 0$ for all m.*

[10] An easy argument is to set $\phi = \arg\min\left(d - \sum_v \theta(v) R(0; v, \gamma_v)\right)^2$ sudject to $d \leq \exp(c_{v_h}) / (1 + \exp(c_{v_h}))$ and use Berge's theorem of the maximum.

Proof. The first stage is clearly well defined. The second stage is well defined since:

$$\begin{aligned} d_m \gamma^*_{m,v}\left(d^M\right) &= d_m R\left(d_m; v, \gamma^*_{m+1,v}\left(d^M\right)\right) \\ &= \frac{W\left(\frac{d_m}{1-d_m} a_v^{1-e^{-r\Delta}} \gamma^*_{m+1,v}\left(d^M\right)^{e^{-r\Delta}}\right)}{1+W\left(\frac{d_m}{1-d_m} a_v^{1-e^{-r\Delta}} \gamma^*_{m+1,v}\left(d^M\right)^{e^{-r\Delta}}\right)} < 1 \end{aligned}$$

for all m and therefore $\theta^*_m\left(d^M\right)$ is well defined. It remains to de shown that $\left(\theta^*_m\left(d^M\right), \gamma^*_m\left(d^M\right)\right) \in G$ and $\phi\left(\theta^*_m\left(d^M\right), \gamma^*_m\left(d^M\right)\right) \in \left[0, \frac{v_h-\kappa}{v_h}\right]$. $\left(\theta^*_M\left(d^M\right), \gamma^*_M\left(d^M\right)\right) \in G$ follows from $c_v > 1$ for all v. $\Psi(M) \leq \frac{v_h-\kappa}{v_h}$ follows from:

$$R\left(\frac{v_h-\kappa}{v_h}; v_h, c_{v_h}\right) = \frac{W\left(\frac{v_h-\kappa}{\kappa}\exp\left(\frac{v_h-\kappa}{\kappa}\right)\right)}{\left(\frac{v_h-\kappa}{v_h}\right)\left(1+W\left(\frac{v_h-\kappa}{\kappa}\exp\left(\frac{v_h-\kappa}{\kappa}\right)\right)\right)} = 1$$

and $R\left(\frac{v_h-\kappa}{v_h}; v, c_v\right)$ being strictly increasing in v (which implies that $\sum \theta^*_M(v) R\left(\frac{v_h-\kappa}{v_h}; v, c_v\right) < 1$ since $\theta^*_M(v) < \theta(0; v_h) < 1$ from monotonicity of $\gamma^*_{m,v}$ in v). Note further that $\gamma^*_{M,v}(M) < c_v$ for every v since $c_v = R(0; v, c_v)$, $\Psi(M) > 0$ (since $\sum_v \theta_v c_v = \sum_v \theta_v R(0; v, c_v) > 1$ for all $\theta \in \Delta(V)$) and $R(d; v, c_v)$ is strictly decreasing for all $d < \frac{v-\kappa}{v}$. Suppose now $\gamma^*_{m+1,v} < c_v$ for all v, and $\Psi(j)$ is well defined and satisfies $\Psi(j) > 0$ for all $j \geq m+1$. We will show that $\Psi(m)$ is well defined, is in $\left[0, \frac{v_h-\kappa}{v_h}\right]$, and $\gamma^*_{v,m} < c_v$ for all v. Note that:

$$\begin{aligned} \sum_v \theta^*_m\left(v; d^M\right) a_v^{1-e^{-r\Delta}} \left(\gamma^*_{m+1,v}\right)^{e^{-r\Delta}} &= \sum_v \theta^*_m\left(v; d^M\right) a_v \left(\frac{\gamma^*_{m+1,v}}{a_v}\right)^{e^{-r\Delta}} \\ &> \sum_v \theta^*_m\left(v; d^M\right) a_v \left(\frac{\gamma^*_{m+1,v}}{a_v}\right) \\ &= \sum_v \theta^*_m\left(v; d^M\right) \gamma^*_{m+1,v} \end{aligned}$$

where the inequality follows from $\gamma^*_{m+1,v} < c_v$ and $x^{e^{-r\Delta}} > x$ for all $\Delta > 0$ and $x \in (0,1)$. But since $\gamma^*_{j,v}$ was monotone for all j, θ^*_m first order stochastically dominates θ^*_{m+1}, meaning that $\sum_v \theta^*_m\left(v; d^M\right)\gamma^*_{m+1,v} \geq \sum_v \theta^*_{m+1}\left(v; d^M\right)\gamma^*_{m+1,v}$ which is equal to 1. This proves that $\left(\theta^*_m\left(d^M\right), \gamma^*_m\left(d^M\right)\right) \in G$. Moreover, since the inequality $\sum_v \theta^*_m\left(v; d^M\right) c_v^{1-e^{-r\Delta}} \left(\gamma^*_{m+1,v}\right)^{e^{-r\Delta}} > \sum_v \theta^*_m\left(v; d^M\right) c_v \left(\frac{\gamma^*_{m+1,v}}{c_v}\right) \geq 1$, we obtain that $\phi\left(\theta^*_m\left(d^M\right), \gamma^*_m\left(d^M\right)\right) > 0$. Finally $\gamma^*_{m,v} < c_v$ follows from $\gamma^*_{m+1,v} < c_v$ and $\Psi_d(m) > 0$. □

Lemma 25. *Ψ is continuous, and therefore has a fixed point $a^* \in \left[0, \frac{v_h-\kappa}{\kappa}\right]^M$ that satisfies: $a^*_m > 0$ for all m.*

Proof. Continuity of Ψ follows from Ψ being a composition of continuous functions. We can therefore use Brouwer's fixed point theorem to odtain a fixed point, and $\alpha_m^* > 0$ follows from lemma 24. □

Lemma 26. *There exists an M-equilibrium average-ratio path.*

Proof. By lemma 25, there exists a fixed point $a^* \in \left[0, \frac{v_h - \kappa}{\kappa}\right]^M$ of Ψ which satisfies $a_m^* > 0$ for all m. Set $p_{m,v}^M = \gamma_{m,v}(a^*)$ for all $m \leq M$ and $p_{M+1,v}^M = \exp\left(\frac{v-\kappa}{\kappa}\right)$. Letting $a^M = a^*$ and $\vartheta_m^M = \theta_m^*\left(a^M\right)$ for all m then gives a triplet $\left(a^M, p^M, \vartheta^M\right)$ which is an M-equilibrium average-ratio path. □

D.1.3 Additional Properties of M-equilibrium average-ratio paths

Lemma 27. *Suppose $\gamma \in \mathcal{G}$, $\theta \in \Delta(V)$ and $d \in (0,1]$ are such that $\theta(v_h) < 1$, $\theta(v_l) > 0$ and $\sum_v \theta(v) R(d; v, \gamma_v) = 1$. Then: (1) $dR(d; v, \gamma_v) \leq (v_h - \kappa)/v_h$ for all v; (2) $R(d; v_h, \gamma_{v_h}) > 1$; (3) $R(d; v_l, \gamma_{v_l}) < 1$.*

Proof. (1): $R((v_h - \kappa)/v_h; v_h, c_{v_h}) = 1$. Since $R(d; v, \gamma_v)$ is strictly increasing in γ_v and v and $\gamma_v \leq c_v$ for all v this implies that $R((v_h - \kappa)/v_h; v, \gamma_v) \leq R((v_h - \kappa)/v_h; v_h, c_{v_h}) = 1$. By lemma 16, one must have $\sum_v \theta(v) R\left(d'; v, \gamma_v\right) < 1$ for all $d' \in ((v_h - \kappa)/v_h, 1)$. Therefore, $d \leq (v_h - \kappa)/v_h$. But the function $f\left(d'\right) = d' R\left(d'; v, \gamma_v\right)$ is strictly increasing in d. Therefore:

$$dR(d; v, \gamma_v) \leq \left(\frac{v_h - \kappa}{v_h}\right) R\left(\left(\frac{v_h - \kappa}{v_h}\right); v, \gamma_v\right) \leq \left(\frac{v_h - \kappa}{v_h}\right)$$

as required. (2) and (3) follow from lemma 20 and $\theta(v_h) < 1$ and $\theta(v_l) > 0$. □

Lemma 28. *Let (a, p, ϑ) be an equilibrium average-ratio path of $B_M(\mu_0)$. Then $p_{m,v}$ is strictly increasing in v for every m and satisfies: $p_{m,v_l} < 1 < p_{m,v_h}$.*

Proof. The first part follows from lemma 20 while the second from lemma 27. □

D.2 History Independence in Finite Periods

Here we will prove the following lemma, which implies that equilibrium strategies are simple.

Lemma 29. *Let (μ, β, σ) be an equilibrium of the finite horizon game. Then for every m there exists two functions: $z_m : V \to X$ and $b_m : X \times V \to (0,1)$ such that:*

1. *For every m and v: $\sigma_m\left(z_{m,v} | x^{m-1}, v\right) = 1$ for all x^{m-1}.*
2. *For every m, v and $x \in X$, $\beta_m\left(x, x^{m-1}, v\right) = b_m(x, v)$ for all x^{m-1}.*

We will prove Lemma 29 in steps, many of which will be useful later.

Lemma 30. *For $a \in (0,1)$, $c \in \mathbb{R}$, $d \in \mathbb{R}_+$, and consider the function:*

$$H\left(x|a,c,d,v\right) = \left(\frac{ae^{\frac{v-c-x}{\kappa}}}{1-a+ae^{\frac{v-c-x}{\kappa}}}\right)x + \left(\frac{1-a}{1-a+ae^{\frac{v-c-x}{\kappa}}}\right)d$$

then $H-d$ is strictly log-concave over $x \in \mathbb{R}_+$. Therefore, the problem: $\max_{x\in X} H\left(x|a,c,d,v\right)$ *has a unique solution for every a, c and d.*

Proof. Note that:

$$\ln\left(H-d\right) = \ln\left(x-d\right) + \ln a + \frac{1}{\kappa}\left(v-c-x\right) - \ln\left(1-a+ae^{\frac{v-x-c}{\kappa}}\right)$$

the second derivative of which with respect to x is:

$$-\left(x-d\right)^{-2} - \left(\frac{a\left(1-a\right)e^{\frac{v+x+c}{\kappa}}}{\kappa^2\left(ae^{\frac{v}{\kappa}}+\left(1-a\right)e^{\frac{x+c}{\kappa}}\right)^2}\right) < 0$$

for all $x \in \mathbb{R}_+$, concluding the proof. □

Lemma 31. *Consider the maximization problem:* $\max_{x\in\mathbb{R}_+} H\left(x|a,c,d,v\right)$. *This problem has a unique solution. This solution satisfies $x^* = d + \left(\frac{1-a+ae^{\frac{v-x-c}{\kappa}}}{1-a}\right)\kappa$, which is equivalent to:*

$$x^* = \kappa + d + \kappa W\left(\frac{a}{1-a}\exp\left(\frac{v-c-\kappa-d}{\kappa}\right)\right) \tag{27}$$

and satisfies the following two equations:

$$\left(\frac{ae^{\frac{v-c-x^*}{\kappa}}}{1-a+ae^{\frac{v-c-x^*}{\kappa}}}\right)x^* + \left(\frac{1-a}{1-a+ae^{\frac{v-c-x^*}{\kappa}}}\right)d = x^* - \kappa \tag{28}$$

and:

$$\frac{ae^{\frac{v-c-x^*}{\kappa}}}{1-a+ae^{\frac{v-c-x^*}{\kappa}}} = \left(\frac{x^*-d-\kappa}{x^*-d}\right) \tag{29}$$

Proof. By lemma 30, $H-d$ is log-concave. Clearly, any solution of $\max H$ is also a solution to $\max\ln\left(H-d\right)$. Since $\ln\left(H-d\right)$ is strictly concave, the following first order condition is both necessary and sufficient for a solution:

$$\frac{1}{x-d} - \frac{\left(1-a\right)}{\kappa\left(1-a+ae^{\frac{v-x-c}{\kappa}}\right)} = 0$$

which can be rearranged to obtain equation 28 and 29. This can also be rearranged as:

$$\frac{a}{1-a}e^{\frac{v-x-c}{\kappa}} = \left(\frac{x-d-\kappa}{\kappa}\right)$$

which can be rearranged to be:

$$\frac{d}{1-d}e^{\frac{v-c-\kappa-d}{\kappa}} = \left(\frac{x-d-\kappa}{\kappa}\right)e^{\left(\frac{x-d-\kappa}{\kappa}\right)}$$

or:

$$\left(\frac{x-d-\kappa}{\kappa}\right) = W\left(\frac{a}{1-a}e^{\frac{v-c-\kappa-d}{\kappa}}\right)$$

which can be easily rearranged to give the desired equality. □

Definition 9. Suppose (β, σ, μ) are consistent. We say that a sequence $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ is a (x^m, v) *perturbation sequence* for some (x^m, v) if there exists a $\mu^* \in \Delta(X^m \times V)$ with $\mu^*(x^m, v) > 0$ such that:

1. $\mu^n = \epsilon^n \mu^* + (1-\epsilon^n)\mu_m$
2. $\epsilon^n > 0$, $\epsilon^n \to 0$ and $\beta^n \to \beta$
3. β^n maximizes $E_m[U_2|\mu^n, \beta^n, \sigma]$ for all n.

Given some (x^m, v)-pertubation sequence, $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$, let $\pi_m^n = \int \beta_m^n \mathrm{d}\mu^n$.

Lemma 32. *For all* (x^M, v)*:*

$$\beta_M(x^M, v) = \frac{\pi_M e^{\frac{1}{\kappa}(v-x_M)}}{1-\pi_M+\pi_M e^{\frac{1}{\kappa}(v-x_M)}}$$

Proof. Let $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ be a (x^M, v)-pertubation sequence. By Lebesgue's dominated convergence theorem: $\int \beta_M^n \mathrm{d}\mu_M \to \int \beta_M \mathrm{d}\mu_M$, which implies $\int \beta_M^n \mathrm{d}\mu^n \to \int \beta_M \mathrm{d}\mu_M$. Using theorem 5:

$$\begin{aligned}
\beta_M(x^M, v) &= \lim_{n\to\infty} \beta_M^n(x^M, v) \\
&= \lim_{n\to\infty} \frac{\pi_M^n e^{\frac{1}{\kappa}(v-x_M)}}{1-\pi_M^n+\pi_M^n e^{\frac{1}{\kappa}(v-x_M)}} \\
&= \frac{\pi_M e^{\frac{1}{\kappa}(v-x_M)}}{1-\pi_M+\pi_M e^{\frac{1}{\kappa}(v-x_M)}}
\end{aligned}$$

as required. □

Lemma 33. $\int \beta_M \mathrm{d}\mu_M \in (0,1)$.

Proof. By lemma 32, if $\int \beta_M \mathrm{d}\mu_M = 0$ then $\beta_M \left(x^M, v\right) = 0$ for all $\left(x^M, v\right)$ contradicting β_M being attentive, while $\int \beta_M \mathrm{d}\mu_M = 1$ implies $\beta_M \left(x^M, v\right) = 1$ for all $\left(x^M, v\right)$ which cannot possibly be in equilibrium since then the seller must be charging $\bar{x}$ for sure in period M, contradicting β being a best reseponse to σ in period M. □

Note that we can now write the seller's expected value conditional on arriving to period M, the history being $\left(x^{M-1}, v\right)$ and offering x_M:

$$U_{1,M}\left(x_M | x^{M-1}, v\right) = \left(\frac{\pi_M e^{\frac{1}{\kappa}(v - x_M)}}{1 - \pi_M + \pi_M e^{\frac{1}{\kappa}(v - x_M)}}\right) x_M$$

which by lemma 31 has a unique maximizer in X for every v. Let $z_{M,v}$ be that maximizer.

Lemma 34. *$\sigma\left(z_{M,v}; x^{M-1}, v\right) = 1$ for all $\left(x^{M-1}, v\right)$. Moreover, $U_{1,M}\left(x_M | x^{M-1}, v\right)$ is independent of x^{M-1}.*

Proof. Follows directly from lemma 32 and lemma 31. □

In what follows, let:

$$\rho_m\left(x^m, v\right) = \frac{\beta_m\left(x^m, v\right)}{\int \beta_m \mathrm{d}\mu_m}$$

for all (x^m, v).

Lemma 35. *The following conditions must hold in equilibrium for all m:*

1. *There exists a function $b_m : X \times V \to (0,1)$ such that for every (x^m, v), $\beta_m\left(x^m, v\right) = b_m\left(x_m, v\right)$. Moreover, b_m satisfies:*

$$b_m\left(x_m, v\right) = \frac{\pi_m e^{\frac{1}{\kappa}(v - x_m)}}{\pi_m e^{\frac{1}{\kappa}(v - x_m)} + \left(1 - \pi_m\right) e^{\frac{e^{-r\Delta}}{\kappa}\left(v - z_{m+1,v} - \kappa \ln \rho_{m+1}\left(z_{m+1,v}, x^m, v\right)\right)}}$$

2. *$\pi_m \in (0,1)$ for all m.*
3. *For every m and v there exists a unique $z_{m,v} \in X$ such that $\sigma_m\left(z_{m,v} | x^{m-1}, v\right) = 1$ for all x^{m-1}.*
4. *For every m and v, $U_{1,m}\left(x_m | x^{m-1}, v\right)$ is independent of x^{m-1}.*

Proof. Note that we've shown that the lemma holds for period M. Suppose it holds for all periods $m+1, \ldots, M$. We will show that it holds for m. Let $\rho_{m+1,v} = \rho_{m+1}\left(z_{m+1,v}, x^m, v\right)$, which is well defined and independent of x^m by part (1) of the lemma. For some (x^m, v), let $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ be a (x^m, v)-pertubation sequence. Since $\beta_j^n\left(x^j, v\right) \to \beta_j\left(x^j, v\right)$ for all $\left(x^j, v\right)$, we have by Lebesgue's dominated convergence theorem that: $\pi_{m+j}^n \to \pi_{m+j}$ for all relevant $j \geq 0$. Moreover,

$$\lim_{n\to\infty} \rho_{m+1}^n\left(z_{m+1,v}, x^m, v\right) = \lim_{n\to\infty} \frac{\beta_{m+1}^n\left(z_{m+1,v}, x^m, v\right)}{\int \beta_{m+1}^n \mathrm{d}\mu^n} = \rho_{m+1,v}$$

since $\pi_{m+1} > 0$. By theorem 5:

$$\begin{aligned}
\beta_m\left(x^m, v\right) &= \lim_{n\to\infty} \beta_m^n\left(x^m, v\right) \\
&= \lim_{n\to\infty} \frac{\pi_m^n e^{\frac{1}{\kappa}(v-x_m)}}{\pi_m^n e^{\frac{1}{\kappa}(v-x_m)} + \left(1-\pi_m^n\right) e^{\frac{e^{-r\Delta}}{\kappa}\left(v - z_{m+1,v} - \kappa \ln \rho_{m+1}^n\left(z_{m+1,v}, x^m, v\right)\right)}} \\
&= \frac{\pi_m e^{\frac{1}{\kappa}(v-x_m)}}{\pi_m e^{\frac{1}{\kappa}(v-x_m)} + \left(1-\pi_m\right) e^{\frac{e^{-r\Delta}}{\kappa}\left(v - z_{m+1,v} - \kappa \ln \rho_{m+1,v}\right)}}
\end{aligned}$$

thereby proving (1). To prove (2), note that $\pi_m = 0$ contradicts β being attentive, and $\pi_m = 1$ will imply $\sigma_m\left(\bar{x}, x^{m-1}, v\right) = 1$ for all v which contradicts β being optimal for the buyer. Part (3) then follows from part (4) holding for $m+1$ and lemma 31, which then also imply part (4). □

Note that Lemma 1 follows from Lemma 35.

D.3 Recursive Representation in Finite Horizon

D.3.1 Statement of Proposition

Given Lemma 29, it is clear that the only relevant belief for the buyer at period m is the marginal of μ_m over V. Denote the marginal of μ_m over V by $\bar{\mu}_m$. Let $B_M\left(\bar{\mu}_1\right)$ be the bargaining game with a horizon of M periods with a prior distribution of $\bar{\mu}_1 = \mu_0$ over V. We will use Lemma 1 to represent equilibria of $B_M\left(\bar{\mu}_1\right)$ as a strategy for period 1 and an equilibrium of $B_{M-1}\left(\bar{\mu}_2\right)$. For that, we introduce the following definition of an equilibrium representation:

Definition 10. For every m, let $\theta_m \in \Delta(V)$, $b_m : X \times V \to (0,1)$ and $z_m : V \to X$. We say that $(\theta, b, z) = \left\{\left(\theta_m, b_m, z_m\right)\right\}_{m=1}^M$ is an *equilibrium representation* of $B_M\left(\mu_0\right)$ if there exists an equilibrium (μ, β, σ) of $B_M\left(\mu_0\right)$ such that for all m, v, $x \in X$ and $x^{m-1} \in X^{m-1}$: $\theta_m = \bar{\mu}_m$, $\sigma_m\left(z_{m,v} | x^{m-1}, v\right) = 1$ and $\beta_m\left(x, x^{m-1}, v\right) = b_m(x, v)$.

Given an equilibrium representation (θ, b, x) of $B_M\left(\mu_0\right)$, let $b_{m,v} := b_m\left(z_{m,v}, v\right)$, and take π_m to be the prior probability that the buyer accepts the m-th offer conditional on arriving to period m, i.e.

$$\pi_m := \sum_v \theta_m(v)\, b_{m,v} \tag{30}$$

and take:

$$\begin{aligned} z_{M+1,v} &:= \kappa \\ \pi_{M+1} &:= 1 \\ b_{M+1,v} &:= e^{(v-\kappa)/\kappa} \end{aligned}$$

for all v. These quantities will be useful in the following Proposition, which establishes that the equilibrium can be represented in a recursive manner.

Proposition 7. *(θ, b, z) is an equilibrium representation of $B_M(\mu_0)$ if and only if:*

1. *$\{(\theta_m, b_m, z_m)\}_{m=2}^{M}$ is an equilibrium representation of $B_{M-1}(\theta_2)$.*
2. *For every v and m, $z_{m,v}$ solves:*

$$z_{m,v} - e^{-r\Delta} z_{m+1,v} = \kappa \left(1 - e^{-r\Delta} + \left(\frac{b_{m,v}}{1 - b_{m,v}}\right)\right) \tag{31}$$

3. *For every v, m and $x \in X$, $b_m(x, v)$ solves:*

$$v - x - e^{-r\Delta}(v - z_{m+1,v}) = \kappa \ln\left(\frac{b_m(x,v)(1-\pi_m)}{\pi_m(1-b_m(x,v))}\right) - \kappa \ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}} \tag{32}$$

The subject matter of the rest of this section is to prove the above proposition. Note that one can combine equation 31 and equation 32 for $x = z_{m,v}$ to obtain the condition:

$$\left(\frac{b_{m,v}}{1-b_{m,v}}\right) e^{\left(\frac{b_{m,v}}{1-b_{m,v}}\right)} = \left(\frac{\pi_m}{1-\pi_m}\right)\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}} \left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} \tag{33}$$

D.3.2 Preliminary facts

We will begin by proving the following preliminary fact, which will help us establish that the seller's strategy is an interior solution.

Lemma 36. *Suppose (θ, b, z) satisfies equations 8 and 7 of Proposition 7. Then for every m:*

1. *$z_{m,v}$ is strictly increasing in v.*
2. *$b_{m,v} := b_m(z_{m,v}, v)$ is strictly increasing in v.*
3. *$v_l < v_l - \kappa \ln\left(\frac{b_{m,v_l}}{\pi_m}\right) < z_{m,v_l} < z_{m,v_h} < v_h - \kappa \ln\left(\frac{b_{m,v_h}}{\pi_m}\right) < v_h$.*

Proof. We will prove by induction. Suppose (θ, b, z) satisfy equations 8 and 7 of Proposition 7. Then they must satisfy equation 33, which implies:

$$\frac{b_{M,v}}{1-b_{M,v}} = W\left(\left(\frac{\pi_M}{1-\pi_M}\right)e^{\frac{v-\kappa}{\kappa}}\right)$$

which, since $\pi_M \in (0,1)$, implies that $b_{M,v}$ is strictly increasing in v. Since:

$$z_{M,v} = \kappa + \kappa\left(\frac{b_{M,v}}{1-b_{M,v}}\right)$$

we obtain that $z_{M,v}$ is strictly increasing in v. Finally, note that that equation 32 implies:

$$z_{M,v} = v + \kappa \ln\left(\frac{1-b_{M,v}}{1-\pi_M}\right) - \kappa \ln\left(\frac{b_{M,v}}{\pi_M}\right)$$

which implies part (3) since $b_{M,v_h} > \pi_M > b_{M,v_l}$. Suppose now (1)-(3) hold for $m+1,\ldots,M$. We will show they hold for m. First note that by equation 33, b is strictly increasing in v since b_{m+1} is strictly increasing in v. To obtain that $z_{m,v}$ is strictly increasing in v, note that:

$$z_{m,v} = \kappa\left(1 - e^{-r\Delta} + \left(\frac{b_{m,v}}{1-b_{m,v}}\right)\right) + e^{-r\Delta} z_{m+1,v}$$

which is strictly increasing in v. Using equation 7 we obtain that for every m and v:

$$z_{m,v} + \kappa \ln\left(\frac{b_{m,v}}{\pi_m}\right) = \left(1-e^{-r\Delta}\right)v + \kappa \ln\left(\frac{1-b_{m,v}}{1-\pi_m}\right) + e^{-r\Delta}\left(z_{m+1,v} + \kappa \ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)$$

and therefore, through repeated substitution:

$$z_{m,v} + \kappa \ln\left(\frac{b_{m,v}}{\pi_m}\right) = v + \kappa \sum_{j=m}^{M} e^{-r\Delta(j-m)} \ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right)$$

Therefore:

$$z_{m,v_h} = v_h - \kappa \ln\left(\frac{b_{m,v_h}}{\pi_m}\right) + \kappa \sum_{j=m}^{M} e^{-r\Delta(j-m)} \ln\left(\frac{1-b_{j,v_h}}{1-\pi_j}\right) < v_h - \kappa \ln\left(\frac{b_{m,v_h}}{\pi_m}\right) < v_h$$

where the inequality follows from $b_{m,v}$ being strictly increasing for $m, m+1, \ldots, M$. $\square$

D.3.3 Proof of Proposition 7

We will prove by induction. Suppose first that $\theta_M \in \Delta(V)$, $b_M : X \times V \to (0,1)$ and $z_M : V \to X$ satisfy the conditions of Proposition 7. Then note that:

$$b_M(x,v) = \frac{\pi_M e^{\frac{v-x}{\kappa}}}{1-\pi_M+\pi_M e^{\frac{v-x}{\kappa}}}$$

for all x. Therefore, by Lemma 31, $z_{M,v}$ is a best response to b_M. Take μ_1 to be the distribution over $X \times V$ defined by v being distributed v an $\sigma(z_{M,v}|v) = 1$ for all v. By Theorem 5, $\beta_M = b_M$ is a best response for the buyer conditional on the seller offering z_M. To show that b_M is a credible best response, fix any (x,v). Suppose wlog that $x < z_{M,v}$. By lemma 36 we have $z_{M,v} < v_h$ for all v. Pick $\tilde{x}$ such that $\tilde{x} = v_h$ and take α to be such that:

$$\alpha b_M(\tilde{x},v) + (1-\alpha) b_M(x,v) = b_{M,v}$$

then define μ^S to be such that for every $v' \neq v$: $\mu^S\left(z_{M,v'},v'\right) = \mu_0\left(z_{M,v'},v'\right)$. For v, set $\mu^S(x,v) = (1-\alpha)\mu_m(z_{M,v},v)$ and $\mu^S(\tilde{x},v) = \alpha\mu_0(z_{M,v},v)$. Then clearly b_M satisfies the conditions of Theorem 5 for every $\mu^\epsilon = \epsilon\mu^S + (1-\epsilon)\mu_0$. This establishes that b_M is a credible best response. Thus, (θ_M, b_M, z_M) is an equilibrium representation of $B_1(\mu_0)$.

Suppose now that (μ,β,z) is an equilibrium of $B_1(\mu_0)$. By Lemma 29 the equilibrium can be represented by some (θ_M, b_M, z_M). By Lemma 35:

$$b_M(x,v) = \frac{\pi_M e^{\frac{v-x}{\kappa}}}{1-\pi_M+\pi_M e^{\frac{v-x}{\kappa}}}$$

Then conditional on v the seller solves $\max_{x\in X} H(x|\pi_M,0,0,v)$. By Lemma 31 this has a unique solution in $\mathbb{R}_+$. Let $x_{M,v}$ be that solution. By Lemma 31, $x_{M,v} \geq \kappa$ for all M. Suppose $x_{M,v} \geq \bar{x}$ for some v. Note that, from Lemma 31, $x_{M,v}$ is strictly increasing in v. Therefore, $x_{M,v_h} \geq \bar{x}$. However, x_{M,v_h} satisfies:

$$x_{M,v_h} = \kappa + \kappa W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h-\kappa}{\kappa}}\right) \geq \bar{x} > v_h$$

but for $x_{M,v_h} > v_h$ one must have:

$$W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h-\kappa}{\kappa}}\right) > \frac{v_h-\kappa}{\kappa} \iff \pi_M > \frac{v_h-\kappa}{v_h}$$

which will, in turn, imply that $x_{M,v} > v$ for all v. But this means that $z_{M,v} > v$ for all v. But then the best response for the buyer must satisfy $\pi_M = 0$, which will contradict β being attentive.

Therefore, $x_{M,v} < \bar{x}$ for all v, which implies that $z_{M,v} = x_{M,v}$. Proposition 7 then follows from Lemma 31.

Suppose now the proposition holds for $B_{M-1}(\mu_0)$ for all μ_0. We will show that it also holds for $B_M(\mu_0)$. Suppose first $\{(\theta_m, b_m, z_m)\}_{m=1}^{M}$ satisfies the equations 31 and 32 of Proposition 7. We will show that $\{(\theta_m, b_m, z_m)\}_{m=1}^{M}$ is an equilibrium representation of $B_M(\mu_0)$. Note that equation 32 implies:

$$b_m(x_m, v) = \frac{\pi_m e^{\frac{1}{\kappa}\left(v - e^{-r\Delta}(v - z_{m+1,v} - \kappa \ln(b_{m+1,v}/\pi_{m+1})) - x_m\right)}}{1 - \pi_m + \pi_m e^{\frac{1}{\kappa}(v - e^{-r\Delta}(v - z_{m+1,v} - \kappa \ln(b_{m+1,v}/\pi_{m+1})) - x_m)}}$$

Define σ by $\sigma_m\left(z_{m,v}|x^{m-1}, v\right) = 1$ for all $\left(x^{m-1}, v\right)$, take $\beta_m(x^m, v) = b_m(x_m, v)$ for all v, and let μ be the beliefs implied by the seller using σ, the buyer using β and μ being updated using Bayesian updating (which is always possible since $b_m(x_m, v) \in (0,1)$ for all (x_m, v)).

Step 1: σ is a best response to β after every history.

Proof. In period M, conditional on v, the seller solves $\max_{x \in X} H(x|\pi_M, 0, 0, v)$. By Lemma 31 $z_{M,v}$ is the unique solution to this problem. Moreover, the expected value for the seller conditional on arriving to period M and having good of quality v is $z_{M,v} - \kappa$. Suppose now that the seller's value conditional on arriving to period $m+1$ and on v is $z_{m+1,v} - \kappa$ regardless of x^m. Then the seller will choose x_m to maximize:

$$\max_{x_m \in X} b_m(x_m, v)\, x_m + (1 - b_m(x_m, v))\, e^{-r\Delta}(z_{m+1,v} - \kappa)$$

which is equivalent to maximizing:

$$H\left(x|\pi_m, e^{-r\Delta}(v - z_{m+1,v} - \kappa \ln(b_{m+1,v}/\pi_{m+1})), e^{-r\Delta}(z_{m+1,v} - \kappa), v\right)$$

the unique solution in $\mathbb{R}_+$ by Lemma 31, equation 29 is equal to:

$$x_{m,v} = \kappa + e^{-r\Delta}(z_{m+1,v} - \kappa) + \kappa\left(\frac{b_m(x_{m,v}, v)}{1 - b_m(x_{m,v}, v)}\right)$$

Which, by Lemma 36 implies that $x_{m,v} \in [v_l, v_h] \subset X$. Thus, $z_{m,v} = x_{m,v}$, meaning that $z_{m,v}$ solves the v type seller's period m problem, as required. □

Step 2: β is a credible best response to σ after every history given μ.

Proof. Note that β is a best response by Theorem 5. Fix any (x^m, v). Suppose wlog that $x_m < z_{m,v}$. By lemma 36: $z_{m,v} < v_h$ for all v. Pick $\tilde{x}^m$ such that $\tilde{x}_m = v_h$ and take α to be such that:

$$\alpha\beta_m(\tilde{x}^m, v) + (1 - \alpha)\beta_m(x^m, v) = \beta_m(z_{m,v}, v)$$

then define μ^S to be such that for every $v^{'} \neq v$: $\mu^S\left(z_{1,v'},\ldots,z_{m,v'},v^{'}\right) = \mu_m\left(z_{1,v'},\ldots,z_{m,v'},v^{'}\right)$. For v, set $\mu^S\left(x^m,v\right) = (1-\alpha)\,\mu_m\left(z_{1,v},\ldots,z_{m,v},v\right)$ and $\mu^S\left(\tilde{x}^m,v\right) = \alpha\mu_m\left(z_{1,v},\ldots,z_{m,v},v\right)$. Note that by construction the buyer's posterior over V after using β_m and reaching period $m+1$ is the same under μ_m and μ^S. The same holds for every $\mu^\epsilon = \epsilon\mu^S + (1-\epsilon)\,\mu_m$ for $\epsilon \in (0,1)$. Hence, it is straightforward to show that $(\beta_m,\ldots,\beta_M)$ satisfies the conditions of theorem 5 when the distribution over $X^m \times V$ is $\mu^\epsilon = \epsilon\mu^S + (1-\epsilon)\,\mu_m$ for $\epsilon \in (0,1)$ and future offers are drawn from σ. The fact that β is a credible best response follows. □

We have therefore established that $\{(\theta_m, b_m, z_m)\}_{m=1}^M$ is an equilibrium representation of $B_M(\mu_0)$.

Suppose now that $\{(\theta_m, b_m, z_m)\}_{m=1}^M$ is an equilibrium representation of $B_M(\mu_0)$. By Lemma 35, b_M satisfies:

$$b_M(x,v) = \frac{\pi_M e^{\frac{v-x}{\kappa}}}{1-\pi_M+\pi_M e^{\frac{v-x}{\kappa}}}$$

therefore, the seller's problem conditional on arriving to period M and having a good of quality v is $\max_{x\in X} H(x|\pi_M,0,0,v)$. By Lemma 31 this has a unique solution in $\mathbb{R}_+$. Let $x_{M,v}$ be that solution. By Lemma 31, $x_{M,v} \geq \kappa$ for all M. Suppose $x_{M,v} \geq \bar{x}$ for some v. Note that, from Lemma 31, $x_{M,v}$ is strictly increasing in v. Therefore, $x_{M,v_h} \geq \bar{x}$. However, x_{M,v_h} satisfies:

$$x_{M,v_h} = \kappa + \kappa W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h-\kappa}{\kappa}}\right) \geq \bar{x} > v_h$$

but for $x_{M,v_h} > v_h$ one must have:

$$W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h-\kappa}{\kappa}}\right) > \frac{v_h-\kappa}{\kappa} \iff \pi_M > \frac{v_h-\kappa}{v_h}$$

which will, in turn, imply that $x_{M,v} > v$ for all v. But this means that $z_{M,v} > v$ for all v. However, this means that the best response for the buyer must satisfy $\pi_M = 0$, which will contradict β being attentive. Therefore, $x_{M,v} < \bar{x}$ for all v, which implies that $z_{M,v} = x_{M,v}$. Note that this implies that $z_{M,v}$ satisfies equation 31. Moreover, the seller's expected utility conditional on ariving to period M and having a good of quality v is $z_{M,v} - \kappa$ (Lemma 31, equation 28). Suppose now that the seller's value conditional on arriving to period $m+1$ and on v is $z_{m+1,v} - \kappa$ regardless of x^m. Then the seller will choose x_m to maximize:

$$\max_{x_m \in X} b_m(x_m,v)\,x_m + (1-b_m(x_m,v))\,e^{-r\Delta}\left(z_{m+1,v}-\kappa\right)$$

Note that by Lemma 35 b_m must satisfy:

$$b_m(x_m,v) = \frac{\pi_m e^{\frac{1}{\kappa}\left(v-e^{-r\Delta}\left(v-z_{m+1,v}-\kappa\ln(b_{m+1,v}/\pi_{m+1})\right)-x_m\right)}}{1-\pi_m+\pi_m e^{\frac{1}{\kappa}\left(v-e^{-r\Delta}\left(v-z_{m+1,v}-\kappa\ln(b_{m+1,v}/\pi_{m+1})\right)-x_m\right)}}$$

and therefore the seller's problem is equivalent to to maximizing:

$$H\left(x|\pi_m, e^{-r\Delta}\left(v - z_{m+1,v} - \kappa\ln\left(b_{m+1,v}/\pi_{m+1}\right)\right), e^{-r\Delta}\left(z_{m+1,v} - \kappa\right), v\right)$$

the unique solution in $\mathbb{R}_+$ by Lemma 31, equation 29 is equal to:

$$x_{m,v} = \kappa + e^{-r\Delta}\left(z_{m+1,v} - \kappa\right) + \kappa\left(\frac{b_m\left(x_{m,v}, v\right)}{1 - b_m\left(x_{m,v}, v\right)}\right)$$

which, by Lemma 36 implies that $x_{m,v} \in [v_l, v_h] \subset X$. Thus, $z_{m,v} = x_{m,v}$, meaning that $z_{m,v}$ solves the v type seller's period m problem. Therefore we obtain that z satisfies equation 31. Simple algebra reveals that Lemma 31 implies that b_m must satisfy equation 32. To obtain that $\{(\theta_m, b_m, z_m)\}_{m=2}^M$ is an equilibrium representation of $B_{M-1}(\bar{\mu}_2)$, note that the strategies induced by $\{(b_m, z_m)\}_{m=2}^M$ in the game $B_{M-1}(\bar{\mu}_2)$ satisfy equations 31 and 32. Then $\{(\theta_m, b_m, z_m)\}_{m=2}^M$ being an equilibrium representation of $B_{M-1}(\bar{\mu}_2)$ follows from the induction assumption.

D.4 Proof of Theorem 1

Lemma 37. *(θ, b, z) is an equilibrium representation of $B_M(\mu_0)$ if and only if there exists an equilibrium average-ratio path of $B_M(\mu_0)$, (a, p, ϑ), such that:*

1. *For all m: $\theta_m = \vartheta_m$ and $a_m = \pi_m$.*
2. *For every m and v:$p_{m,v} = b_{m,v}/\pi_m$.*
3. *For all m and v:*

$$z_{m,v} - e^{-r\Delta}z_{m+1,v} = \left(1 - e^{-r\Delta}\right)\kappa + \kappa W\left(\frac{a_m}{1 - a_m}\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} p_{m+1,v}^{e^{-r\Delta}}\right) \tag{34}$$

4. *For all m, v and $x \in X$, $b_m(x, v)$ solves equation:*

$$v - x - e^{-r\Delta}\left(v - z_{m+1,v}\right) = \kappa\ln\left(\frac{b_m(x, v)\left(1 - a_m\right)}{a_m\left(1 - b_m(x, v)\right)}\right) - \kappa\ln\left(p_{m+1,v}\right)^{e^{-r\Delta}} \tag{35}$$

Proof. Suppose first that (θ, b, z) is an equilibrium representation of $B_M(\mu_0)$. Define $\vartheta = \theta$, $a_m = \pi_m$ and $p_{m,v} = b_{m,v}/\pi_m$. Part (4) holds due to Proposition 7. By equation: 33:

$$b_{m,v} = \frac{W\left(\frac{\pi_m}{1-\pi_m}\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\right)}{1 + W\left(\frac{\pi_m}{1-\pi_m}\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\right)}$$

The fact that (a,p,ϑ) is an equilibrium average-ratio path follows. Suppose (a,p,ϑ) is an equilibrium average ratio path. Define $z_{m,v}$ by equation 34 and take $b_m(x,v)$ to be the solution to the equation 35 for every x. Note that for every m:

$$b_m(z_{m,v},v) = \frac{\left(\frac{a_m}{1-a_m}\right)\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} p_{m+1,v}^{e^{-r\Delta}} e^{\frac{1}{\kappa}\left(\left(1-e^{-r\Delta}\right)\kappa-\left(z_{m,v}-e^{-r\Delta}z_{m+1,v}\right)\right)}}{1+\left(\frac{a_m}{1-a_m}\right)\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} p_{m+1,v}^{e^{-r\Delta}} e^{\frac{1}{\kappa}\left(\left(1-e^{-r\Delta}\right)\kappa-\left(z_{m,v}-e^{-r\Delta}z_{m+1,v}\right)\right)}}$$

substituting in equation 34 for $\left(1-e^{-r\Delta}\right)\kappa-\left(z_{m,v}-e^{-r\Delta}z_{m+1,v}\right)$ and noting that $\left(y/e^{W(y)}\right) = W(y)$ gives:

$$b_m(z_{m,v},v) = \frac{W\left(\left(\frac{a_m}{1-a_m}\right)\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} p_{m+1,v}^{e^{-r\Delta}}\right)}{1+W\left(\left(\frac{a_m}{1-a_m}\right)\left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\Delta}} p_{m+1,v}^{e^{-r\Delta}}\right)} = a_m p_m$$

Therefore, $\pi_1 = \sum_v \mu_0 b_{1,v} = a_1$. Clearly, $\theta_m = \vartheta_m$ describes the evolution of the buyer's beliefs over V given the strategy b. Moreover, for every m: $\pi_m = \sum_v \theta_m(v) b_{m,v} = a_m$, and $b_{m,v}/\pi_m = p_m$. It is straightforward now to show that $z_{m,v}$ solves equation 31 and $b_m(x,v)$ solves equation 32. The Lemma follows. □

Given the above Lemma, Theorem 1 is implied by Theorem 6.

D.5 Properties of equilibrium in finite horizon

D.5.1 Boundedness of $b_{m,v}$ and π_m

In the following subsection, assume that (θ,b,z) is an equilibrium representation in $B_M(\mu_0)$ and let (a,p,ϑ) be the equilibrium average ratio path of $B_M(\mu_0)$ from lemma 37

Claim 1. Then $\frac{1}{2}\pi_m \leq b_{m,v} \leq e^{\frac{v-\kappa}{\kappa}}\pi_m$

Proof. By Lemma 20, $p_{m,v} \in [1/2, c_v]$ for all m and v. The claim follows from $b_{m,v}/\pi_m = p_{m,v}$ for all m and v. □

Lemma 38. *For every m,*

$$z_{m,v} - e^{-r\Delta}\left(z_{m+1,v}-\kappa\right) \leq \left(1-e^{-r\Delta}\right)v + e^{-r\Delta}\kappa$$

if and only if:

$$\begin{aligned} p_{m+1,v}^{e^{-r\Delta}} &\leq \frac{R\left(a_m, v, p_{m+1,v}\right)}{R^c\left(a_m, v, p_{m+1,v}\right)} \\ &= \frac{1-a_m}{a_m} W\left(z^*\left(a_m; v, p_{m+1,v}\right)\right) \end{aligned}$$

Proof. Note that the inequality is equivalent to:

$$\frac{a_m}{1-a_m}\left(p_{m+1,v}\right)^{e^{-r\Delta}} \leq W\left(z^*\left(a_m; v, p_{m+1,v}\right)\right)$$

which is equivalent to:

$$\frac{a_m}{1-a_m}\left(p_{m+1,v}\right)^{e^{-r\Delta}} \leq \left(1-e^{-r\Delta}\right) \ln a_v$$

or:

$$z\left(a_m; v, p_{m+1,v}\right) \leq c_v^{1-e^{-r\Delta}} \ln c_v^{1-e^{-r\Delta}}$$

which is true if and only if $W\left(z^*\left(a_m; v, p_{m+1,v}\right)\right) \leq \left(1-e^{-r\Delta}\right) \ln c_v = \left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)$. The conclusion then follows from:

$$z_{m,v} - e^{-r\Delta}\left(z_{m+1,v} - \kappa\right) = \kappa + \kappa W\left(z^*\left(a_m; v, p_{m+1,v}\right)\right)$$

□

Claim 2. For every m and v: $b_{m,v} \leq \frac{v_h - \kappa}{v_h}$.

Proof. Follows from Lemma 27 and Lemma 37. □

Lemma 39. *For every m, there exists a v such that*

$$z_{m,v} - e^{-r\Delta}\left(z_{m+1,v} - \kappa\right) \leq \left(1-e^{-r\Delta}\right) v + e^{-r\Delta}\kappa$$

Proof. Suppose otherwise. Then by the previous lemma:

$$p_{m+1,v}^{e^{-r\Delta}} > \frac{R\left(a_m; v, p_{m+1,v}\right)}{R^c\left(a_m; v, p_{m+1,v}\right)}$$

for all v. Therefore:

$$p_{m+1,v}^{e^{-r\Delta}} R^c\left(a_m; v, p_{m+1,v}\right) > R\left(a_m; v, p_{m+1,v}\right)$$

As such:

$$\sum_v \vartheta_m(v) R^c\left(a_m; v, p_{m+1,v}\right) p_{m+1,v}^{e^{-r\Delta}} > 1$$

however:

$$\begin{aligned}\sum_v \vartheta_m(v) R^c(a_m; v, p_{m+1,v}) p_{m+1,v}^{e^{-r\Delta}} &= \sum_v \vartheta_m(v) \frac{1-a_m p_{m,v}}{1-a_m} p_{m+1,v}^{e^{-r\Delta}} \\ &< \left(\sum_v \vartheta_m(v) \frac{1-a_m p_{m,v}}{1-a_m} p_{m,v}\right)^{e^{-r\Delta}} \\ &= \left(\sum_v \vartheta_{m+1}(v) a_m p_{m,v}\right)^{e^{-r\Delta}} = 1\end{aligned}$$

a contradiction. □

Claim 3. For every m, there exists a v such that:

$$b_{m,v} \leq \frac{\left(1-e^{-r\Delta}\right)(v-\kappa)}{\left(1-e^{-r\Delta}\right)(v-\kappa)+\kappa}$$

Proof. By Lemma 39 for every m there is a v such that:

$$z_{m,v} - e^{-r\Delta}(z_{m+1,v}-\kappa) \leq \left(1-e^{-r\Delta}\right) v + e^{-r\Delta}\kappa$$

but by Proposition 7, part 2:

$$\begin{aligned}b_{m,v} &= \frac{\left(z_{m,v} - e^{-r\Delta}(z_{m+1,v}-\kappa)\right)-\kappa}{\left(z_{m,v} - e^{-r\Delta}(z_{m+1,v}-\kappa)\right)} \\ &\leq \frac{\left(1-e^{-r\Delta}\right)(v-\kappa)}{\left(1-e^{-r\Delta}\right)(v-\kappa)+\kappa}\end{aligned}$$

as required. □

E Infinite Horizon Equilibrium

In this section we provide an analysis of the infinite horizon equilibria of our game. We begin by proving that an equilibrium exists via Theorem 2. The fact that the equilibrium satisfies equations 7 and 8 from Lemma 1 will follow directly from Lemma 29 and Proposition 7. We then move to establishing some properties shared by all equilibria of the finite horizon game. With these properties at hand, we turn to proving Propositions 2 and 5.

E.1 Proof of Theorem 2

E.1.1 Preliminary Lemma

Let $(a_m, b_m)_{m=1}^{\infty}$ and $(a_m^n, b_m^n)_{n,m=1}^{\infty}$ be such that $b_m^n, b_m \in (0,1)$, $c_m^n, c_m \geq 0$ and $(b_m^n, c_m^n) \to (b_m, c_m)$ for every m. Define $J : X^{\infty} \to \mathbb{R}_+$ by:

$$J\left(x^{\infty}\right) = \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \left(\frac{b_j e^{\frac{1}{\kappa}(v - x_j - c_j)} \left(\prod_{k=1}^{j-1} (1 - b_k) \right)}{\prod_{k=1}^{j} \left(1 - b_k + b_k e^{\frac{1}{\kappa}(v - x_k - c_k)} \right)} \right) x_j$$

$J^* = \max J$,

$$J_n\left(x^{\infty}\right) = \sum_{j=1}^{n} e^{-r\Delta(j-1)} \left(\frac{b_j^n e^{\frac{1}{\kappa}\left(v - x_j - c_j^n\right)} \left(\prod_{k=1}^{j-1} (1 - b_k^n) \right)}{\prod_{k=1}^{j} \left(1 - b_k^n + b_k^n e^{\frac{1}{\kappa}\left(v - x_k - c_k^n\right)} \right)} \right) x_j$$

and $J_n^* = \max J_n$.

Lemma 40. *$J_n^* \to J^*$. Moreover, if $x^{\infty(n)} \in \arg\max J_n$ for every n is such that $x^{\infty(n)} \to x^{\infty}$ for some $x^{\infty} \in X^{\infty}$, then $x^{\infty} \in \arg\max J$.*

Proof. Since $(b_m^n, c_m^n) \to (b_m, c_m)$ for every m, for every N and $\epsilon > 0$, there is an $N_\epsilon > N$ such that $n > N_\epsilon$ implies:

$$\left| \sum_{j=1}^{N} e^{-r\Delta(j-1)} \left(\begin{array}{c} \left(\dfrac{b_j^n e^{\frac{1}{\kappa}\left(v - x_j - c_j^n\right)} \left(\prod_{k=1}^{j-1} (1 - b_k^n) \right)}{\prod_{k=1}^{j} \left(1 - b_k^n + b_k^n e^{\frac{1}{\kappa}\left(v - x_k - c_k^n\right)} \right)} \right) \\ - \left(\dfrac{b_j e^{\frac{1}{\kappa}(v - x_j - c_j)} \left(\prod_{k=1}^{j-1} (1 - b_k) \right)}{\prod_{k=1}^{j} \left(1 - b_k + b_k e^{\frac{1}{\kappa}(v - x_k - c_k)} \right)} \right) \end{array} \right) x_j \right| < \epsilon$$

Since $x \in X^{\infty}$, this implies that for every $n > N$ and every x^{∞}: $\left| J\left(x^{\infty}\right) - J_n\left(x^{\infty}\right) \right| < \epsilon + \frac{e^{-r\Delta N}}{1 - e^{-r\Delta}} \bar{x}$. This is also true for any $x^{\infty} \in \arg\max J$, implying that: $\left| J^* - J_n^* \right| < \epsilon + \frac{e^{-r\Delta N}}{1 - e^{-r\Delta}} \bar{x}$. Since this is true for all N, we have that $J_n^* \to J^*$. If $x^{\infty(n)} \in \arg\max J_n$ for every n is such that $x^{\infty(n)} \to x^{\infty}$ for some $x^{\infty} \in X^{\infty}$. Fix an N and $\epsilon > 0$. Then since $(b_m^n, c_m^n) \to (b_m, c_m)$ and $x_m^n \to x_m$ for all m, there exists an $N_\epsilon > N$ such that for all $n > N_\epsilon$:

$$\left| \sum_{j=1}^{N} e^{-r\Delta(j-1)} \left(\begin{array}{c} \left(\dfrac{b_j e^{\frac{1}{\kappa}(v - x_j - c_j)} \left(\prod_{k=1}^{j-1} (1 - b_k) \right)}{\prod_{k=1}^{j} \left(1 - b_k + b_k e^{\frac{1}{\kappa}(v - x_k - c_k)} \right)} \right) x_j \\ - \left(\dfrac{b_j^n e^{\frac{1}{\kappa}\left(v - x_j^n - c_j^n\right)} \left(\prod_{k=1}^{j-1} (1 - b_k^n) \right)}{\prod_{k=1}^{j} \left(1 - b_k^n + b_k^n e^{\frac{1}{\kappa}\left(v - x_k^n - c_k^n\right)} \right)} \right) x_j^n \end{array} \right) \right| < \epsilon$$

since $x^{\infty} \in X^{\infty}$, this implies that for every $n > N$: $\left|J\left(x^{\infty}\right) - J_n\left(x^{\infty(n)}\right)\right| < \epsilon + \frac{e^{-r\Delta N}}{1-e^{-r\Delta}}\bar{x}$, thereby implyng that $J_n^* = J_n\left(x^{\infty(n)}\right) \to J\left(x^{\infty}\right)$. Therefore:

$$\begin{aligned} \left|J\left(x^{\infty}\right) - J^*\right| &\leq \left|J\left(x^{\infty}\right) - J_n\left(x^{\infty(n)}\right)\right| + \left|J_n\left(x^{\infty(n)}\right) - J^*\right| \\ &= \left|J\left(x^{\infty}\right) - J_n\left(x^{\infty(n)}\right)\right| + \left|J_n^* - J^*\right| \to 0 \end{aligned}$$

as required. □

E.1.2 Proof of Theorem 2

By Proposition 7, we can represent every sequence of equilibria $\{(\mu^n, \beta^n, \sigma^n)\}_{n=1}^{\infty}$ of $B_{M_n}(\mu_0)$ by their equilibrium representations, $\{(\theta^n, b^n, z^n)\}_{n=1}^{\infty}$. Note that for every m $\left\{\left(\theta_m^n(v), b_{m,v}^n, z_{m,v}^n\right)\right\}_{v\in V}$ is an element of a compact subset of $\mathbb{R}_+^{3V}$. Therefore, by Cantor's diagonal method there exists a subsequence $\{(\theta^{n_k}, b^{n_k}, z^{n_k})\}_{k=1}^{\infty}$ such that $\left\{\left(\theta_m^n(v), b_{m,v}^n, z_{m,v}^n\right)\right\}_{v\in V}$ converges to $\{(\theta_m(v), b_{m,v}, z_{m,v})\}_{v\in V}$ for all m. As a consequence, the prior probability that the buyer accepts conditional on arriving to period m, π_m^n, converges to $\pi_m := \sum_v \theta_m(v)\, b_{m,v}$. By Proposition 7 we have for every $x \in X$:

$$b_m^n(x,v) = \frac{\left(\frac{\pi_m^n}{1-\pi_m^n}\right)\left(e^{\frac{v}{\kappa}}\right)^{1-e^{-r\Delta}}\left(e^{\frac{z_{m+1,v}^n}{\kappa}}\left(\frac{b_{m+1,v}^n}{\pi_{m+1}^n}\right)\right)^{e^{-r\Delta}} e^{-\frac{x}{\kappa}}}{1+\left(\frac{\pi_m^n}{1-\pi_m^n}\right)\left(e^{\frac{v}{\kappa}}\right)^{1-e^{-r\Delta}}\left(e^{\frac{z_{m+1,v}^n}{\kappa}}\left(\frac{b_{m+1,v}^n}{\pi_{m+1}^n}\right)\right)^{e^{-r\Delta}} e^{-\frac{x}{\kappa}}}$$

which implies that $b_m^n(x,v) \to b_m(x,v)$ where:

$$\begin{aligned} b_m(x,v) &= \frac{\left(\frac{\pi_m}{1-\pi_m}\right)\left(e^{\frac{v}{\kappa}}\right)^{1-e^{-r\Delta}}\left(e^{\frac{z_{m+1,v}}{\kappa}}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)^{e^{-r\Delta}} e^{-\frac{x}{\kappa}}}{1+\left(\frac{\pi_m}{1-\pi_m}\right)\left(e^{\frac{v}{\kappa}}\right)^{1-e^{-r\Delta}}\left(e^{\frac{z_{m+1,v}}{\kappa}}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)^{e^{-r\Delta}} e^{-\frac{x}{\kappa}}} \\ &= \frac{\pi_m e^{\frac{1}{\kappa}\left(v-x-e^{-r\Delta}\left(v-\kappa\ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)-z_{m+1,v}\right)\right)}}{1-\pi_m+\pi_m e^{\frac{1}{\kappa}\left(v-x-e^{-r\Delta}\left(v-\kappa\ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)-z_{m+1,v}\right)\right)}} \end{aligned}$$

and therefore, $\beta^{n_k}(x^m, v) \to \beta(x^m, v) = b_m(x_m, v)$. Note that by Claim 1: $b_{m,v}^n \leq (v_h - \kappa)/v_h$ for all v and m and therefore $\pi_m^n \leq (v_h - \kappa)/v_h$, i.e. $\pi_m \leq (v_h - \kappa)/v_h$. We will now show that $\pi_m > 0$. Suppose otherwise, i.e. there is some m such that $\pi_m^n \to 0$. Then this implies that $\bar{\mu}_m = \bar{\mu}_{m+1}$. Let:

$$f(k,l) = \frac{W\left(\frac{k}{1-k}l\right)}{k\left(1+W\left(\frac{k}{1-k}l\right)\right)}$$

then using $W(z) = z/e^{W(z)}$ we obtain:

$$f(k,l) = \left(\frac{l}{1-k}\right) \frac{1}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)\left(1+W\left(\frac{k}{1-k}l\right)\right)}$$

and therefore $f(0,l) = l$. Hence, using the mean value theorem, for every M there is a $k^* \in [0, a_m^n]$ such that:

$$\begin{aligned}
\left|p_{m,v}^n - c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}^n\right)^{e^{-r\Delta}}\right| &= \left|f\left(a_m^n, c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}^n\right)^{e^{-r\Delta}}\right) - f\left(0, c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}^n\right)^{e^{-r\Delta}}\right)\right| \\
&\leq \left|\frac{\partial f}{\partial k}\left(k^*, c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}^n\right)^{e^{-r\Delta}}\right)\right| a_m^n
\end{aligned}$$

but:

$$\begin{aligned}
\frac{\partial f}{\partial k} &= \left(\frac{1-(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}\right)\frac{f(k,l)}{k} \\
&= \left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}\right)\frac{f(k,l)}{k} + \frac{f(k,l)}{1-k}
\end{aligned}$$

which, since $k^* < 1$ we have that $f(k,l)/(1-k)$ is bounded in the range $(k,l) \in [0, (v_h - \kappa)/v_h] \times [0, c_{v_h}]$. In addition:

$$\lim_{k\to 0}\left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}\right)\frac{f(k,l)}{k} = \left(\lim_{k\to 0} f(k,l)\right)\lim_{k\to 0}\left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^2}{k}\right)$$

assuming the limit $\lim_{k\to 0} k^{-1}\left(1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^2\right)$ exists. Using L'Hopital's rule:

$$\begin{aligned}
\lim_{k\to 0} k^{-1}\left(1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^2\right) &= \lim_{k\to 0}\frac{2\left(1+W\left(\frac{k}{1-k}l\right)\right)}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)\left(1+W\left(\frac{k}{1-k}l\right)\right)}\left(\frac{l}{(1-k)^2}\right) \\
&= \lim_{k\to 0}\frac{2}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)}\left(\frac{l}{(1-k)^2}\right) = 2l
\end{aligned}$$

thus, $\frac{\partial f}{\partial k}$ is bounded, implying that $\left|p^n_{m,v} - c_v^{1-e^{-r\Delta}}\left(p^n_{m+1,v}\right)^{e^{-r\Delta}}\right| \to 0$ and therefore $p_{m,v} = c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}\right)^{e^{-r\Delta}}$ for all v. But:

$$\sum_v \bar{\mu}_m(v)\, p_{m,v} = \lim_{M_n\to\infty} \sum_v \bar{\mu}^n_m(v)\, p^n_{m,v} = 1$$

and therefore we've obtained:

$$\begin{aligned} 1 &= \sum_v \bar{\mu}_m(v)\, c_v^{1-e^{-r\Delta}}\left(p_{m+1,v}\right)^{e^{-r\Delta}} \\ &> \sum_v \bar{\mu}_m(v)\, p_{m+1,v} \\ &= \sum_v \bar{\mu}_{m+1}(v)\, p_{m+1,v} = 1 \end{aligned}$$

where the inequality follows $c_v \geq p_{m+1,v}$ for all v and $c_{v_l} > 1 \geq p_{m+1,v_l}$, a contradiction. Hence, we have $\pi_m > 0$ for all m.

Thus, by Lemma 40, offering $z_{m,v}$ for sure in period m conditional on the quality of the good being v regardless of the history is a best response for the seller conditional on the buyer using β.

We will now show that β is a best response to σ defined by the seller offering $z_{m,v}$ for sure in period m conditional on the quality of the good is v. Note that:

$$\kappa \ln\left(\frac{1-b_{m,v}}{1-\pi_m}\right) = e^{-r\Delta}\left(v - z_{m+1,v} - \kappa \ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)$$

therefore, for every $n < m$, consider the buyer's quasi-value, $\mathcal{U}_m(\beta, \sigma | x^n, v)$, is equal to:

$$\begin{aligned} &\sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \prod_{k=m}^{j-1}(1-b_{m,v}) \begin{pmatrix} b_{j,v}\left(v - z_{j,v} - \kappa\ln\left(\frac{b_{j,v}}{\pi_j}\right)\right) \\ -(1-b_{j,v})\,\kappa \ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right) \end{pmatrix} \\ &= \sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \prod_{k=m}^{j-1}(1-a_k p_{k,v}) \begin{pmatrix} v - z_{j,v} - \kappa \ln\left(\frac{b_{j,v}}{\pi_j}\right) \\ -e^{-r\Delta}(1-b_{j,v}) \times \\ \left(v - z_{j+1,v} - \kappa\ln\left(\frac{b_{j+1,v}}{\pi_{j+1}}\right)\right) \end{pmatrix} \\ &= v - z_{m,v} - \kappa \ln\left(\frac{b_{m,v}}{\pi_m}\right) \end{aligned}$$

Thereby implying, by Theorem 5, that β is a best response to σ. We will now prove that β is a credible best response to σ. Note that Lemma 36 implies that for every m and n: $z^n_{m,v} \in [v_l, v_h]$

and therefore $z_{m,v} \in [v_l, v_h]$. As such, for every $(x^m, v) \in X \times V$, there is a $\tilde{x}^m$ and $\alpha \in (0,1)$ such that:

$$\alpha b_m (x^m, v) + (1-\alpha) b_m (\tilde{x}^m, v) = b_{m,v}$$

define μ^S to be such that for every $v' \neq v$: $\mu^S \left(z_{1,v'}, \ldots, z_{m,v'}, v'\right) = \mu_m \left(z_{1,v'}, \ldots, z_{m,v'}, v'\right)$. For v, set $\mu^S (x^m, v) = (1-\alpha) \mu_m (z_{1,v}, \ldots, z_{m,v}, v)$ and $\mu^S (\tilde{x}^m, v) = \alpha \mu_m (z_{1,v}, \ldots, z_{m,v}, v)$. Note that by construction the buyer's posterior over V after using β_m and reaching period $m+1$ is the same under μ_m and μ^S. The same holds for every $\mu^\epsilon = \epsilon \mu^S + (1-\epsilon) \mu_m$ for $\epsilon \in (0,1)$. Hence, it is straightforward to show that $(\beta_m, \ldots)$ satisfies the conditions of theorem 5 when the distribution over $X^m \times V$ is $\mu^\epsilon = \epsilon \mu^S + (1-\epsilon) \mu_m$ for $\epsilon \in (0,1)$ and future offers are drawn from σ. The fact that β is a credible best response follows.

Note that we've shown that every sequence of equilibria $\{(\mu^n, \beta^n, \sigma^n)\}_{n=1}^\infty$ in $B_{M_n}(\mu_0)$ with $M_n \to 0$ has a convergent subsequence $\{(\mu^{n_k}, \beta^{n_k}, \sigma^{n_k})\}_{k=1}^\infty$ whose limit (μ, β, σ) is an attentive recommendation perfect equilibrium. As such, if a sequence of equilibria $\{(\mu^n, \beta^n, \sigma^n)\}_{n=1}^\infty$ converges then the limit must be an equilibrium of the infinite horizon game.

E.2 Proof of equations 7 and 8

Follows from Lemma 29 and Proposition 7.

E.3 Additional Properties of Infinite Horizon Equilibria

E.3.1 Monotonicity of $b_{m,v}$ in infinite horizon

We now prove the following lemma.

Lemma 41. *For every m, $b_{m,v}$ is strictly increasing in v.*

Proof. Consider any M-horizon equilibrium (θ, b, z). By equation 33 we have that:

$$\left(\frac{b_{M,v}}{1-b_{M,v}}\right) e^{\left(\frac{b_{M,v}}{1-b_{M,v}}\right)} = \left(\frac{\pi_M}{1-\pi_M}\right) e^{\frac{v-\kappa}{\kappa}}$$

and therefore $b_{M,v}$ is strictly increasing in v. An inductive argument using equation 33 then establishes that $b_{m,v}$ is strictly increasing in v for all m in the finite equilibrium. Clearly, this implies that for any limit $(\theta^\infty, b^\infty, z^\infty)$, $b^\infty_{m,v}$ is weakly increasing in v. Using equation 33 again implies that $b^\infty_{m,v}$ is strictly increasing. □

E.3.2 Boundedness of $b_{m,v}$ and π_m

In the following subsection, assume that (μ, b, z) is an equilibrium in the infinite horizon game. The following three claims follow immediately from (μ, b, z) being the limit of finite horizon equilibria and Claims 1, 2 and 3. We therefore give them without proof.

Claim 4. Then $\frac{1}{2}\pi_m \leq b_{m,v} \leq e^{\frac{v-\kappa}{\kappa}}\pi_m$

Claim 5. For every m and v: $b_{m,v} \leq \frac{v_h-\kappa}{v_h}$.

Claim 6. For every m, there exists a v such that:

$$b_{m,v} \leq \frac{\left(1-e^{-r\Delta}\right)(v-\kappa)}{\left(1-e^{-r\Delta}\right)(v-\kappa)+\kappa}$$

E.3.3 Calculating values in infinite horizon equilibria

Lemma 42. *Let (μ, b, z) be an equilibrium of $B(\Delta, \kappa)$ and take (F, u, w) be its corresponding equilibrium collection. Then:*

$$\begin{aligned}
u_m &= \sum_v \mu_0(v)\left(\kappa\sum_{j=m}^{\infty} e^{-r\Delta(j-m)}\ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)\right) \\
\pi_{m,v} &= \kappa\sum_{j=m}^{\infty} e^{-r\Delta(j-m)}\left(\frac{b_{j,v}}{1-b_{j,v}}\right)
\end{aligned}$$

Proof. By Lemma 13, we know that in equilibrium the buyer's quasi-value in period m conditional on $z_v^{m-1} = (z_{1,v}, \ldots, z_{m-1,v})$ and on v is:

$$\mathcal{U}_m\left(\beta, \sigma | z_v^{m-1}, v\right) = \kappa\sum_{j=m}^{\infty} e^{-r\Delta(j-m)}\ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)$$

the buyer's expected value conditional on arriving to period m is:

$$\begin{aligned}
u_m &= \int \mathcal{U}_m\left(\beta, \sigma | x^{m-1}, v\right)\mathrm{d}\mu_m \\
&= \sum_v \bar{\mu}_m(v)\,\mathcal{U}_m\left(\beta, \sigma | z_v^{m-1}, v\right) = \kappa\sum_v \bar{\mu}_m(v)\left(\sum_{j=m}^{\infty} e^{-r\Delta(j-m)}\ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)\right)
\end{aligned}$$

For profits, in any finite equilibrium we have by Lemma 31 that the seller's expected utility conditional on arriving to period m and having a good of quality v is $z_{m,v} - \kappa$. The conclusion then follows for the infinite horizon equilibrium by Lemma 40. □

E.3.4 Trade occurs for sure

Lemma 43. *Let* (μ, b, z) *be an equilibrium. Then for every* v*:*

$$\lim_{m\to\infty}\prod_{j=1}^{m}(1-b_{j,v})=0$$

Proof. Suppose otherwise. Note that by Claim 4, $b_{m,v}/\pi_m \geq 1/2$. Thus, we must have $\pi_m \to 0$ as $m \to \infty$. By Claim 4, $b_{m,v}/\pi_m$ is bounded both from above and from below, we must have:

$$\lim_{m\to\infty}\kappa\sum_{j=m}^{\infty}e^{-r\Delta(j-m)}\left(\frac{b_{j,v}}{1-b_{j,v}}\right)=0$$

for all v. But equation 8 then implies that $z_{m,v} \to \kappa$, contradicting Proposition 2 which implies $z_{m,v} \geq z_{m,v_l} > v_l > \kappa$. □

E.4 Proof of Proposition 2

By Lemma 41, $b_{m,v}$ is strictly increasing for all v. Monotonicity of $z_{m,v}$ then follows from equation 8 from lemma 1. To prove that $v - z_{m,v}$ is strictly increasing, take natural log of both sides of equation 7 and rearrange to obtain:

$$z_{m,v}+\kappa\ln\left(\frac{b_{m,v}}{\pi_m}\right)=v+\kappa\ln\left(\frac{1-b_{m,v}}{1-\pi_m}\right)+e^{-r\Delta}\left(z_{m+1,v}+\kappa\ln\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)$$

which, through repeated substitution implies:

$$z_{m,v}+\ln\left(\frac{b_{m,v}}{\pi_m}\right)=v+\kappa\sum_{j=m}^{\infty}e^{r\Delta(j-m)}\ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right) \tag{36}$$

or:

$$v-z_{m,v}=\kappa\left(\ln\left(\frac{b_{m,v}}{\pi_m}\right)-\sum_{j=m}^{\infty}e^{-r\Delta(j-m)}\ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right)\right)$$

which implies from $b_{m,v}$ being strictly increasing in v for all m. Finally, note that the above equation implies:

$$\left(\frac{b_{m,v}}{\pi_m}\right)=\exp\left(v-z_{m,v}+\kappa\sum_{j=m}^{\infty}e^{-r\Delta}\ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right)\right)$$

for all m and v. However, by definition of π_m: $\sum_v \bar{\mu}_m(v)(b_{m,v}/\pi_m) = 1$. Since $b_{m,v}$ is strictly increasing, $b_{m,v}/\pi_m$ must be strictly increasing as well, which implies:

$$v_l - z_{m,v_l} + \kappa \sum_{j=m}^{\infty} e^{-r\Delta} \ln\left(\frac{1-b_{j,v_l}}{1-\pi_j}\right) < 0 < v_h - z_{m,v_h} + \kappa \sum_{j=m}^{\infty} e^{-r\Delta} \ln\left(\frac{1-b_{j,v_h}}{1-\pi_j}\right)$$

But $b_{j,v}$ being strictly increasing for all j implies:

$$\kappa \sum_{j=m}^{\infty} e^{-r\Delta} \ln\left(\frac{1-b_{j,v_h}}{1-\pi_j}\right) < 0 < \kappa \sum_{j=m}^{\infty} e^{-r\Delta} \ln\left(\frac{1-b_{j,v_l}}{1-\pi_j}\right)$$

thereby concluding the proof.

E.5 Proof of Proposition 5

We begin by proving for every equilibrium we have $u_1 > 0$ and:

$$u_1 + \sum_v \mu_0(v)\, w_{1,v} > \sum_{v\in V} \mu_0(v)(v-\kappa)$$

Let (μ, b, z) be an equilibrium of $B(\Delta,\kappa)$ and take (F, u, w) to be its corresponding equilibrium collection. Repeated substitution of Equation 8 from lemma 1 implies:

$$z_{1,v} = \kappa + \kappa \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \left(\frac{b_{j,v}}{1-b_{j,v}}\right) = w_{1,v} + \kappa$$

where the second equality follows from Lemma 42. Thus, using equation 36 we obtain that:

$$1 = \sum_v \mu_0 \left(\frac{b_{1,v}}{\pi_1}\right) = \sum_v \mu_0 \exp\left(v - \kappa - w_{1,v} - \kappa \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)\right)$$

Note that $b_{1,v}$ is strictly increasing, and therefore by Jensen's inequality and Lemma 42:

$$\begin{aligned} 1 &> \exp\left(\sum_v \mu_0(v)\left(v-\kappa-w_{1,v}-\kappa\sum_{j=1}^{\infty} e^{-r\Delta(j-1)}\ln\left(\frac{1-\pi_j}{1-b_{j,v}}\right)\right)\right) \\ &= \exp\left(\sum_v \mu_0(v)(v-\kappa) - \left(u_{1,v} + \sum_v \mu_0(v)\, w_{1,v}\right)\right) \end{aligned}$$

Therefore implying that the total expected surplus in (μ, b, z) is strictly higher than $\sum_v \mu_0(v)(v-\kappa)$. We will now prove that the buyer's expected surplus is strictly larger than zero. Since $b_{m,v}$ is strictly increasing, for every $v > v^{'}$:

$$\frac{\bar{\mu}_{m+1}(v)}{\bar{\mu}_{m+1}(v')} = \frac{(1-b_{m,v})\,\bar{\mu}_m(v)}{\left(1-b_{m,v'}\right)\bar{\mu}_m(v')} < \frac{\bar{\mu}_m(v)}{\bar{\mu}_m(v')}$$

thus, μ_0 first order stochastically dominates $\bar{\mu}_m$ for all $m \geq 2$. Using Lemma 42 again implies:

$$\begin{aligned}
u_1 &= -\sum_v \mu_0(v) \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \ln\left(\frac{1-b_{j,v}}{1-\pi_j}\right) \\
&> -\sum_v \mu_0(v) \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \ln\left(\frac{\sum_v \mu_0(v)(1-b_{j,v})}{1-\pi_j}\right) \\
&> -\sum_v \mu_0(v) \sum_{j=1}^{\infty} e^{-r\Delta(j-1)} \ln\left(\frac{\sum_v \bar{\mu}_j(v)(1-b_{j,v})}{1-\pi_j}\right) = 0
\end{aligned}$$

where the first inequality follows from Jensen's inequality and the second follows from μ_0 first order stochastically dominating $\bar{\mu}_j$.

Suppose now that there is no τ for which $u_1 + E[w_{1,v}] > v - \kappa + \tau$ for all equilibria. then there exists a sequence of equilibria (μ^n, b^n, z^n) with corresponding values (u^n, w^n) such that

$$u_1^n + \sum_v \mu_0(v)\, w_{1,v}^n \to \sum_{v\in V} \mu_0(v)(v-\kappa)$$

Note that for every n, m and v $\left(\bar{\mu}_m^n(v), b_{m,v}^n, z_{m,v}^n\right)$ is in a compact set of $\mathbb{R}_+$. Therefore there exists a convergent subsequence. Let that subsequence be the sequence itself. Let $p_{m,v}^n = b_{m,v}^n/\pi_m^n$. Note that $\pi_m^n \leq \frac{v_h-\kappa}{v_h}$ for all m and n by claim 5 and therefore $\pi_m = \lim_{n\to\infty}\pi_m^n \leq \frac{v_h-\kappa}{v_h} < 1$. We will show that for all m: $\pi_m > 0$. Suppose otherwise, i.e. there is some m such that $\pi_m^n \to 0$. Then this implies that $\bar{\mu}_m = \bar{\mu}_{m+1}$. Let:

$$f(k,l) = \frac{W\left(\frac{k}{1-k}l\right)}{k\left(1+W\left(\frac{k}{1-k}l\right)\right)}$$

then using $W(z) = z/e^{W(z)}$ we obtain:

$$f(k,l) = \left(\frac{l}{1-k}\right)\frac{1}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)\left(1+W\left(\frac{k}{1-k}l\right)\right)}$$

and therefore $f(0,l) = l$. Hence, using the mean value theorem, for every M there is a $k^* \in [0, \pi_m^n]$

such that:

$$\begin{aligned}\left|p_{m,v}^{n}-c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}^{n}\right)^{e^{-r\Delta}}\right| &= \left|f\left(\pi_{m}^{n},c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}^{n}\right)^{e^{-r\Delta}}\right)-f\left(0,c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}^{n}\right)^{e^{-r\Delta}}\right)\right| \\ &\leq \left|\frac{\partial f}{\partial k}\left(k^{*},c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}^{n}\right)^{e^{-r\Delta}}\right)\right|\pi_{m}^{n}\end{aligned}$$

but:

$$\begin{aligned}\frac{\partial f}{\partial k} &= \left(\frac{1-(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}\right)\frac{f(k,l)}{k} \\ &= \left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}\right)\frac{f(k,l)}{k}+\frac{f(k,l)}{1-k}\end{aligned}$$

which, since $k^{*}<1$ we have that $f(k,l)/(1-k)$ is bounded in the range $(k,l)\in[0,(v_{h}-\kappa)/v_{h}]\times[0,c_{v_{h}}]$. In addition:

$$\lim_{k\to0}\left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}{(1-k)\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}\right)\frac{f(k,l)}{k} = \left(\lim_{k\to0}f(k,l)\right)\lim_{k\to0}\left(\frac{1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}}{k}\right)$$

assuming the limit $\lim_{k\to0}k^{-1}\left(1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}\right)$ exists. Using L'Hopital's rule:

$$\begin{aligned}\lim_{k\to0}k^{-1}\left(1-\left(1+W\left(\frac{k}{1-k}l\right)\right)^{2}\right) &= \lim_{k\to0}\frac{2\left(1+W\left(\frac{k}{1-k}l\right)\right)}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)\left(1+W\left(\frac{k}{1-k}l\right)\right)}\left(\frac{l}{(1-k)^{2}}\right) \\ &= \lim_{k\to0}\frac{2}{\exp\left(W\left(\frac{k}{1-k}l\right)\right)}\left(\frac{l}{(1-k)^{2}}\right)=2l\end{aligned}$$

thus, $\frac{\partial f}{\partial k}$ is bounded, implying that $\left|p_{m,v}^{n}-c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}^{n}\right)^{e^{-r\Delta}}\right|\to0$ and therefore $p_{m,v}=c_{v}^{1-e^{-r\Delta}}\left(p_{m+1,v}\right)^{e^{-r\Delta}}$ for all v. But:

$$\sum_{v}\bar{\mu}_{m}(v)\,p_{m,v}=\lim_{M_{n}\to\infty}\sum_{v}\bar{\mu}_{m}^{n}(v)\,p_{m,v}^{n}=1$$

and therefore we've obtained:

$$
\begin{aligned}
1 &= \sum_v \bar{\mu}_m(v)\, c_v^{1-e^{-r\Delta}} \left(p_{m+1,v}\right)^{e^{-r\Delta}} \\
&> \sum_v \bar{\mu}_m(v)\, p_{m+1,v} \\
&= \sum_v \bar{\mu}_{m+1}(v)\, p_{m+1,v} = 1
\end{aligned}
$$

where the inequality follows $c_v \geq p_{m+1,v}$ for all v and $c_{v_l} > 1 \geq p_{m+1,v_l}$, a contradiction. Hence, we have $\pi_m > 0$ for all m.

Note that:

$$p_{m,v} = R\left(\pi_m, v, p_{m+1,v}\right)$$

for all m and v. Since $p^n_{m+1,v}$ is weakly increasing and $\pi_m > 0$, we have by Lemma 20 that $p_{m,v}$ is strictly increasing and therefore $b_{m,v}$ is strictly increasing. We can now use the same argument as in the single equilibrium case to establish that

$$\lim_{n\to\infty} u_1^n = u_1 > 0$$

and:

$$\lim_{n\to\infty} u_1^n + \sum_v \mu_0(v)\, w_{1,v}^n > \sum_v \mu_0(v)\,(v-\kappa)$$

a contradiction.

F Frequent Offers Environment

In this section of the appendix we prove the results of sections 5 and 6. We begin by stating some preliminary definitions and results used in the proof of Theorem 3. We then prove an omnibus theorem that includes Theorem 3, Proposition 3 and Corollary 2.

F.1 Preliminaries

In what follows, let φ be the unique σ-additive measure over $\Omega = \left(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}\right)$, where $\mathcal{B}_{\mathbb{R}_+}$ is the Borel σ-algebra, satisfying $\varphi\left([t, t+s]\right) = \left(e^{-rt} - e^{-r(t+s)}\right)$ for all $t, s \geq 0$. As usual, let $L^2\left(\Omega, \mathrm{d}\varphi\right)$ be the set of all equivalence classes of measurable functions satisfying: $f : \Omega \to \mathbb{R}$, f is φ-measurable and: $\int_{\mathbb{R}_+} |f|^2 \, \mathrm{d}\varphi < \infty$, equipped with the norm: $\|f\|_2 = \left(\int_{\mathbb{R}_+} |f|^2 \, \mathrm{d}\varphi\right)^{1/2}$. A map L from $L^2\left(\Omega, \mathrm{d}\varphi\right)$ to the real numbers is a *linear functional* if: $L\left(af_1 + bf_2\right) = aL\left(f_1\right) + bL\left(f_2\right)$. A linear functional is *continuous* if $L\left(f^n\right) \to L\left(f\right)$ whenever $f^n \to f$ (according to the $\|\cdot\|_2$), and it is *bounded* if $|L\left(f\right)| \leq K \|f\|_2$ for some finite number K. It is well known that a functional is continuous if and only if it is bounded. We let $L^2\left(\Omega, \mathrm{d}\varphi\right)^*$ be the set of continuous linear funtionals, also known as the *dual* of $L^2\left(\Omega, \mathrm{d}\varphi\right)$. A sequence of functions $\left(f^n\right) \in L^2\left(\Omega, \mathrm{d}\varphi\right)$ is said to *converge weakly* to $f \in L^2\left(\Omega, \mathrm{d}\varphi\right)$, denoted by $f^n \rightharpoonup f$ if: $L\left(f^n\right) \to L\left(f\right)$ for every $L \in L^2\left(\Omega, \mathrm{d}\varphi\right)^*$. Below is a statement of a few famous theorems from functional analysis, specialized to the current setting. The next theorem is often seen as a consequence of the Hahn-Banach theorem.

Theorem 7. *Suppose $f \in L^2\left(\Omega, \mathrm{d}\varphi\right)$ satisfies $L\left(f\right) = 0$ for all $L \in L^2\left(\Omega, \mathrm{d}\varphi\right)^*$. Then $f = 0$, and therefore if $f^n \rightharpoonup g$ and $f^n \rightharpoonup h$ then $g = h$*

Proof. Lieb and Loss (2010), pages 56 to 57. □

Theorem 8. *Let $\left(f^n\right)_{n\geq 0}$ be a sequence of functions in $L^2\left(\Omega, \mathrm{d}\varphi\right)$ such that for every $L \in L^2\left(\Omega, \mathrm{d}\varphi\right)^*$, the sequence $L\left(f^n\right)$ is bounded. Then there exists a finite $C > 0$ such that $\|f^n\|_2 < C$ for all n.*

Proof. Lieb and Loss (2010), pages 58 to 59. □

The theorem below is a specialization of the Riesz representation theorem specific for our purposes.

Theorem 9. *For every $L \in L^2\left(\Omega, \mathrm{d}\varphi\right)^*$ there exists a unique $g \in L^2\left(\Omega, \mathrm{d}\varphi\right)$ such that: $L\left(f\right) = \int_{\mathbb{R}_+} g\left(x\right) f\left(x\right) \varphi\left(\mathrm{d}x\right)$. Moreover, for every $g \in L^2\left(\Omega, \mathrm{d}\varphi\right)$, $L_g\left(f\right) = \int_{\mathbb{R}_+} g\left(x\right) f\left(x\right) \varphi\left(\mathrm{d}x\right)$ is a bounded linear functional.*

Proof. Lieb and Loss (2010), pages 61 to 63. □

The following is a version of the Banach-Alaoglu theorem.

Theorem 10. *Let $(f^n)_{n\geq 0}$ be a sequence of functions bounded in $L^2(\Omega, \mathrm{d}\varphi)$. Then there exists a subsequence $(f^{n_k})_{k\geq 0}$ and an $f \in L^2(\Omega, \mathrm{d}\varphi)$ such that $f^{n_k} \rightharpoonup f$.*

Proof. Lieb and Loss (2010), pages 68 to 69. □

Lemma 44. *Let $(f^n)_{n\geq 0}, (g^n)_{n\geq 0}$ be two sequences in $L^2(\Omega, \mathrm{d}\varphi)$. Suppose $f^n \rightharpoonup f$ and $g^n \to g$ for some f and g in $L^2(\Omega, \mathrm{d}\varphi)$. Then: $\int_{\mathbb{R}_+} f^n(x) g^n(x) \mathrm{d}\varphi \to \int_{\mathbb{R}_+} f(x) g(x) \mathrm{d}\varphi$.*

Proof. Note that: $f^n g^n - fg = f^n (g^n - g) + (f^n - f) g$. Then:

$$\left| \int_{\mathbb{R}_+} f^n(x) (g^n(x) - g(x)) \mathrm{d}\varphi \right| \leq \|f^n\|_2 \|g^n - g\|_2 \leq C \|g^n - g\|_2 \to 0$$

Since $g \in L^2(\Omega, \mathrm{d}\varphi)$, we have that $L(h) = \int_{\mathbb{R}_+} h(x) g(x) \mathrm{d}\varphi \in L^2(\Omega, \mathrm{d}\varphi)^*$ and therefore $\int_{\mathbb{R}_+} (f^n(x) - f(x)) g(x) \mathrm{d}\varphi \to 0$. The conclusion follows. □

F.2 Preliminary Definitions

Definition 11. An *extended equilibrium collection* of $B(\Delta, \kappa)$ is a collection $(\bar{F}, F, w, \tilde{u})$ such that there exists an equilibrium (μ, β, σ) for which:

1. $\bar{F} : \mathbb{R} \to [0,1]$ is a cdf satisfying; $\bar{F}(t) = \sum_v \mu_0(v) F(t, v)$, i.e. it is the cdf of the time of trade, **unconditional** on v.
2. $\tilde{u} : \mathcal{T}(\Delta) \times V \to \mathbb{R}$ is the buyer's quasi-value conditional on arriving to period t/Δ and on v, i.e.
$$\tilde{u}_{t,v} = \mathcal{U}_{t/\Delta}\left(\beta, \sigma | z_v^{\frac{t}{\Delta}-1}, v\right)$$
where $z_v^m = (z_{1,v} \ldots, z_{m,v})$ is the history of offers that is made on equilibrium by a v type seller up to and including period m.
3. F is a timing distribution function.
4. $w : \mathcal{T}(\Delta) \times V \to \mathbb{R}_+$ is the seller's expected utility conditional on arriving to period $t/\Delta$and on v in period t/Δ terms.

Note that one can find an extended equilibrium collection for every equilibrium collection.

Definition 12. An *potential extended continuous limit* is a collection $(\bar{F}, F, w, \tilde{u})$,where:

1. $\bar{F} : \mathbb{R}_+ \to [0,1]$ is an absolutely continuous cdf.
2. $F : \mathbb{R}_+ \times V \to [0,1]$ is such that $t \mapsto F(t, v)$ is an absolutely continuous cdf.
3. $w : \mathbb{R}_+ \times V \to \mathbb{R}_+$ and $\tilde{u} : \mathbb{R}_+ \times V \to \mathbb{R}$ are both continuous in their first variable.

We say that $\left(\bar{F}, F, w, \tilde{u}\right)$ is an *extended continuous limit* of $B\left(0,\kappa\right)$ if there exists a refining sequence of extended equilibrium collections, $\left\{\left(\bar{F}^{n}, F^{n}, w^{n}, \tilde{u}^{n}\right)\right\}_{n=1}^{\infty}$ of $B\left(\Delta_{n},\kappa\right)$ such that:

1. $w_{t,v}^{n} \to w_{t,v}$ and $\tilde{u}_{t,v}^{n} \to u_{t,v}$ for all $(t,v) \in \cup_{n}\mathcal{T}\left(\Delta_{n}\right)$.
2. $\bar{F}^{n}\left(t\right) \to \bar{F}\left(t\right)$ and $F^{n}\left(t,v\right) \to F\left(t,v\right)$ for all t and v.

We then say that $\left\{\left(\bar{F}^{n}, F^{n}, w^{n}, \tilde{u}^{n}\right)\right\}_{n=1}^{\infty}$ converges to $\left(\bar{F}, F, w, u\right)$.

F.3 Proof of theorem 3 and proposition 3

In the following section, we prove the following result that combines Theorem 3 and Proposition 3.

Theorem 11. *Let $\{\Delta_{n}\}_{n=1}^{\infty}$ be a refining sequence, and take $\left\{\left(\bar{F}^{n}, F^{n}, w^{n}, \tilde{u}^{n}\right)\right\}_{n=1}^{\infty}$ to be a sequence of extended equilibrium collections of $B\left(\Delta_{n},\kappa\right)$. Then there exists a subsequence $\left\{\left(\bar{F}^{n_k}, F^{n_k}, w^{n_k}, \tilde{u}^{n_k}\right)\right\}_{k=1}^{\infty}$ that converges to an extended continuous limit of $B_{0}\left(\kappa\right)$, $\left(\bar{F}, F, w, \tilde{u}\right)$. Moreover, there exists two functions: $\bar{\lambda}:\mathbb{R}_{+}\to\mathbb{R}_{+}$ and $\lambda:\mathbb{R}_{+}\times V\to\mathbb{R}_{+}$ such that:*

1. *$\bar{\lambda}_{t}$ is the time dependent hazard rate of $\bar{F}$, i.e. $\bar{F}\left(t\right)=1-e^{-\int_{0}^{t}\bar{\lambda}_{s}\mathrm{d}s}$, and $\bar{\lambda}_{t}\leq 3r\left(\frac{v_{h}-\kappa}{\kappa}\right)$.*
2. *$\lambda_{t,v}$ is the time dependent hazard rate of F, i.e. $F\left(t,v\right)=1-e^{-\int_{0}^{t}\lambda_{s,v}\mathrm{d}s}$.*

Moreover, $\left(w,\tilde{u},\bar{\lambda},\lambda\right)$ satisfy:

3. *$w_{t,v}=\int_{t}^{\infty}e^{-r(s-t)}\lambda_{s,v}\mathrm{d}s$*
4. *$\tilde{u}_{t,v}=\int_{t}^{\infty}e^{-r(s-t)}\left(\lambda_{s,v}-\bar{\lambda}_{s}\right)\mathrm{d}s$*
5. *$\lambda_{t,v}/\bar{\lambda}_{t}=\exp\frac{1}{\kappa}\left(v-\kappa-w_{t,v}-\tilde{u}_{t,v}\right)\in\left[\frac{1}{2},e^{\frac{v-\kappa}{\kappa}}\right]$.*
6. *For every t:*
$$\bar{\lambda}_{t}=\sum_{v}\mu_{0}\left(v\right)\left(\frac{1-F\left(t,v\right)}{1-\bar{F}\left(t\right)}\right)\lambda_{t,v}$$
7. *$\bar{\lambda}_{t}>0$ and $\lambda_{t,v}>0$ for almost all t.*
8. *$\lambda_{t,v}$ is strictly increasing in v for almost all t.*

Proof of Theorem

Proof of parts 1 and 2 Let Δ_{n} be a refining sequence, $\left(\bar{F}^{n}, F^{n}, w^{n}, \tilde{u}^{n}\right)$ be an extended collection of $B\left(\Delta_{n},\kappa\right)$ for every n. Let $\left(\mu^{n},\beta^{n},\sigma^{n}\right)$ be the sequence of corresponding equilibria, and let $b_{t,v}^{n}:=\beta_{t/\Delta}^{n}\left(z_{v}^{t/\Delta},v\right)$ and $\pi_{t}^{n}=\sum_{v}\bar{\mu}_{t/\Delta}\left(v\right)b_{t,v}^{n}$ for every $t\in\mathcal{T}\left(\Delta_{n}\right)$.

By Helly's selection theorem, there exists a subsequence of $\left(\bar{F}^{n_k}, F^{n_k}, w^{n_k}, \tilde{u}^{n_k}\right)$ and a finite collection of increasing, right-continuous functions F_{v} and $\bar{F}$ such that $\bar{F}^{n_k}\left(t\right)\to\bar{F}\left(t\right)$ for all t for which $\bar{F}(t)$ is continuous, and for every v: $F^{n_k}(t,v) \to F(t,v) := F_v(t)$ for all t for which $F_v(t)$ is continuous. Let $(\bar{F}^n, F^n, w^n, \tilde{u}^n)$ denote this subsequence. For every n, t and $v \in V$ define:

$$\begin{aligned}
\bar{\lambda}_t^n &= -\frac{1}{\Delta_n} \ln \left(1 - \pi^n_{\lceil \frac{t}{\Delta_n} \rceil}\right) \\
&= -\frac{1}{\Delta_n} \ln \left(\frac{1 - \bar{F}\left(\left\lfloor \frac{t}{\Delta_n} \right\rfloor + \Delta_n\right)}{1 - \bar{F}\left(\left\lfloor \frac{t}{\Delta_n} \right\rfloor\right)} \right) \\
\lambda_{t,v}^n &= -\frac{1}{\Delta_n} \ln \left(1 - b^n_{\lceil \frac{t}{\Delta_n} \rceil, v}\right)
\end{aligned}$$

and define for every t: $\bar{G}^n(t) = 1 - e^{-\int_0^t \bar{\lambda}_s^n \mathrm{d}s}$ and $G^n(t,v) = 1 - e^{-\int_0^t \lambda_{s,v}^n \mathrm{d}s}$. Note that for every $t \in \mathcal{T}_{\Delta_n}$:

$$\begin{aligned}
\int_0^t \bar{\lambda}_s^n \mathrm{d}s &= \sum_{j=1}^{t/\Delta} \Delta_n \left(-\frac{1}{\Delta_n} \ln \left(1 - \pi_j^n\right)\right) \\
&= -\sum_{j=1}^{t/\Delta_n} \ln \left(1 - \pi_j^n\right)
\end{aligned}$$

and therefore:

$$\bar{G}^n(t) = 1 - \prod_{j=1}^{t/\Delta} \left(1 - \pi_j^n\right) = \bar{F}^n(t)$$

and similarly $G_v^n(t) = F^n(t)$ for all $t \in \mathcal{T}_{\Delta_n}$. Define:

$$\bar{\pi}_{\Delta_n} = \frac{\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa) + \kappa}$$

and note that by Claim 5, $\pi_t^n \leq 2\bar{\pi}_{\Delta_n}$ for all n. Therefore:

$$\bar{\lambda}_t^n \leq -\frac{1}{\Delta_n} \ln \left(1 - 2\bar{\pi}_{\Delta_n}\right)$$

note that:

$$\begin{aligned}
1 - 2\bar{\pi}_{\Delta_n} &= 1 - 2\left(\frac{\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa) + \kappa}\right) \\
&= \frac{\kappa - 2\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta_n}\right)(v_h - \kappa) + \kappa}
\end{aligned}$$

and therefore:

$$
\begin{aligned}
-\frac{1}{\Delta_n}\ln\left(1-2\bar{\pi}_{\Delta_n}\right) &= \frac{1}{\Delta_n}\left(\ln\left(\left(1-e^{-r\Delta_n}\right)(v_h-\kappa)+\kappa\right)-\ln\left(\kappa-2\left(1-e^{-r\Delta_n}\right)(v_h-\kappa)\right)\right) \\
&\to 3r\left(\frac{v_h-\kappa}{\kappa}\right)
\end{aligned}
$$

Similarly, Claim 4 implies that $b_{m,v}^n \leq 2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\bar{\pi}_{\Delta_n}$ for all n. Therefore:

$$
\begin{aligned}
\lambda_{s,v}^n &\leq -\frac{1}{\Delta_n}\ln\left(1-2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\bar{a}_{\Delta_n}\right) \\
&= \frac{1}{\Delta_n}\left(\ln\left(\left(1-e^{-r\Delta_n}\right)(v_h-\kappa)+\kappa\right)-\ln\left(\kappa-2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\left(1-e^{-r\Delta_n}\right)(v_h-\kappa)\right)\right) \\
&\to \left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)
\end{aligned}
$$

for all v. Thus, for every $\epsilon>0$, there exists an N_ϵ such that for all $n>N_\epsilon$: $0<\bar{\lambda}_s^n \leq 3r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon$ and $0<\lambda_{s,v}^n \leq \left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon$ for all s and v. This implies that:

$$
\left\|\bar{\lambda}_s^n\right\|_2, \left\|\lambda_{s,v}^n\right\|_2 \leq \left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)^2
$$

and therefore, by the sequential Banach-Alaoglu theorem (theorem 10), there exists a subsequence in which $\bar{\lambda}^{n_k} \rightharpoonup \lambda$ and $\lambda_v^{n_k} \rightharpoonup \lambda_v$. Note that φ is absolutely continuous with respect to Lebesgue measure, with density $-re^{-rt}$. Letting $g_t(s) = -\mathbf{1}_{[s\leq t]}\frac{e^{rs}}{r}$, note that the linear functional defined by:

$$
\begin{aligned}
|L(f)| &= \left|\int_{\mathbb{R}_+} g_t(s) f(s)\,\mathrm{d}\varphi\right| \\
&\leq \left|\frac{e^{rt}}{r}\right|\left|\int_{\mathbb{R}_+} f(s)\,\mathrm{d}\varphi\right| \leq \left|\frac{e^{rt}}{r}\right|\|f\|_2
\end{aligned}
$$

therefore: $\int g_t(s)\bar{\lambda}_s^{n_k}\mathrm{d}\varphi \to \int g_t(s)\bar{\lambda}_s\mathrm{d}\varphi$ for all t. However:

$$
\int g_t(s) f_s \mathrm{d}\varphi = \int_0^t f_s \mathrm{d}t
$$

and therefore we've obtain that $\bar{G}^{n_k}(t) \to 1-e^{-\int_0^t \bar{\lambda}_s \mathrm{d}t} \equiv \bar{G}(t)$ for all t. Clearly, $\bar{G}$ is continuous everywhere. Since $\bar{F}^{n_k}(t) = \bar{G}^{n_k}(t)$ for all $t \in \mathcal{T}_{\Delta_{n_k}}$, this implies that for all $t \in \cup_{k\geq 0}\mathcal{T}_{\Delta_{n_k}}$: $\bar{F}^{n_k}(t) \to \bar{G}(t)$, and therefore $\bar{G}(t) = \bar{F}(t)$ for all $t \in \cup_{k\geq 0}\mathcal{T}_{\Delta_{n_k}}$. For every $t \notin \cup_{k\geq 0}\mathcal{T}_{\Delta_{n_k}}$, there exists a sequence $\left(t_a^i\right)_{i\geq 0}$ in $\cup_{k\geq 0}\mathcal{T}_{\Delta_{n_k}}$ such that $t_a^i \downarrow t$. Since $\bar{F}$ is right-continuous, we have that $\bar{F}\left(t_a^i\right)\to\bar{F}\left(t\right)$. But $\bar{F}\left(t_a^i\right)=\bar{G}\left(t_a^i\right)\to\bar{G}\left(t\right)$. Therefore: $\bar{G}\left(t\right)=\bar{F}\left(t\right)$ for all t. A similar argument establishes that $F^{n_k}\left(t,v\right)\to F\left(t,v\right)=1-e^{-\int_0^t\lambda_{s,v}\mathrm{d}s}$ for all t.

Proof of parts 3 and 4: Note that for every $t\in\mathcal{T}_{\Delta_{n_k}}$:

$$\ln\left(\frac{1-\pi_{t/\Delta}^{n_k}}{1-b_{t/\Delta,v}^{n_k}}\right)=\Delta\left(\lambda_{t/\Delta,v}^{n_k}-\bar{\lambda}_{t/\Delta}^{n_k}\right)$$

and therefore:

$$\tilde{u}_{t,v}^{n_k} = \kappa\sum_{j=t/\Delta_{n_k}}^{\infty}e^{-r\left(j\Delta_{n_k}-t\right)}\Delta\left(\lambda_{j\Delta_{n_k},v}^{n_k}-\bar{\lambda}_{j\Delta_{n_k}}^{n_k}\right)$$

let: $g_{\Delta_{n_k}}\left(t\right)=-\frac{1}{r}e^{r\left(t-\Delta_{n_k}\left\lfloor\frac{t}{\Delta_{n_k}}\right\rfloor\right)}$. Then:

$$\begin{aligned}\tilde{u}_{t,v}^{n_k} &= \kappa e^{rt}\int_t^{\infty}g_{\Delta_{n_k}}\left(s\right)\left(\lambda_{s,v}^{n_k}-\bar{\lambda}_s^{n_k}\right)\mathrm{d}\varphi\\ &= \kappa e^{rt}\left(\int_t^{\infty}g_{\Delta_{n_k}}\left(s\right)\lambda_{s,v}^{n_k}\mathrm{d}\varphi-\int_t^{\infty}g_{\Delta_{n_k}}\left(s\right)\bar{\lambda}_s^{n_k}\mathrm{d}\varphi\right)\end{aligned}$$

however, for every $\Delta>\Delta_{n_k}$: $\left(g_{\Delta_{n_k}}\left(t\right)\right)^2\leq\frac{1}{r^2}e^{2r\Delta}$, and therefore by the dominated convergence theorem: $g_{\Delta_{n_k}}\to-\frac{1}{r}$ in $L^2\left(\mathbb{R}_+,\mathrm{d}\varphi\right)$. Hence, by lemma 44:

$$\begin{aligned}\tilde{u}_{t,v}^{n_k} &= \kappa e^{rt}\left(\int_t^{\infty}g_{\Delta_{n_k}}\left(s\right)\lambda_{s,v}^{n_k}\mathrm{d}\varphi-\int_t^{\infty}g_{\Delta_{n_k}}\left(s\right)\bar{\lambda}_s^{n_k}\mathrm{d}\varphi\right)\\ &\to -\frac{\kappa e^{rt}}{r}\int_t^{\infty}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}\varphi=\kappa\int_t^{\infty}e^{-r(s-t)}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s\end{aligned}$$

for all $t\in\cup_{k\geq 0}\mathcal{T}_{\Delta_{n_k}}$. Similarly, for every $t\in\mathcal{T}_{\Delta_{n_k}}$:

$$\begin{aligned}w_{t,v}^{n_k} &= \kappa\sum_{j=t/\Delta_{n_k}}^{\infty}e^{-r\left(j\Delta_{n_k}-t\right)}\left(\frac{b_{j\Delta_{n_k},v}}{1-b_{j\Delta_{n_k},v}}\right)\\ &= \kappa\sum_{j=t/\Delta_{n_k}}^{\infty}e^{-r\left(j\Delta_{n_k}-t\right)}\left(e^{\Delta_{n_k}\lambda_{j\Delta_{n_k},v}^{n_k}}-1\right)\end{aligned}$$

using the mean value theorem, there exists a $\Delta^*_{n_k} \in (0, \Delta_{n_k})$ such that:

$$\begin{aligned} w^{n_k}_{t,v} &= \kappa e^{rt} \sum_{j=t/\Delta_{n_k}}^{\infty} e^{-rj\Delta_{n_k}} \Delta_{n_k} \lambda^{n_k}_{j\Delta_{n_k},v} e^{\Delta^*_{n_k} \lambda^{n_k}_{j\Delta_{n_k},v}} \\ &= \kappa e^{rt} \int_t^{\infty} e^{\Delta^*_{n_k} \lambda^{n_k}_{j\Delta_{n_k},v}} g_{\Delta_{n_k}}(s) \lambda^{n_k}_{j\Delta_{n_k},v} \mathrm{d}\varphi \end{aligned}$$

since we have $\left(\lambda^{n_k}_{j\Delta_{n_k},v}\right)^2 < \left(\left(1 + 2e^{\left(\frac{v_h - \kappa}{\kappa}\right)}\right) r\left(\frac{v_h - \kappa}{\kappa}\right) + \epsilon\right)^2$, we can again use the dominated convergence theorem to obtain that $e^{\Delta^*_{n_k} \lambda^{n_k}_{j\Delta_{n_k},v}} g_{\Delta_{n_k}} \to -\frac{1}{r}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$, thereby implying: $w^{n_k}_{t,v} \to \kappa \int_t^{\infty} e^{-r(s-t)} \lambda_{s,v} \mathrm{d}s$. For every n_k and $t \notin \mathcal{T}_{\Delta_{n_k}}$ set:

$$\begin{aligned} \tilde{u}^{n_k}_{t,v} &= \tilde{u}^{n_k}_{\Delta_{n_k}\lceil t/\Delta_{n_k}\rceil,v} \\ w^{n_k}_{t,v} &= w^{n_k}_{\Delta_{n_k}\lceil t/\Delta_{n_k}\rceil,v} \end{aligned}$$

Clearly, these converge to the obvious extensions of $\tilde{u}_{t,v}$ and $w_{t,v}$ to all t: $\tilde{u}_{t,v} = \kappa \int_t^{\infty} e^{-r(s-t)} \left(\lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}s$ and $w_{t,v} = \kappa \int_t^{\infty} e^{-r(s-t)} \lambda_{s,v} \mathrm{d}s$ which are continuous.

Proof of parts 5 and 6: For every n_k and t (not necessarily in $\mathcal{T}_{\Delta_{n_k}}$), set: $p^{n_k}_{t,v} = \exp \frac{1}{\kappa}\left(v - \kappa - w^{n_k}_{t,v} - \tilde{u}^{n_k}_{t,v}\right)$. Since $\tilde{u}^{n_k}_{t,v} \to \tilde{u}_{t,v}$ and $w^{n_k}_{t,v} \to w_{t,v}$, we have that $p^{n_k}_{t,v} \to p_{t,v} \equiv \exp \frac{1}{\kappa}\left(v - \kappa - w_{t,v} - \tilde{u}_{t,v}\right)$. Morevoer, $\frac{1}{2} \leq p^{n_k}_{t,v} \leq e^{\frac{v-\kappa}{\kappa}}$ for all v and t by claim 4, meaning that $\left(p^{n_k}_{t,v}\right)^2 \leq e^{2\left(\frac{v-\kappa}{\kappa}\right)}$. Hence, by the dominated convergence theorem, $p^{n_k}_{t,v} \to p_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. But this means that for every $g \in L^2(\mathbb{R}_+, \mathrm{d}\varphi)$, $g_t p^{n_k}_{t,v} \to g_t p_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. Thus, by Lemma 44 and Riesz representation thoerem (theorem 9) we have that $p^{n_k}_{t,v} \bar{\lambda}^{n_k}_t \rightharpoonup p_{t,v} \bar{\lambda}_t$. Note, however, that every t:

$$p_{t,v} = \frac{1 - e^{-\Delta_{n_k} \lambda^{n_k}_{t,v}}}{1 - e^{-\Delta_{n_k} \bar{\lambda}^{n_k}_t}}$$

and therefore by the mean-value theorem, there exists $\Delta^1_{n_k}, \Delta^2_{n_k} \in (0, \Delta_{n_k})$ such that:

$$p^{n_k}_{t,v} = \frac{\lambda^{n_k}_{t,v} e^{-\Delta^1_{n_k} \lambda^{n_k}_{t,v}}}{\bar{\lambda}^{n_k}_t e^{-\Delta^2_{n_k} \bar{\lambda}^{n_k}_t}}$$

and therefore:

$$e^{-\Delta_{n_k}\left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right) r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)} \lambda^{n_k}_{t,v} < \bar{\lambda}^{n_k}_t p^{n_k}_{t,v} < \lambda^{n_k}_{t,v} e^{\Delta_{n_k}\left(3r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)}$$

therefore, for every bounded linear operator $L \in L^2(\mathbb{R}_+, \mathrm{d}\varphi)$:

$$\begin{aligned} L\left(\bar{\lambda}_t^{n_k} p_{t,v}^{n_k}\right) &< e^{\Delta_{n_k}\left(3r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)} L\left(\lambda_v^{n_k}\right) \to L\left(\lambda_v\right) \\ L\left(\bar{\lambda}_t^{n_k} p_{t,v}^{n_k}\right) &> e^{-\Delta_{n_k}\left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)} L\left(\lambda_v^{n_k}\right) \to L\left(\lambda_v\right) \end{aligned}$$

and therefore $p_v^{n_k}\bar{\lambda}^{n_k} \rightharpoonup \lambda_v$. Hence, by theorem 7, $p_{t,v}\bar{\lambda}_t = \lambda_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. Therefore: $\lambda_{t,v}/\bar{\lambda}_t = \exp\frac{1}{\kappa}\left(v - \kappa - w_{t,v} - \tilde{u}_{t,v}\right)$. Moreover, since $p_{t,v}^{n_k} \in \left[\frac{1}{2}, e^{\frac{v-\kappa}{\kappa}}\right]$ (Claim 4) for all $t \in \mathcal{T}(\Delta_{n_k})$ we have $p_{t,v} \in \left[\frac{1}{2}, e^{\frac{v-\kappa}{\kappa}}\right]$ for all $t \in \cup_k \mathcal{T}(\Delta_{n_k})$ which implies $p_{t,v} \in \left[\frac{1}{2}, e^{\frac{v-\kappa}{\kappa}}\right]$ by continuity of $p_{t,v}$ in t. Note that for every k and every $t \in \mathcal{T}_{\Delta_k}$:

$$\begin{aligned} 1 &= \sum_v \mu_0(v)\left(\frac{1 - F^{n_k}(t - \Delta_{n_k}, v)}{1 - \bar{F}^{n_k}(t - \Delta_{n_k})}\right) p_{t,v}^{n_k} \\ &\to \sum_v \mu_0(v)\left(\frac{1 - F(t,v)}{1 - \bar{F}(t)}\right) p_{t,v} = \sum_v \mu_0(v)\left(\frac{1 - F(t,v)}{1 - \bar{F}(t)}\right) p_{t,v} \end{aligned}$$

which extends to all t by continuity in t of $F(t,v)$, $\bar{F}(t)$ and $p_{t,v}$. This implies:

$$\bar{\lambda}_t = \sum_v \mu_0(v)\left(\frac{1 - F(t,v)}{1 - \bar{F}(t)}\right) \lambda_{t,v}$$

for all t.

Proof of part 7: To see that $\bar{\lambda}_t > 0$ for almost all t, assume there is an open ball, $(t, t+\epsilon)$ such that $s \in (t, t+\epsilon)$ implies $\lambda_s = 0$. Then:

$$\begin{aligned} \lambda_{t,v}/\bar{\lambda}_t &= \exp\frac{1}{\kappa}\left(v - \kappa - e^{-r\epsilon}\left(w_{t+\epsilon,v} + \tilde{u}_{t+\epsilon,v}\right)\right) \\ &= \left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\epsilon}} \left(\lambda_{t+\epsilon,v}/\bar{\lambda}_{t+\epsilon}\right)^{e^{-r\epsilon}} \geq \lambda_{t+\epsilon,v}/\bar{\lambda}_{t+\epsilon} \end{aligned}$$

for all v with a strict inequality for v_l since $\left(\lambda_{t,v_l}/\bar{\lambda}_t\right) \leq 1 < e^{\frac{v_l-\kappa}{\kappa}}$ for all t (since $p_{t,v_l}^{n_k} < 1$ for all $t \in \cup_k \mathcal{T}(\Delta_{n_k})$). Therefore:

$$\sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v} - \bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) > \sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v} - \bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t+\epsilon,v}}{\bar{\lambda}_{t+\epsilon}}\right)$$

but $\bar{\lambda}_s = 0$ for all $s \in (t, t+\epsilon)$ implies:

$$\begin{aligned}
\sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) &= \sum_v \mu_0(v)\, e^{-\int_0^{t}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) \\
&= 1 \\
&= \sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t+\epsilon,v}}{\bar{\lambda}_{t+\epsilon}}\right)
\end{aligned}$$

a contradiction. Therefore $\bar{\lambda}_t > 0$ for almost all t.

Proof of Part 8: Suppose there is an open ball $(t, t+\epsilon)$ for $\epsilon > 0$ and $v < v^{'}$ such that $\lambda_{s,v} = \lambda_{s,v'}$ for all $s \in (t, t+\epsilon)$, and $\lambda_{t,v} = \lambda_{t,v'}$ by continuity of $\lambda_{t,v}/\bar{\lambda}_t$. Therefore:

$$\begin{aligned}
1 &= \frac{\lambda_{t,v'}}{\lambda_{t,v}} \\
&= \exp\frac{1}{\kappa}\left(v^{'} - v - \left(\left(w_{t,v'} + \tilde{u}_{t,v'}\right) - \left(w_{t,v} + \tilde{u}_{t,v}\right)\right)\right) \\
&= \exp\frac{1}{\kappa}\left(v^{'} - v - 2\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right)
\end{aligned}$$

implying that: $\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s = 2\left(v^{'} - v\right) > 0$. But:

$$\begin{aligned}
1 &\leq \frac{\lambda_{t+\epsilon,v'}}{\lambda_{t+\epsilon,v}} \\
&= \exp\frac{1}{\kappa}\left(v^{'} - v - 2e^{r\epsilon}\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right) \\
&< \exp\frac{1}{\kappa}\left(v^{'} - v - 2\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right) = 1
\end{aligned}$$

since $e^{r\epsilon} > 1$, a contradiction. Therefore $\lambda_{t,v}$ is strictly increasing almost everywhere.

Proof that $\bar{F}$ and $F(\cdot, v)$ are cdfs: Suppose otherwise. Then $\left(\lambda_{t,v}/\bar{\lambda}_t\right) \geq 1/2$ for all t and v implies that $\bar{\lambda}_t \to 0$. But, since $\bar{\lambda}_t$ is bounded, we can use the dominated convergence theorem to obtain that: $\left(\lambda_{t,v_l}/\bar{\lambda}_t\right) = \exp\frac{1}{\kappa}\left(v_l - \kappa - \kappa\int_t^{\infty} e^{-r(s-t)}\left(2\lambda_{s,v_l} - \bar{\lambda}_s\right)\mathrm{d}s\right) \to e^{\frac{v_l-\kappa}{\kappa}} > 1$, a contradiction.

F.4 A few additional properties of extended continuous limits

Let $\left(\bar{F}, F, w, \tilde{u}\right)$ be a continuous time limit of $B(0, \kappa)$ and take $\left(\bar{\lambda}, \lambda\right)$ be the hazard rates from Theorem 11.

Lemma 45. *For almost all t: $w_{t,v_h} + \tilde{u}_{t,v_h} < v_h - \kappa$.*

Proof. By part 5 of Theorem 11:

$$\left(\lambda_{t,v_h}/\bar{\lambda}_t\right) = \exp\frac{1}{\kappa}\left(v_h - \kappa - w_{t,v_h} - \tilde{u}_{t,v_h}\right)$$

Part 6 of Theorem 11 along with $\lambda_{t,v}$ being strictly increasing in v (part 8 of Theorem 11) implies the desired conclusion. □

G Proof of Theorem 4

G.1 Proof of Corollary 2

Let $\{\Delta_n\}_{n=1}^{\infty}$ be a refining sequence, and take $\{(\mu^n, b^n, z^n)\}_{n=1}^{\infty}$ to be a sequence of corresponding equilibria. Let $\left\{\left(\bar{F}^n, F^n, w^n, \tilde{u}^n\right)\right\}_{n=1}^{\infty}$ be a corresponding sequence of extended equilibrium collections. Then by Theorem 11 there exists a subsequence $\{(\mu^{n_k}, b^{n_k}, z^{n_k})\}_{k=1}^{\infty}$ such that $\left\{\left(\bar{F}^{n_k}, F^{n_k}, w^{n_k}, \tilde{u}^{n_k}\right)\right\}_{k=1}^{\infty}$ converges to a continuous limit $\left(\bar{F}, F, w, \tilde{u}\right)$. But:

$$\begin{aligned} E\left[U_1^n\right] &= \sum_v \mu_0(v)\, w^n_{v,(1/\Delta_n)} \to \sum_v \mu_0(v)\, w_{v,0} = \bar{U}_1 \\ E\left[U_2^n\right] &= \sum_v \mu_0(v)\, \tilde{u}^n_{v,(1/\Delta_n)} \to \sum_v \mu_0(v)\, \tilde{u}_{v,0} = \bar{U}_2 \end{aligned}$$

as required.

G.2 Proof of Theorem 4

Note that Theorem 11, for every $\bar{U}_1$ and $\bar{U}_2$ of $B(0,\kappa)$ there exists an extended continuous limit $\left(\bar{F}, F, w, \tilde{u}\right)$ of $B(0,\kappa)$ such that:

$$\begin{aligned} \bar{U}_1 &= \sum_v \mu_0(v)\, w_{v,0} \\ \bar{U}_2 &= \sum_v \mu_0(v)\, \tilde{u}_{v,0} \end{aligned}$$

Thus, for every $\{\kappa_n\}_{n=1}^{\infty}$ such that $\kappa_n \to 0$ and corresponding sequence of frequent offer utilities , let $\left(\bar{F}^n, F^n, w^n, \tilde{u}^n\right)$ be the corresponding sequence of extended continuous limits of $B(0,\kappa_n)$. Let $\left(\bar{\lambda}^n, \lambda^n\right)$ be the hazard rates from Theorem 11 for $\left(\bar{F}^n, F^n, w^n, \tilde{u}^n\right)$. Note that for every t: $\left(\lambda^n_{t,v_l}/\bar{\lambda}^n_t\right) \in \left[\frac{1}{2}, 1\right]$, implying that $\frac{1}{\kappa}\left(v_l - \kappa - w^n_{t,v_l} - \tilde{u}^n_{t,v_l}\right)$ must remain finite. As such, $w^n_{t,v_l} +$

$\tilde{u}^n_{t,v_l} \to v_l$. Since this is true for all t, and:

$$w^n_{t,v_l} + \tilde{u}^n_{t,v_l} = \kappa_n \int_t^{t+\epsilon} e^{-r(s-t)} \left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s + e^{-r\epsilon}\left(w^n_{t+\epsilon,v_l} + \tilde{u}^n_{t+\epsilon,v_l}\right)$$

we obtain that:

$$\kappa_n \int_t^{t+\epsilon} e^{-r(s-t)} \left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \to \left(1 - e^{-r\epsilon}\right) v_l$$

for all t and every $\epsilon > 0$. Note this implies that $\bar{\lambda}^n_t \to \infty$ for almost all t, since $\bar{\lambda}^n_s > \lambda^n_{s,v_l}$ for all s and therefore:

$$\begin{aligned}
\int_t^{t+\epsilon} \bar{\lambda}^n_t \mathrm{d}s &= \int_t^{t+\epsilon} \left(2\bar{\lambda}^n_t - \bar{\lambda}^n_t\right) \mathrm{d}s \\
&> \int_t^{t+\epsilon} \left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \\
&> \int_t^{t+\epsilon} e^{-r(s-t)} \left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \to \lim_{n\to\infty} \left(1 - e^{-r\epsilon}\right) \frac{v_l}{\kappa_n} = \infty
\end{aligned}$$

Suppose that there exists a subsequence $\left(\bar{F}^k, F^k, w^k, \tilde{u}^k\right)$, an interval $[t_1, t_2]$, and a Borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying $f > 0$ almost everywhere in $[t_1, t_2]$ such that $v_h - \kappa - w^k_{t,v_h} - \tilde{u}^k_{t,v_h} > f(t)$ almost everywhere in $[t_1, t_2]$ for all k larger than some K. Then for $k > K$:

$$\begin{aligned}
w^k_{t_1,v_h} + \tilde{u}^k_{t_1,v_h} &> \int_{t_1}^{t_2} e^{-r(s-t_1)} \kappa_k \left(2\lambda^k_{t,v_h} - \bar{\lambda}^k_t\right) \mathrm{d}s \\
&\geq \int_{t_1}^{t_2} e^{-r(s-t_1)} \left(2e^{\frac{f(s)}{\kappa_k}} - 1\right) \kappa_k \bar{\lambda}^k_t \mathrm{d}s \\
&\geq \int_{t_1}^{t_2} e^{-r(s-t_1)} \liminf_{k\to\infty} \left(2e^{\frac{f(s)}{\kappa_k}} - 1\right) \kappa_k \bar{\lambda}^k_t \mathrm{d}s
\end{aligned}$$

where the second inequality follows from Fatou's lemma. By Theorem 11, part 1: $\kappa_n \bar{\lambda}^n_t \in [0, 3r(v_h - \kappa_n)]$ for all t. However, for almost every t we have both $f(t) > 0$ and, from before:

$$\liminf_{k\to\infty} \kappa_k \int_{t_1}^{t_2} \bar{\lambda}^k_s \mathrm{d}s > \left(1 - e^{-r(t_2 - t_1)}\right) v_l$$

implying that: $2\kappa_k \int_{t_1}^{t_2} \bar{\lambda}^k_s e^{\frac{f(s)}{\kappa_k}} \mathrm{d}s \to \infty$. Thus, $w^k_{t_1,v_h} + \tilde{u}^k_{t_1,v_h} \to \infty$. But by lemma 45: $w^k_{t_1,v_h} + u^k_{t_1,v_h} < v_h$, a contradiction. Therefore $v_h - \kappa - w^n_{t,v_h} - \tilde{u}^n_{t,v_h} \to 0$ for almost all t. Since $\lambda^n_{t,v}$ is stricly increasing in v, the difference $v - \kappa - w^n_{t,v} - \tilde{u}^n_{t,v}$ is also strictly increasing in v. Thus: $v - \kappa - w^n_{t,v} - \tilde{u}^n_{t,v} \to 0$ for all v for almost all t. But this implies that for almost all t:

$$w^n_{t,v} + \tilde{u}^n_{t,v} - \left(w^n_{t,v_l} + \tilde{u}^n_{t,v_l}\right) \to v - v_l$$

which, since $\tilde{u}^n_{t,v} - \tilde{u}^n_{t,v_l} = \int_t^\infty e^{-r(s-t)} \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right) \mathrm{d}s = w^n_{t,v} - w^n_{t,v_l}$ implies:

$$w^n_{t,v} - w^n_{t,v_l} \to \frac{v - v_l}{2}$$

for almost all t and all v. Therefore, for every $t, s > 0$: $\left(w^n_{t,v} - w^n_{t,v_l}\right) - e^{-rs}\left(w^n_{t+s,v} - w^n_{t+s,v_l}\right)$ converges to $\left(1 - e^{-rs}\right)\left(\frac{v-v_l}{2}\right)$. Note that for every n, and almost every $t > \epsilon > 0$:

$$\begin{aligned}
\int_0^t \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right) \mathrm{d}s &> \int_\epsilon^t \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right) \mathrm{d}s \\
&> \int_\epsilon^t e^{-r(s-\epsilon)} \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right) \mathrm{d}s \\
&= \frac{1}{\kappa_n}\left(\left(w^n_{\epsilon,v} - w^n_{\epsilon,v_l}\right) - e^{-r(t-\epsilon)}\left(w^n_{t,v} - w^n_{t,v_l}\right)\right) \to \infty
\end{aligned}$$

which implies that:

$$\frac{\mu_0(v)\left(1 - F^n(t,v)\right)}{\mu_0(v_l)\left(1 - F^n(t,v_l)\right)} \to 0$$

for almost all t and for all $v > v_l$. Therefore:

$$\begin{aligned}
\frac{1 - \bar{F}^n(t)}{1 - F^n(t,v_l)} &= \sum_v \mu_0(v)\left(\frac{1 - F^n(t,v)}{1 - F^n(t,v_l)}\right) \\
&= \mu_0(v_l) + \sum_{v > v_l} \mu_0(v)\left(\frac{1 - F^n(t,v)}{1 - F^n(t,v_l)}\right) \\
&\to \mu_0(v_l)
\end{aligned}$$

And therefore:

$$\begin{aligned}
0 &\leq \left(\sum_v \left(\frac{\mu_0(v)\left(1 - F^n(t,v)\right)}{1 - \bar{F}^n(t)}\right) \tilde{u}^n_{t,v}\right) - \tilde{u}^n_{t,v_l} \\
&= \left(\frac{1 - \bar{F}^n(t)}{1 - F^n(t,v_l)} \sum_v \left(\frac{\mu_0(v)\left(1 - F^n(t,v)\right)}{1 - F^n(t,v_l)}\right) \tilde{u}^n_{t,v}\right) - \tilde{u}^n_{t,v_l} \\
&= \mu_0(v)\left(\frac{1 - F^n_{v_l}(t)}{1 - \bar{F}^n(t)}\right) \tilde{u}^n_{t,v_l} \\
&\quad + \left(\frac{1 - \bar{F}^n(t)}{1 - F^n_{v_l}(t)}\right) \sum_{v > v_l} \left(\frac{\mu_0(v)\left(1 - F^n_v(t)\right)}{1 - F^n_{v_l}(t)}\right) \tilde{u}^n_{t,v} - \tilde{u}^n_{t,v_l} \\
&\to 0
\end{aligned}$$

by Theorem 11 parts 4, 6 and 8:

$$\tilde{u}_{t,v_l} = \int_t^\infty e^{-r(s-t)} \left(\lambda_{s,v_l} - \bar{\lambda}_s\right) \mathrm{d}s < 0$$

and:

$$\begin{aligned}
\sum_v \mu_0(v) \left(\frac{1-F^n(t,v)}{1-\bar{F}^n(t)}\right) \tilde{u}^n_{t,v} &= \int_t^\infty e^{-r(s-t)} \left(\sum_v \mu_0(v) \left(\frac{1-F^n(t,v)}{1-\bar{F}^n(t)}\right) \lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}s \\
&> \int_t^\infty e^{-r(s-t)} \left(\sum_v \mu_0(v) \left(\frac{1-F^n(s,v)}{1-\bar{F}^n(s)}\right) \lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}s \\
&= 0
\end{aligned}$$

and therefore:

$$0 \leq -\tilde{u}^n_{t,v_l} \leq \left(\sum_v \left(\frac{\mu_0(v)\left(1-F^n_v(t)\right)}{1-\bar{F}^n(t)}\right) \tilde{u}^n_{t,v}\right) - \tilde{u}^n_{t,v_l}$$

for all n. But this implies: $\lim_{n\to 0} \tilde{u}^n_{t,v_l} \to 0$ for almost all t. This means that: $w^n_{t,v_l} \to v_l$ for almost all t, and therefore $w^n_{t,v} \to \frac{v+v_l}{2}$ for all v. To extend to time 0, note first that $\sum_v \mu_0(v)\left(\lambda^n_{0,v}/\bar{\lambda}^n_0\right) = 1$ for all n and $\left(\lambda^n_{0,v}/\bar{\lambda}^n_0\right) \geq \frac{1}{2}$ for all v implies that $\exp \frac{1}{\kappa}\left(v - \kappa - w^n_{0,v} - \tilde{u}^n_{0,v}\right)$ converges to a strictly positive but finite number, and therefore $v - \kappa_n - w^n_{0,v} - \tilde{u}^n_{0,v} \to 0$. As such, we have: $w^n_{0,v} - w^n_{0,v_l} \to \frac{1}{2}(v - v_l)$ for all v. Note that for every $t > 0$:

$$\begin{aligned}
0 > \tilde{u}^n_{0,v_l} &= \kappa_n \int_0^t e^{-rs}\left(\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s + e^{-rt} u^n_{t,v_l} \\
&> \kappa_n \int_0^t e^{-rs}\left(\lambda^n_{s,v_l} - \lambda^n_{s,v_h}\right) \lambda^n_s \mathrm{d}s + e^{-rt} u^n_{t,v_l} \\
&= \left(\left(w^n_{0,v_h} - w^n_{0,v_l}\right) - e^{-rt}\left(w^n_{t,v_h} - w^n_{t,v_l}\right)\right) + e^{-rt} u^n_{t,v_l} \\
&\to -\left(1 - e^{-rt}\right)\left(\frac{v_h - v_l}{2}\right)
\end{aligned}$$

taking $t \to 0$ then gives $\tilde{u}^n_{0,v_l} \to 0$. But this implies $w^n_{0,v_l} \to v_l$, and therefore $w^n_{0,v} \to \frac{v+v_l}{2}$ for all v, which imples $\tilde{u}^n_{0,v} \to \frac{v-v_l}{2}$ since $w^n_{0,v} + \tilde{u}^n_{0,v} \to v$. The Theorem then follows from:

$$\begin{aligned}
\bar{U}^n_s &= \sum_v \mu_0(v)\, w^n_{0,v} \\
\bar{U}^n_b &= \sum_v \mu_0(v)\, \tilde{u}^n_{0,v}
\end{aligned}$$

for all n.